Reductive groups defined over an arbitrary field

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Throughout, G denotes a connected, reductive algebraic defined over an arbitrary field K. As we go along, we recall the definitions of various objects which have been studied in earlier lectures. References are sometimes made to particular sections of Springer's text.

Introduction

We have already seen the structure and classification of connected, reductive groups over algebraically closed fields. In a nutshell, here is a quick recall. There exist Borel subgroups (all of them are conjugate) and maximal tori (all of them are conjugate). The root system (or more generally, the root datum) of the group can be abstractly characterized and not only determines the group but the group itself can be built from an abstract root datum. For instance, if G is semisimple, T is a maximal torus, then the character group X of T has a finite generating subset Φ of roots (consisting of those nontrivial characters α which act on an additive one-parameter subgroup of G via α) satisfying :
there exists a map $\alpha \mapsto \alpha^\vee$ from Φ to the $\mathbf{Z}$-dual $X^\vee$ such that $< \alpha, \alpha^\vee >= 2$ and the reflection $\tau_\alpha : x \mapsto x- < x, \alpha^\vee > \alpha$ leaves Φ stable. The semisimple groups are completely classified by the pair (X, Φ); in particular, the classification (remarkably) does not depend on the (algebraically closed field) K one is working over. In between, one had a Bruhat decomposition, a description of all parabolic subgroups (closed subgroups containing some Borel subgroup) and a Levi decomposition for each parabolic subgroup. In what follows, we have a connected, reductive group defined over some arbitrary field K. The above model generalizes (thanks to Borel-Tits structure theory) to yield the analogous model where Borel subgroups are replaced by

minimal parabolic subgroups defined over K and maximal tori are replaced by maximal K-split tori. There are analogous Bruhat and Levi decompositions. The root datum over $\bar{K}$ - the algebraic closure of K - gets jacked up to an 'indexed' root datum called the Tits index. That is, one has an action of the Galois group of the separable algebraic closure of K on the root datum. Roughly speaking, each orbit of simple roots over $\bar{K}$ gives one simple K-root and the classification is akin to that of quadratic forms over an arbitrary field. There is an 'anisotropic kernel' similar to the anisotropic part of a quadratic form and the classification is done modulo this anisotropic kernel which can be described in most cases explicitly as a sort of unitary group of some σ-Hermitian form over a division algebra. Similar to the Dynkin diagram, one has the Tits index - a diagram from which one can determine all the corresponding objects like K-root system etc. Roughly speaking, one starts with the Dynkin diagram (over K_{sep}), and considers the orbits of the simple roots under the action of $\mathrm{Gal}(K_{sep}/K)$. Orbits of those simple roots which restrict nontrivially on a maximal K-split subtorus (that is give a K-root) are circled. Each such circled orbit gives one simple K-root. Also, the roots in an orbit are shown close by putting one below the other. Galois cohomology will have a role to play.

1 Levi subgroups, Bruhat decomposition over K

Theorem 1.
(i) *Let P, Q be nontrivial parabolic subgroups defined over K. Then, $P \cap Q$ is defined over K and contains a maximal K-split torus of G.*
(ii) *For any parabolic subgroup P defined over K, the unipotent radical $R_u(P)$ is automatically defined over K and K-split as well. Moreover, P contains Levi subgroups defined over K and any two Levi subgroups of P defined over K are conjugate by a unique element of $R_u(P)(K)$.*

Look back.
Recall certain subgroups $P(\mu)$ defined over K which arise as 'contracting' elements with respect to some cocharacter μ of G defined over K. That is, $P(\mu)$ consists of all those x for which the morphism $\phi : \mathbf{G}_m \to G; a \mapsto \mu(a)x\mu(a)^{-1}$ extends to a morphism $\tilde{\phi}$ on $\mathbf{A}^1$. One writes $lim_{a \to 0}\mu(a)x\mu(a)^{-1} = \tilde{\phi}(0)$. For instance, if $G = GL_n$ and $\mu : a \mapsto diag(a, 1, 1, \cdots, 1)$, then $P(\mu)$ consists of all matrices X in GL_n with $x_{21} = x_{31} = \cdots = x_{n1} = 0$. We have seen earlier (while studying reductive groups over algebraically closed fields) that these are parabolic subgroups. Indeed, $P(\mu)$ contains the Borel B which corresponds to a set of positive roots such which contains all roots α with $< \alpha, \mu >> 0$. To obtain such a positive system, one chooses some λ close to μ in the real vector space V generated by the cocharacters such that $(\alpha, \lambda) \neq 0$ for any root α - this is possible because there are only finitely many roots and the subspaces of V orthogonal to the roots cannot make up the whole of V. Then, one takes $R^+ = \{\alpha : (\alpha, \lambda) > 0\}$.

Further, any parabolic subgroup of G (over the algebraic closure of K) arises as above by means of some cocharacter μ of G. In fact, such a parabolic containing the above B, must be of the form P_I where I is the set of all simple roots orthogonal to μ; that is, $P_I = P(\mu)$. It was also proved (§ 13.4) that when μ is defined over K, then so are $P(\mu)$ and its reductive subgroup $Z(\mu) :=$ centralizer of $Im(\mu)$ (note that centralizers of tori in reductive groups are reductive). The definability over K mentioned above was checked directly for GL_n first and deduced for general G by computing the Lie algebra and using the fact that the intersection of two K-groups is defined over K if the Lie algebra of the intersection is the intersection of the Lie algebras. The fact that parabolic subgroups defined over K are precisely the K-groups $P(\mu)$ arising from cocharacters μ defined over K was proved in earlier lectures - this used the fact over the algebraic closure along with the action of the Galois group on the character and cocharacter groups of a maximal torus in P defined over K. He also proved in the same lemma 15.1.2 that the unipotent radical $R_u(P)$ of a parabolic subgroup P defined over K is a connected K-group which is K-split. Note that if $P = P(\mu)$, then $R_u(P)$ consists of all those elements $x \in P$ for which the map $a \mapsto \mu(a)x$ for $a \in \mathbf{G}_m$ and $0 \mapsto 1$ is a morphism from $\mathbf{A}^1$ to G. The result (theorem 13.4.2) quoted above shows that the product map $Z(\mu) \times R_u(P) \to P(\mu)$ is a K-isomorphism of varieties.

