manuscripta math. 75, 97 - 102 (1992)

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ARITHMETIC GROUPS AND SALEM NUMBERS

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We show that the existence of a sequence of elements from cocompact torsion-free arithmetic subgroups of $SL(2,\mathbf{R})$ converging to the identity is equivalent to the density of Salem numbers in $[1,\infty)$.

1 Introduction

In this note we show that a natural question on arithmetic subgroups of $SL(2,\mathbf{R})$, is essentially equivalent to one studied by number theorists working on the set of algebraic integers called Salem numbers. More precisely, we show that an affirmative answer for either of the following two questions implies one for the other.

Question 1 : Does there exist a neighbourhood W of the identity in $SL(2,\mathbf{R})$ such that for every cocompact torsion-free arithmetic subgroup Γ, we have $\Gamma \bigcap W = \{e\}$?

Question 2 : Does there exist a number $\epsilon > 0$ such that any Salem number $\tau > 1 + \epsilon$?

We point out that, as of now, neither of the questions has been answered. Actually Question 2 is expected to have an affirmative answer. We note that

Question 1 has a negative answer if either the assumption of cocompactness or of arithmeticity is dropped. Cocompactness is necessary for otherwise there are unipotent elements, which can be conjugated to get close to the identity. Necessity of arithmeticity is forced since Thurston ([5], Ch.8) has constucted, for any cocompact lattice, a sequence of noncocompact lattices converging to it. We recall a theorem of Kazhdan-Margulis ([2]) which asserts that a neighbourhood W of the identity e can be so chosen that for all discrete subgroups Γ, some conjugate $g\Gamma g^{-1}$ intersects W in $\{e\}$. They also show that given a cocompact lattice Γ, there exists a neighbourhood W_Γ of e such that

$$g\Gamma g^{-1} \bigcap W_\Gamma = \{e\} \ \forall \, g \in SL(2,\mathbf{R})$$

Thus, Question 1 is a strengthening of this last assertion.

Question 2 is a particular case of a question of Lehmer ([3]) viz. whether there exists $\epsilon > 0$ such that, for all monic, noncyclotomic integral polynomials P, we have $M(P) := \prod_i \text{Max}\,(\mid \alpha_i \mid) > 1 + \epsilon$, where α_i are the roots of P. In fact, Lehmer picks out the Salem number $\tau \sim 1.176$ corresponding to the polynomial $X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$ as the smallest known $M(P)$. At present, this is still the best result known. If Question 2 has a negative answer, then the set of Salem numbers is dense in $[1, \infty)$ (See remark in section 3). From the work of Boyd ([1]), the set of limit points of the Salem numbers is expected to be much smaller and Question 2 is expected to have an affirmative answer.

I wish to record my thanks to Madhav Nori for his help and encouragement.

2 Arithmetic subgroups of $SL(2, \mathbf{R})$

Definition A discrete subgroup Γ of $SL(2,\mathbf{R})$ is said to be arithmetic, if there exists an algebraic $\mathbf{Q}$-group G such that $G(\mathbf{R}) \cong SL(2,\mathbf{R}) \times H$ where H is a compact group, and such that Γ is commensurable with the projection of $G(\mathbf{Z})$ to $SL(2,\mathbf{R})$.

Remark Cocompact arithmetic subgroups Γ of $SL(2,\mathbf{R})$ are of the form $\mathcal{O}^1$, for some order $\mathcal{O}$ of a quaternion division algebra D (see [6]). Moreover, the center K of D is totally real, $D \bigotimes K_{v_0} \cong M(2,\mathbf{R})$, and $\{v_1, ..., v_r\} \subseteq \mathrm{Ram}(D)$ where $\{v_0, ..., v_r\}$ is the set of archimedean places of K. Here $\mathrm{Ram}(D)$ is the set v of places of K such that $D \bigotimes K_v$ is a division algebra, and $\mathcal{O}^1$ denotes the elements in $\mathcal{O}$ of reduced norm 1.

3 Salem numbers

Definition A real algebraic integer $\tau > 1$ is a Salem number if its conjugates have absolute value ≤ 1 and there is atleast one of absolute value 1.

Remark It is easy to see (For e.g. [4]) that the conjugates of a Salem number τ are $\frac{1}{\tau}, \tau_1, \frac{1}{\tau_1}, ..., \tau_r, \frac{1}{\tau_r}$ where $\mid \tau_i \mid = 1$ for $1 \leq i \leq r$. Thus, τ is a unit in the ring of algebraic integers and the irreducible monic polynomial of τ is a reciprocal poynomial. Let T denote the set of Salem numbers. We note that $\tau \in T \Rightarrow \tau^k \in T \ \forall \ k \in \mathbf{N}$. So, if there exists $r_n = 1 + \epsilon_n \in T$ such that $\epsilon_n \to 0$, then $\forall \ \alpha > 0$, $(1+\epsilon_n)^{[\alpha/\epsilon_n]} \to e^\alpha$. Therefore, if 1 is a limit point of T, then T is dense in $[1, \infty)$. Actually, it is expected ([1]) that the closure of T is $S \bigcup T$. Here S is the (closed) set of Pisot-Vijayaraghavan numbers defined as the set of real algebraic integers $\theta > 1$ such that all other conjugates θ_i of θ have absolute value less than 1. Thus, Question 2 is expected to have an affirmative answer.