Proof of theorem 1.
First, we observe that each minimal parabolic subgroup P_0 defined over K contains a maximal K-split torus of G. Indeed, due to the above identification of P_0 with a $P(\mu)$, it follows that for some maximal K-split torus S of G and a system of positive roots in the K-root system corresponding to S, the minimal parabolic $P(\mu)$ is generated by $Z(S)$ and all $U_{(a)}$ with a, a positive K-root. Now, if P, Q are parabolic subgroups defined over K, then we may deduce that the intersection $P \cap Q$ is again defined over K provided we can check that dim $L(P \cap Q) =$ dim $L(P) \cap L(Q))$. But, as $P \cap Q$ certainly contains a maximal torus of G, the above dimensions match (exercise 8.1.12). Thus, $P \cap Q$ is defined over K. Now, we have observed above that $R_u(P)$ is also defined over K. So, $R := (P \cap Q)R_u(P)$ is a parabolic subgroup which is also defined over K (as $R_u(P)$ and $P \cap Q$ are). By what we proved above, this R must contain a maximal K-split torus; hence, so does the quotient group $R/R_u(P)$. This proves (i) of the theorem.
To prove (ii), note that we have already recalled above for any parabolic subgroup P defined over K that $R_u(P)$ is automatically defined over K and K-split. Also, identifying P with $P(\mu)$, the reductive group $Z(\mu)$ is defined over K as well and the product map $Z(\mu) \times R_u(P(\mu)) \to P(\mu)$ is a K-isomorphism. Thus, $Z(\mu)$ is a Levi subgroup of P by definition (and is defined over K). If L is any *other* Levi subgroup of P defined over K, consider any maximal torus T of G defined over K which is contained in L (this always exists as seen by looking at the subgroup generated by the centralizers of semisimple elements in the Lie algebra - see 13.3.6).

Now, the key point is that T determines the Levi subgroup of P; that is, it does not depend on the Borel subgroup containing T that it contains. This is because any two Borels containing T are conjugate by $W(T)$. From this, it follows that L must also be of the form $Z(\lambda)$ for some cocharacter λ of T. Now, $Z(L)^0$ is a torus contained in the maximal torus T of L. As any torus splits over K_{sep}, the separable closure of K, both the K-torus T and the K_{sep}-torus $Z(L)^0$ are split over K_{sep}. The latter is defined over K too because $Z(L)^0(K_{sep})$ is invariant under the Galois group of K_{sep} over K. Now, by definition, $L = Z(\lambda) = C_G(Im\lambda)$ which is, therefore, a fortiori, equal to $C_G(Z(L)^0)$. Using conjugacy of maximal tori in $P = P(\mu)$ and the uniqueness of the Levi subgroup containing a maximal torus, we have $x \in R_u(P)$ with $Z(\lambda) = L = xZ(\mu)x^{-1}$. If $u \in R_u(P)$ normalizes $Z(\mu)$, it centralizes it because the product map $Z(\mu)\times R_u(P) \to P(\mu)$ is an isomorphism of varieties as we saw. In other words, the above element $x \in R_u(P)$ is unique. As transporters between K-subgroups are defined over K, the element x must be in $G(K)$. The proof is complete.

Bruhat decomposition for $G(K)$
Assume that G contains a proper parabolic subgroup defined over K (equivalently, G contains a noncentral K-split torus; that is, G is K-isotropic as defined in the next section). Let P be a minimal parabolic subgroup defined over K. Let $S \subset P$ be a maximal K-split torus. Consider the K-Weyl group $W_K = N_G(S)/C_G(S)$ - its elements w can be represented by elements $\bar{w}$ of $N_G(S)(K)$ (this essentially follows from the Borel fixed point theorem). Then $G(K)$ is the disjoint union of $P(K)\bar{w}P(K)$ as w varies over W_K.
Proof. Let us first show that $G(K)$ can be expressed as such a union of double cosets; let $g \in G(K)$. Let $T \subset P \cap gPg^{-1}$ be a maximal K-split torus; then $T = x^{-1}Sx$ for some $x \in P(K)$. So, $g^{-1}x^{-1}Sxg = g^{-1}Tg \subset P$. Again, by the $P(K)$-conjugacy of the maximal K-split tori S and $g^{-1}x^{-1}Sxg$, we have $y \in P(K)$ such that $y^{-1}g^{-1}x^{-1}Sxgy = S$; that is, $xgy \in N_G(S)(K)$. In other words, $g = x^{-1}(xgy)y^{-1} = x^{-1}x\bar{g}yy^{-1} \in P(K)x\bar{g}yP(K)$.
To prove uniqueness of the W_K-component, assume

$$P(K)\bar{w}P(K) = P(K)\bar{z}P(K).$$

Now, if T is a maximal torus in P which is defined over K and contains S, then W_K can be considered as a subgroup of $W = N_G(T)/C_G(T)$ - it is generated by the simple reflections with respect to the roots in the K-root system (that is those roots which are nontrivial characters on S). Moreover, in this manner, the elements of W_K normalize the elements of the Weyl group W_L of the Levi subgroup $L = C_G(S)$ of P. Thus, by the Bruhat decomposition of G, we obtain $z \in W_L w W_L = wW_L$ which gives $z^{-1}w \in W_L \cap W_K$. Since $S \subset Z(L)$, we get $z^{-1}w = 1$ in W_K.

2 Indexed root data

Definitions
G is *K-split* if it contains a maximal torus which is K-split. If G contains a noncentral K-split torus, it is said to be *K-isotropic*; otherwise, it is *K-anisotropic*. G is said to be quasi-split over K if there exist a maximal K-split torus S and a maximal torus T defined over K and containing S such that nontrivial roots of T restrict nontrivially on S; this happens iff $C_G(S) = T$ as follows from the fact that global and local centralizers correspond (5.4.7). Clearly, K-split groups are quasi-split over K. These definitions come from the theory of quadratic forms - for $G = SO(f)$ for a nondegenerate quadratic form over K, this means that G is K-isotropic iff f represents zero over K, and G is K-split iff f is a totally isotropic (that is, an orthogonal sum of hyperbolic planes).

Look back.
As we saw, the parabolic subgroups defined over K of a connected reductive group defined over K are the groups $P(\lambda)$ where λ is a cocharacter of G defined over K. Note that $P = P(\lambda) = G$ iff $R_u(P) = \{1\}$ and so $Z(\lambda) = G$. Thus, $Im(\lambda) \leq Z(G)$ which means that noncentral K-split tori exist, if there are proper parabolic subgroups defined over K. The converse is also true because if there are noncentral K-split tori, there exists λ defined over K such that $Z(\lambda) \neq G$ and so, $P(\lambda) \neq G$ as seen by looking at its Lie algebra (13.4.2 (ii)).

Proposition.
G is quasi-split over K iff it contains a Borel subgroup defined over K.
Proof.
If G is quasi-split over K, then there is a maximal K-split torus S whose centralizer $T = C_G(S)$ is a maximal torus of G and is defined over K. Choose a cocharacter λ of S such that $T = C_G(S) = Z(\lambda)$. Thus, $P(\lambda)$ (being isomorphic to $T \times R_u(P)$ under the product map) is a solvable group defined over K; that is, it is a Borel subgroup.
Conversely, if G contains a Borel subgroup B defined over K, this subgroup

must be of the form $B = P(\lambda)$ for some cocharacter λ of G defined over K. Also, $Z(\lambda)$ being a Levi part of $P(\lambda)$ which happens to be solvable, we have that $Z(\lambda)$ is a torus defined over K. Now, $\mathrm{Im}(\lambda)$ is a K-split torus. If S is a maximal K-split torus containing $\mathrm{Im}(\lambda)$, then $C_G(S) = Z(\lambda)$, which is, on the one hand, a torus defined over K and, on the other hand a Levi subgroup of $P(\lambda)$ containing a maximal torus of G defined over K. This means $Z(\lambda)$ is a maximal torus of G and thus G is quasi-split over K.

Prelude.
From the root datum of G (which is an object over K_{sep}), we would like to obtain an object over K which keeps track of the action of the Galois group $\mathrm{Gal}(K_{sep}/K)$ on the various ingredients of the former. After defining such an object, we will show that it breaks up into the corresponding objects for semisimple groups and tori. In the next sections, we will show that these objects can be characterized abstractly just as we did for abstract root data.

Look back.
Let G, S, T be as above. Let $(X, \Phi, X^\vee, \Phi^\vee)$ be the root datum of G corresponding to T. Fix a simple system Δ of Φ. Let $\Delta_0 \subset \Delta$ be the subset of those simple roots which restrict to the trivial map on S. We consider a certain action τ of $\Gamma = Gal(K_{sep}/K)$ on the set Δ which gives a diagram automorphism as follows. Looking at the map $\pi : \Phi \to \Phi_K$ given by restriction to S, it can be seen (15.5.1(ii)) that for any fixed system Φ_k^+ of positive K-roots in Φ_K, there is a system Φ^+ of positive roots in Φ such that π maps exactly the positives to the positives. Fix such a system Φ^+. If $\gamma \in \Gamma$, then $\gamma.\Phi^+$ is also a system of positive roots in Φ. Therefore, there exists an element w_γ in W such that $w_\gamma(\gamma.\Phi^+) = \Phi^+$.

Definitions.
Define, for $\gamma \in \Gamma$, the permutation $\tau(\gamma)$ on Δ given by $\alpha \mapsto w_\gamma(\gamma.\alpha)$. This is a continuous action. One calls (Δ, Δ_0, τ) the *Tits index* of G relative to K, S, T. It turns out (see 15.5.5) that the index does not depend on the choices of S, T. The tuple $(X, \Phi, X^\vee, \Phi^\vee, D_0, \tau)$ is called the *indexed root datum* of G over K.
We now define an abstract indexed root system. Firstly, let $(X, R, X^\vee, R^\vee)$ be an abstract root datum. If D is a basis of R, R^+ is the corresponding positive system, and $D^\vee$ is a basis of $R^\vee$ corresponding to $(R^+)^\vee$, one calls $(X, D, X^\vee, D^\vee)$ a *based root datum.* An *abstract indexed root datum* is a 6-

tuple $(X, D, X^\vee, D^\vee, D_0, \tau)$ where $(X, D, X^\vee, D^\vee)$ is a based root datum, D_0 is a subset of D and τ is a continuous homomorphism from $\Gamma = Gal(K_{sep}/K)$ to $Aut(X)$ stabilizing the subsets D, D_0.