4 Equivalence

Let $\tau \in T$. Then, $\theta = \tau + \tau^{-1}$ is a totally real algebraic integer > 2 such that $\mid \theta_i \mid < 2$ where θ_i are the other conjugates of θ. Let us denote by K the splitting field of θ and by P the monic irreducible polynomial of θ. Let K_θ be the quadratic extension of K given by the polynomial $X^2 - \theta X + 1$ over K, and $\{v_0, ..., v_r\}$ be the archimedean places of K.

Lemma

(i) There exists a quaternion division algebra D over K such that $K_\theta \subseteq D$, $D \bigotimes K_{v_0} \cong M(2, \mathbf{R})$ and $\{v_1, ..., v_r\} \subseteq Ram(D)$.

(ii) There exists an order $\mathcal{O}$ of D and a nontorsion element x in $\mathcal{O}$ of reduced norm 1 *and reduced trace θ.*

Proof It is well-known (For e.g. see [6]) that, corresponding to any nonempty set S of noncomplex places of even cardinality, there exists a quaternion division algebra D over K with Ram$(D) = S$. Moreover, a quadratic extension L of K is contained in D if, and only if, $L \bigotimes K_v$ is a field $\forall\ v \in$ Ram(D). Since $X^2 - \theta_i X + 1$ has two complex conjugate roots for $1 \leq i \leq r$, we have $K_\theta \bigotimes K_{v_i} \cong \mathbf{C}$. Thus, assertion (i) follows.

To prove (ii) we just observe that τ is in the ring of integers of K_θ and is of norm 1 over K.

Now, we can prove the following result.

Theorem

Questions 1 and 2 have the same answer.

Proof By the remark in section 2, any element γ of a cocompact arithmetic subgroup of $SL(2,\mathbf{R})$ arises in the following manner. There is an order $\mathcal{O}$ in a quaternion division algebra D over a totally real number field K, i.e. $D = \frac{(a,b)}{K}$ where the notation means that there exist $a, b \in K^* \setminus (K^*)^2$ such that D is the K-algebra with a basis $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ with the multiplication

$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{k},\ \boldsymbol{i} \cdot \boldsymbol{j} = -\boldsymbol{j} \cdot \boldsymbol{i},\ \boldsymbol{i}^2 = a \cdot 1,\ \boldsymbol{j}^2 = b \cdot 1$$

D has a canonical involution σ which sends $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ to $-\boldsymbol{i}, -\boldsymbol{j}, -\boldsymbol{k}$ respectively. Moreover, if $v_0, \ldots, v_r$ are the archimedean places of K, then $D \bigotimes K_{v_0} \cong M(2,\mathbf{R})$ and $D \bigotimes K_{v_i}$ is a division algebra for $1 \leq i \leq r$. The norm form of D is the quadratic form $X_0^2 - aX_1^2 - bX_2^2 + abX_3^2$ in four variables over K. For $1 \leq i \leq r$, this form is anisotropic i.e. $a, b < 0$ in K_{v_i}. Any γ as above can be written as $\gamma = \gamma_0 + \gamma_1\ \boldsymbol{i} + \gamma_2\ \boldsymbol{j} + \gamma_3\ \boldsymbol{k}$ in D with $\gamma \in K$ and $\gamma.\sigma(\gamma) = \gamma_0^2 - a\gamma_1^2 - b\gamma_2^2 + ab\gamma_3^2 = 1$. The reduced trace of γ is $\mathrm{Tr}(\gamma) = \gamma + \sigma(\gamma) = 2\gamma_0$. But, in $K_{v_i} (1 \leq i \leq r)$, we have $\gamma_0^2 < N(\gamma) = 1$ i.e. $\mid \gamma_0 \mid < 2$, where $N(\gamma)$ denotes the reduced norm of γ. Now, since the elliptic elements in $SL(2,\mathbf{R})$ are of finite order and, by Godement criterion, no nontrivial unipotent elements exist in a cocompact lattice, it follows that, if γ is nontorsion, then it is necessarily hyperbolic. As the reduced trace map on D extends to the trace map on $M(2,\mathbf{R}) = D \bigotimes K_{v_0}$, it follows that $2\gamma_0 > 2$ in K_{v_0} if γ is nontorsion. Calling $2\gamma_0$ as θ for simplicity, we have shown that θ is a totally real algebraic integer such that $\theta > 2$ and all its other conjugates are in $(-2, 2)$. Further, if $\{\gamma_n\}$ is a sequence of nontorsion elements from cocompact arithmetic subgroups of $SL(2, \mathbf{R})$, converging to the identity, then the corresponding traces θ_n converge to 2. Writing $\theta_n = \tau_n + \tau_n^{-1}$ for some Salem numbers τ_n, we see that $\tau_n \to 1$. Conversely, if we have a sequence of Salem numbers $\tau_n \to 1$, then we consider the splitting fields K_n of $\theta_n = \tau_n + \tau_n^{-1}$ and the quadratic extensions of K_n given by the polynomials $X^2 - \theta_n X + 1$. By the above lemma, we can choose quaternion division algebras D_n over K_n, orders $\mathcal{O}_n$ in D_n, and nontorsion elements $x_n \in \mathcal{O}_n$ with reduced norm 1 and such that $\mathrm{Tr}(x_n) = \theta_n$. Since $\theta_n \to 2$, the characteristic polynomials of x_n converge to $X^2 - 2X + 1 = (X-1)^2$. But, since by Godement's criterion, there are no nontrivial unipotent elements in a cocompact lattice, we have $x_n \to 1$. This completes the proof of the theorem.

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(Received June 7, 1991;
in revised form December 30, 1991)