Look back.
Recall that an isogeny θ from G to G_1 is a surjective homomorphism with finite kernel. If T is a maximal torus of G, $T_1 = \theta(T)$ and $\Psi = (X, R, X^\vee, R^\vee)$ and $\Psi_1 = (X_1, R_1, X_1^\vee, R_1^\vee)$ are root data of G and G_1 corresponding to T and T_1, then the induced homomorphism $f(\theta) : X_1 \to X$ is an isomorphism onto a subgroup of finite index. Let $(u_\alpha)_{\alpha \in R}, (u_{\alpha_1})_{\alpha_1 \in R_1}$ be realizations of R, R_1 in G, G_1 respectively. Now, there is a bijection $b : R \to R_1$ such that $\theta(U_\alpha) = U_{b(\alpha)}$. Writing, for each fixed $\alpha \in R$, $\theta(u_\alpha(x)) = u_{b(\alpha)}(h(x))$ for all $x \in K_{sep}$ where h is a polynomial such that $h(\alpha(t)x) = b(\alpha(\theta(t)))h(x)$ for all x - hence h is homogeneous. As h is also evidently additive, there exists a power $q(\alpha)$ of the characteristic exponent of K such that h is the polynomial $T^{q(\alpha)}$ upto a constant multiple. Therefore, we have

$$f(b(\alpha)) = q(\alpha)\alpha \ , \ f^\vee(\alpha^\vee) = q(\alpha)(b(\alpha))^\vee.$$

Definitions.
With $\theta : G \to G_1$ an isogeny and the notations as above, one calls θ a *central isogeny* if $q(\alpha) = 1$ for all α. This is always the case if K has characteristic 0. If θ is also defined over K, it is called a *central K-isogeny.* Let S be a maximal K-split torus of G and $S_1 = \theta(S)$; then S_1 is a maximal K-split torus of G_1. If we view X_1 as a subgroup of X by means of the homomorphism f (which is viewed as the inclusion map). Then $R_1 = R$, $b =$ identity. Look at the indexed root datum of G where we change notation and write it as $(X, D, X^\vee, D^\vee, D_0, \tau)$. Then, the corresponding indexed root datum of G_1 is $(X_1, D, X_1^\vee, D^\vee, D_0, \tau)$. The Galois group Γ which acts continuously on X, stabilizes the subgroup X_1.

Lemma.
Let G, T, X be as before and let X_1 be any subgroup of finite index in X which is stabilized by Γ. Then, there exists a connected reductive group G_1, and a central K-isogeny $\theta : G \to G_1$ such that X_1 arises as above for G_1.
Proof.
Now there is a K-torus T_1 with $X(T_1) = X_1$ and a K-isogeny $\phi : T \to T_1$ such that θ induces the inclusion of X_1 in X. If we prove the lemma in the

case when X/X_1 is an elementary abelian p-group for an arbitrary prime p, then the general case follows by induction. When p is not the characteristic of K, Ker(ϕ) is a finite subgroup A of $T(K_{sep})$ and $A \cong X/X_1$. As $A =$ Ker (ϕ), it is Γ-stable, and hence defined over K (being closed). Being a finite normal subgroup of G, it is central. Then $G_1 = G/A$ with the natural map, works. If p equals the characteristic of K, the Lie algebra $\mathbf{a}$ =Ker ($d\phi$) is a p-subalgebra of the p-Lie algebra Lie(T). As it is defined over K and centralized by Ad (G), there exists a quotient $G/\mathbf{a}$ and a corresponding quotient homomorphism (see 12.2.4). Taking G_1 to be this quotient, we are done.

Corollary.
The based root data of connected reductive K-groups G can be determined from the corresponding data for tori and for semisimple groups.
Proof. $[G, G]$ is a connected semisimple K-group and the radical $R(G)$ is a K-torus. The product map $[G, G] \times R(G) \to G$ is a central K-isogeny.

Behaviour of root data under field extensions

Our groups G over K will split over certain finite extensions L; so, it is useful to have a way of passing from a group over L to one over K. Let L/K be a finite extension. If G_1 is a connected, reductive L-group, one can apply Weil's restriction of scalars (this was done by Weil for the separable case and by Oesterle in general) to get a reductive group over K. This is denoted by $R_{L/K}G_1$. There exists a surjective L-homomorphism $\pi : R_{L/K}G_1 \to G_1$ with the universal property that for any K-group H and a L-homomorphism $\phi : G_1 \to H$, there is a unique K-homomorphism $\psi : H \to R_{L/K}G_1$ such that $\phi = \pi \circ \psi$. More than the definition or the proof of the existence using a universal property, it is useful to know this group through its important property : $(R_{L/K}G_1)(A) = G_1(L \otimes_K A)$ for each K-algebra A. If $\Gamma_1 = Gal(K_{sep}/L)$, then Γ/Γ_1 can be identified with the set Σ of K-embeddings of L in K_{sep}. In fact, $G := R_{L/K}G_1$ is isomorphic to the product G_1^{Σ} and is, therefore, a connected, reductive K-group. Note also that dim $G = [L : K]$dim G_1. If S_1 is a maximal L-split torus, and T_1 is a maximal torus of G_1 defined over L, then $T = R_{L/K}T_1$ is a maximal torus of G defined over K and contains a K-split torus $S :=$ the maximal K-split subtorus of $R_{L/K}S_1$.

Lemma.

With the above notations, S is a maximal K-split torus of G and dim $S =$ dim S_1. Moreover, if $(X_1, D_1, X_1^\vee, D_1^\vee, (D_1)_0, \tau_1)$ is an indexed root datum of G_1, then the corresponding indexed root datum of G satisfies $X = X_1^\Sigma, X^\vee = (X_1^\vee)^\Sigma$ etc.

Proof.
As S_1 is a L-split torus $\mathbf{G}_m^n$, the maximal K-split subtorus S of $R_{L/K}S_1 = R_{L/K}\mathbf{G}_m^n$ is n (see 13.1.5); that is, dim $S =$ dim S_1. Also, $\pi(S) = S_1$ where π is the canonical L-homomorphism from $G = R_{L/K}G_1$ to G_1. If S' is a K-split torus in G containing S, then $\pi(S')$ is an L-split torus of G_1 containing S_1 and, thus, equals it. If $S' \neq S$, then Ker π would contain a nontrivial subtorus of S' defined over K (as Γ acts trivially). But the universal property implies (see 12.4.3) that Ker π cannot contain any nontrivial, closed normal K-subgroups. Thus, $S = S'$; that is, S is maximal K-split torus of G. The last assertions follow from generalities (see 11.4.22) on restriction of scalars.

3 Existence and uniqueness of root data for split groups

Recall that corresponding to any K-root α, we defined a K-split unipotent group $U_{(a)}$ whose Lie algebra is the sum of all weight spaces in Lie (G) whose weights are positive integral multiples of α (the latter are either just α or $\{\alpha, 2\alpha\}$). If G is K-split, then $U_{(\alpha)} = U_\alpha$ for all roots $\alpha \in R$ and these are defined over K. One can choose a maximal torus T which is K-split and one can choose realizations $(u_\alpha)_{\alpha \in R}$ of R in G such that the morphisms u_α are all defined over K. If R^+, D are fixed and B is the corresponding Borel subgroup, it is defined over K as well. The indexed root datum is of the form $(X, D, X^\vee, D^\vee, \emptyset, id)$ and is, thus, determined by the based root datum $(X, D, X^\vee, D^\vee)$ and in turn, by B and T. One usually denotes this by $\Psi(G, B, T, K)$. If G_1 is another K-split connected, reductive group, , then a K-isomorphism θ from G onto G_1 maps the based root datum $\Psi(G, B, T, K)$ isomorphically to the based root datum $\Psi(G_1, \theta(B), \theta(T), K)$ - isomorphism of based root data can be naturally defined. The following result shows that the converse is true (that is, this is a version of the uniqueness theorem over K).

Uniqueness theorem for K-split groups.

Let G, G_1 be K-split connected, reductive groups and $\Psi = \Psi(G,B,T,K)$, $\Psi_1 = \Psi(G_1,B_1,T_1,K)$ be their based root data. Let $f : \Psi_1 \to \Psi$ be an isomorphism of based root data. Then, there exists a K-isomorphism $\theta : G \to G_1$ with $\theta(T) = T_1, \theta(B) = B_1$ such that f is induced by θ. Moreover, the K-isomorphism θ with these properties is unique up to conjugation by an(y) element t of T with $\alpha(t) \in K$ for all $\alpha \in D$.

Proof.

The existence of such a θ has been proved over K_{sep}. That is, if $(u_\alpha)_{\alpha \in R}$ and $(u_{\alpha_1})_{\alpha_1 \in R_1}$ are realizations defined over K, then there exists $\theta : G \to G_1$ such that $\theta \circ u_\alpha = u_{f^{-1}(\alpha)}$ for all $\alpha \in R$. We shall prove that θ is actually defined over K. Now, since u_α's are defined over K, the Weyl group (of (G,T)) elements

$$n_\alpha := u_\alpha(1)u_{-\alpha}(1)^{-1}u_\alpha(1) \in G(K).$$

Thus, $W := W(G,T)$ has representatives in $N_G(T) \cap G(K)$. Let $w = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_r} \in W$ be a reduced expression. Then, the element $\bar{w} := n_{\alpha_1}\cdots n_{\alpha_r} \in G(K)$ is uniquely determined by w (property of the root system as proved in 9.3.2). Now $\theta(\bar{w}) \in G_1(K)$. So, the translates $\bar{w}Bw_0B$ of the big cell are also defined over K for various w in W. As these cover G, and θ restricted to Bw_0B is defined over K, it follows that θ itself is defined over K. To prove the extent of uniqueness of θ, consider any other θ' and get $t \in T$ with $\theta'(g) = \theta(tgt^{-1})$ for all $g \in G$. As Int (t) is defined over K, and as it acts on the K-group U_α by $\alpha(t)$, it follows that $\alpha(t) \in K$ for all α. Of course, t could be arbitrary with $\alpha(t) \in K$ for all $\alpha \in R$ because, in this case, Int (t) restricted to translates $\bar{w}Bw_0B$ of the big cell, is defined over K and, hence it is defined over K as a map on the whole of G.

Existence theorem for the K-split case

Let $\Psi = (X, D, X^\vee, D^\vee)$ be a based root datum. Then, there exists a connected, reductive K-split group G and a Borel B over K and a maximal torus $T \subset B$ which is K-split, such that $\Psi = \Psi(G,B,T,K)$.

Proof. The proof is exactly that of the existence theorem over K_{sep}.

The adjoint group of a K-split group

Let G, B, T be as above; that is, G is K-split by assumption. Look at the subgroup $Q = < D > \leq X$. Then the dual $Q^\vee$ can be identified with $X^\vee/Q^\perp$ where $Q^\perp$ is the annihilator of Q in $X^\vee$ under the pairing between X and $X^\vee$. Now, $G/Z(G)^0$ is a semisimple K-group whose character group is the rational closure $\tilde{Q} := \{x \in X : \mathbf{Z}x \cap Q \neq \{0\}\}$ of Q in X (see 8.1.8). Recall also that the rational closure is "double perp" i.e., $\tilde{Q} = (Q^{\perp})^{\perp}$. Let us see what its based root datum is. We approach this using the existence theorem above. Now, the subtorus of T generated by the images of $\alpha^{\vee}$ is maximal in $[G,G]$, has character group $\cong X/(Q^{\vee})^{\perp}$ and cocharacter group $\widetilde{Q^{\vee}})$, one may identify $D^{\vee}$ with its image in $Q^{\vee}$. By the existence theorem above, there is a K-split connected, semisimple group with based root datum $(Q, D, Q^{\vee}, D^{\vee})$ - this group is denoted by G^{ad} and is called the *adjoint group of* G. Note that semisimplicity of G^{ad} is a consequence of the fact that $Q =< D >$. Let us observe the relationship of the adjoint group with $G/Z(G)^0$ is that there is a K-isogeny $G/Z(G)^0 \to G^{ad}$. Put $\pi : G \to G/Z(G)^0 \to G^{ad}$; this is a K-homomorphism. Then, $\pi(T)$ is a maximal torus in G^{ad} defined over K and $\pi \circ u_{\alpha} = \widetilde{u_{\alpha}}$ is a realization of R in G^{ad} where $(u_{\alpha})_{\alpha \in R}$ is a realization of R in G defined over K.

Lemma.
(i) $\tilde{T} := \{t \in T(\bar{K}) : \alpha(t) \in K \ \forall \ \alpha \in D\}$ *normalizes the group* $G(K)$.
(ii) $\pi(T)(K) = \pi(\tilde{T})$.
(iii) $G^{ad}(K)$ *is generated by* $\pi(G)(K)$ *and* $\pi(\tilde{T})$.
Proof. The first statement follows because $T(K)$ as well as all $U_{\alpha}(K)$ are clearly normalized by $\tilde{T}$. The second statement is a consequence of the fact that π is a surjective homomorphism from T and Q is the character group of $\pi(T)$. Finally, the last assertion follows from applying Bruhat decomposition for G^{ad}.

Lemma.
If (Int G)(K) is the group of inner automorphisms of G which are defined over K, then there is an isomorphism $\Psi(K)$: *(Int G)(K)* $\to G^{ad}(K)$.
Proof.
Fix a Borel B of G and a maximal torus $T \subset B$ both defined over K. Let $\sigma \in$(Int G)(K); then $\sigma(B) = gBg^{-1}$. As any two maximal tori over K are conjugate in $G(K)$, we may assume that $\sigma(T) = gTg^{-1}$. By the normalizer theorem for Borels and the fact that normalizers of tori in connected solvable groups are centralizers, it follows that $\sigma =$Int (gt) for some $t \in T$. As σ,Int g are both defined over K, so is Int t. Looking at its action on the various U_{α}'s, we get $t \in \tilde{T}$. Further, σ determines the pair (g,t) up to the transformations $(g,t) \mapsto (gt_1^{-1}, t_1 t)$ for $t_1 \in T(K)$. Therefore, by (ii) above, $\Psi(\sigma) := \pi(gt)$

is well-defined and lies in $G^{ad}(K)$. If $\sigma' =$Int $(g't')$ is another element, then $\sigma\sigma' =$Int $(gtg't') =$ Int$(g(tg't^{-1})tt')$ means

$$\Psi(\sigma\sigma') = \pi(g(tg't^{-1})tt') = \pi(gt)\pi(g't') = \Psi(\sigma)\Psi(g't')$$

since $tg't^{-1} \in G(K)$ by (i) above. If $\sigma =$Int $gt \in Ker(\Psi)$, then $gt \in T$. So, $g \in T$ and thus we may take $g = e$. So $\alpha(t) = 1$ for each root α means $t \in Z(G)$. Hence $\sigma =$Int $t =$id. Finally, (iii) above shows surjectivity of Ψ.

Diagram automorphisms as automorphisms of G

Let G be a connected reductive K-split group as above. Let T be a maximal torus which is K-split and let B be a Borel subgroup containing T. Also, R, D as before, are the corresponding roots and simple roots respectively. Consider the Dynkin diagram $\mathcal{D}$ defined by D and the corresponding (finite) group A of its automorphisms. This finite group has a subgroup A_0 which consists of those elements which leave B, T invariant and fix $R(G) = Z(G)^0$ pointwise. If G is an adjoint group, then since D is a basis of X, we have $A = A_0$. If G is semisimple and simply-connected, then also one has $A_0 = A$ but, in general, they can be different. For instance, when G is semisimple of type D_{2n} with $n \geq 2$, one can choose X such that $A_0 \neq A$.

Lemma.
Assume that G is semisimple (in addition to being K-split as before). We have the following :
(i) *Corresponding to any $\sigma \in A_0$, there is a unique K-automorphism $\tilde{\sigma}$ of G satisfying*

$$\tilde{\sigma}(u_\alpha(x)) = u_{\sigma(\alpha)}(x) \;\; \forall \;\; \alpha \in D, x \in K.$$

As a consequence, $\sigma \mapsto \tilde{\sigma}$ is a homomorphism from A_0 to $Aut(G)(K)$.
(ii) *Each K-automorphism a of G fixing the center is of the form $a' \circ \tilde{\sigma}$ for a unique $a' \in Int(G)(K)$ and $\sigma \in A_0$.*
Proof.
By the uniqueness theorem in the split case, we have that each $\sigma \in A_0$ gives rise to constants c_α's in K^* such that $u_\alpha(x) \mapsto u_{\sigma(\alpha)}(c_\alpha x)$ is a K-automorphism. We may choose a suitable $t \in \tilde{T} := \{t \in T(\bar{K}) : \alpha(t) \in$

$K \; \forall \; \alpha \in D\}$ so that when we conjugate the above automorphism by t, the new c_α's are all 1; call this $\tilde{\sigma}$. The uniqueness of $\tilde{\sigma}$ with the property

$$\tilde{\sigma}(u_\alpha(x)) = u_{\sigma(\alpha)}(x) \;\; \forall \;\; \alpha \in D, x \in K$$

follows from the fact that U_α's generate G and each $\tilde{\sigma} \circ u_\alpha$ is uniquely determined in U_α (corresponding to any nontrivial element $u \in U_\alpha$, there is a unique nontrivial element $u' \in U_{-\alpha}$ such that $uu'u \in N_G(T)$ as we saw in 8.1.4). This proves (i).
To prove (ii), let B be a fixed Borel containing T and defined over K. Call D to be the basis of R determined by B. Exactly as in the proof of the previous lemma, we see that there exists $g \in G(K)$ with $Int(g) \circ a$ is an automorphism fixing both B and T; call this a'. Thus, again there is $t \in \tilde{T} := \{t \in T(\bar{K}) : \alpha(t) \in K \; \forall \; \alpha \in D\}$ so that $Int(t) \circ a' = \tilde{\sigma}$ for some $\sigma \in A_0$.

Remarks.
Let G be semisimple, K-split as in the last lemma. The lemma gives us (when applied for the adjoint group of G) that A_0 acts on G^{ad} as a group of K-automorphisms. Denoting by $Aut(G)$ to be the corresponding semidirect product $A_0 \propto G^{ad}$, it is clear that $Aut(G)$ is a K-algebraic group with $(Aut(G))^0 = G^{ad}$. The part (ii) of the lemma shows that $Aut(G)(K)$ is exactly the group of K-automorphisms of G. We write $Int(G)$ for G^{ad} from now onwards and call it the group of algebraic inner K-automorphisms of G. It should be noted that for K-tori of dimension bigger than 1, the group of automorphims cannot have the structure of an algebraic group.

4 Uniqueness theorem in general

We saw that the based root datum for a connected, reductive group G with $B \geq T$ as before, can be determined from those for the semisimple group $[G, G]$ and the torus $R(G)$ by means of the central isogeny $[G, G] \times R(G) \to G$ given by the product map. Let us recall some finer details here. The radical $R(G)$ is a K-subtorus of T. As it is the intersection of Ker α for all the various roots α, it follows that its character group is isomorphic to $X/tildeQ$ where $\tilde{Q}/Q$ is the torsion subgroup of X/Q. Here, of course, Q is the subgroup of X spanned by R. Thus, $R(G)$ is generated by the various Im y, as y varies in its cocharacter group $Q^\perp = \{y \in X^\vee :< Q, y >= 0\}$. The subgroup

T_1 of T generated by Im $\alpha^\vee$ as α varies in R, is a maximal torus of $[G,G]$ and $X^*(T_1) \cong X/(Q^\vee)^\perp$ and $X_*(T_1) \cong \tilde{Q}^\vee$ (see 8.1.8). The root datum $(X^*(T_1), R, X_*(T_1), R^\vee)$ of $[G,G]$ is therefore $(X/(Q^\vee)^\perp, R, \tilde{Q}^\vee, R^\vee)$. Thus, the root datum of the product $[G,G] \times R(G)$ with respect to the maximal torus $T_1 \times R(G)$ is

$$(X/(Q^\vee)^\perp \oplus X/\tilde{Q}, (R, \{0\}), \tilde{Q}^\vee \oplus Q^\perp, (R^\vee, \{0\})).$$

The product map to G induces the homomorphism of character groups $X \to X/(Q^\vee)^\perp \oplus X/\tilde{Q}$ whose image consists of pairs of cosets $(y + (Q^\vee)^\perp, z + \tilde{Q})$ with $y - z \in (Q^\vee)^\perp \oplus \tilde{Q}$.

Now, in recalling these, we have not used the fact that G is defined over K. Using this, we have that the above maximal torus T_1 of $[G,G]$ obtained from T which was a fixed maximal torus of G defined over K is also defined over K. The maximal K-split subtorus S_1 of T_1 must be then a subtorus of S. Thus, S_1 is a maximal K-split torus of $[G,G]$ as seen by going over to $G/R(G)$. Now, as we saw with S,T for G, we see that S_1, T_1 determine an indexed root datum of $[G,G]$, viz., $(X/(Q^\vee)^\perp, D, \tilde{Q}^\vee, D^\vee, D_0, \tau)$ (the sets R, D as well as D_0 are the same as for G). The centralizer $H = C_{[G,G]}(S_1)$ is generated by T_1 and all those U_α's for which α runs through the root system R_{D_0} with basis D_0. Note that H is K-anisotropic; it is called the *anisotropic kernel* of G over K. Now $T = T_1 R(G)$ and the character groups X, X_1, Y of these three tori $T, T_1, R(G)$ are Γ-modules and X is of finite index in $X_1 \oplus Y$. *Our aim now is to abstractly characterize these properties and that would help in formulating and proving the uniqueness theorem.*

Now, we know as recalled from 8.1.8, that $X_1 = X/(Q^\vee)^\perp$. Also, the indexed root datum of H is $(X_1, D_0, X_1^\vee, D_0^\vee, D_0, \tau)$. In general, let us consider 3-tuples (Ψ, H, C) where $\Psi = (X, D, X^\vee, D^\vee, D_0, \tau)$ is an abstract indexed root datum, H is *any* connected reductive K-group, and C is a K-torus. Let us define such a 3-tuple to be *admissible* if H has an indexed root datum $(X_1, D_0, X_1^\vee, D_0^\vee, D_0, \tau)$ such that X is a Γ-stable subgroup of finite index in $X_1 \oplus X(C)$, so that the projections induce isomorphisms $(X + X(C))/X(C) \cong X_1, (X + X_1)/X_1 \cong X(C)$. Note that our discussion above shows that a connected, reductive K-group G gives rise to admissible 3-tuples relative to choices of S, T, B. If G, G_1 are two connected, reductive K-groups giving rise to admissible 3-tuples (Ψ, H, C) and (Ψ_1, H_1, C_1), then evidently a K-isomorphism θ from G to G_1 (if one exists) which maps S, T, B isomorphically onto S_1, T_1, B_1 induces an isomorphism of admissible 3-tuples $f(\theta) : (\Psi_1, H_1, C_1) \to (\Psi, H, C)$. We have :

Uniqueness theorem - akin to Witt's theorem
Let $f : (\Psi_1, H_1, C_1) \to (\Psi, H, C)$ *be an isomorphism of admissible 3-tuples. That is,* Ψ, Ψ_1 *are indexed root data of* G, G_1 *(w.r.t.* S, T, S_1, T_1*) respectively,* $H = C_G(S), H_1 = C_G(S_1)$ *are their respective anisotropic kernels,* $C = Z(G)^0, C_1 = Z(G_1)^0$*. Then, there is an isomorphism* $\theta : G \to G_1$ *mapping isomorphically* S *onto* S_1*,* T *onto* T_1 *and* B *onto* B_1 *such that* $f = f(\theta)$*.*
Proof.
The proof (indeed such uniqueness proofs are mostly the consequences of 'book-keeping'. That is, the given representative objects (here admissible 3-tuples) are so representative of the object they represent (here, a reductive group with given S, T) that an isomorphism between the former automatically allows one to define an isomorphism between the latter (albeit step-by-step). Let us see this somewhat more precisely.
Recall first that $H = C_G(S)$ is generated by T and the U_α's with $\alpha \in R_{D_0}$. As we know how the based root datum for a connected reductive group G can be described in terms of the semisimple group $[G, G]$ and the central torus $R(G) = Z(G)^0$ and, as it is easy to prove the theorem for a torus, we assume that G, G_1 are connected semisimple groups. Now, from the split case considered earlier, one will have a K_{sep}-defined isomorphism $\theta : G \to G_1$ with $\theta(S) = S_1, \theta(T) = T_1, \theta(B) = B_1$. In fact, that result gives us such an K_{sep}-isomorphism which restricts to K-isomorphisms on T and on H.
The idea of the rest of the proof is to show there is some $t \in T$ *such that the isomorphism* $g \mapsto \theta(tgt^{-1})$ *is defined over* K*. The method to do this is to use the result that* K*-forms of certain objects are given by elements of the first cohomology set of the Galois group acting on the automorphisms of that object.*
The given isomorphism f from R_1 to R implies that for realizations $(u_\alpha)_{\alpha \in R}, (u'_{\alpha_1})_{\alpha_1 \in R_1}$ and the corresponding unipotent subgroups U_α, U'_{α_1} in G and G_1, one has

$$\theta(U_{f(\alpha_1)}) = U'_{\alpha_1} \ \ \forall \ \alpha_1 \in R_1.$$

Recall that the element w_γ in the Weyl group W_0 of H with respect to T (corresponding to any $\gamma \in \Gamma$) satisfies $w_\gamma^{-1}(D_0) = \gamma D_0$. In fact, we saw that $w_\gamma^{-1}(\tau(\gamma)(\alpha)) = \gamma\alpha$ for every root $\alpha \in R$ (see 15.5.3). One has analogous elements in the Weyl group $(W_0)_1$ of (H_1, T_1). Under the given isomorphism f, these Weyl groups are isomorphic and these two elements correspond for each γ. Now, to choose an appropriate "K-form" for θ, consider

$$c(\gamma) = \theta^{-1} \circ \gamma \circ \theta \circ \gamma^{-1}$$

for each $\gamma \in \Gamma$. Of course, this is a K_{sep}-automorphism of G. By the observations above, we know that it fixes all elements of T and leaves stable each U_α. By the uniqueness (up to a torus element) of the isomorphism of the uniqueness theorem for split groups (in this case G^{ad} over K_{sep}), it follows that there is some $t_\gamma \in \pi(T)(K_{sep})$ with $c(\gamma) = Int(t_\gamma)$. Here, $\pi : G \to G^{ad} = Int(G)$ is the natural map and takes T to the maximal torus $\pi(T)$ of G^{ad} defined over K, as before. Now, $X^*(\pi(T)) = Q :=< R >$. As θ gives a K-isomorphism on the anisotropic kernel H of G, its action on the U_α's with $\alpha \in R_{D_0}$ respects the Γ-action (this is a general fact - a K_{sep}-isomorphism respecting the Γ-actions is defined over K). Since any t acts on $u_\alpha(x) \in U_\alpha$ by taking it to $u_\alpha(\alpha(t)x)$, it follows that our element t_γ satisfies

$$\alpha(t_\gamma) = 1 \ \ \forall \ \ \alpha \in D_0.$$

So, $t_\gamma \in T' := (\bigcap_{\alpha \in D_0} Ker(\alpha))^0$, which is a torus with character group Q' (which is the sublattice of $Q = X^*(\pi(T))$) with basis $(\alpha)_{D-D_0}$. Also, $c(\gamma) = \theta^{-1} \circ \gamma \circ \theta \circ \gamma^{-1}$ can be thought of as a 1-cocycle in $Z^1(K, T')$ (as we mentioned above K-forms correspond to first cohomology of automorphisms). We have seen earlier in earlier lectures while defining the index, w_γ etc. that γ acts on Q' via $\tau(\gamma)$ (15.5.4 shows that the action on $\alpha \in D - D_0$ has some D_0-components and the $D - D_0$ component is just $\tau(\alpha)$). Therefore, the character group Q' of T' is a direct sum of free abelian groups on bases corresponding to roots in the various Γ-orbits in the basis $D - D_0$ of Q'. In other words, the torus T' is a direct product of K-tori, one for each Γ-orbit on $D - D_0$; thus, T' is a product of tori of the form $\prod_{L/K} \mathbf{G}_m$ where L runs through some finite subextensions of K_{sep}. By Hilbert 90, $H^1(K, T') = 1$ which means there exists $t \in T'(K_{sep})$ such that $t_\gamma = t.\gamma t^{-1}$. Hence $\theta \circ Int(t) : G \to G_1$ is a K-isomorphism.

Remarks.
Let H be any connected, semisimple K-group with a maximal torus T defined over K. If $\Psi = (X, R, X^\vee, R^\vee)$ is the corresponding root datum, then, by the existence theorem for K-split groups, we have a connected, semisimple, K-**split** group G with the same root datum. Hence H is a K_{sep}-form of G and by the bijection between (K-isomorphism classes of) connected, reductive K-groups with root datum Ψ and the set $H^1(K, Aut(G))$. This bijection is set up by what is known as a twisting procedure. What is twisting? Firstly, for a Galois extension L of K, a K-group G is said to be a L/K-form of

another K-group G_0 if these two groups are L-isomorphic. If we fix an L-isomorphism, the two abstract groups $G(L), G_0(L)$ can be identified as the same group with two different actions of $\mathrm{Gal}(L/K)$. That is, there are two homomorphisms θ, θ_0 of $\mathrm{Gal}(L/K)$ into Aut $(G_0(L))$. The quotient of these two is the map $\beta(x) = \theta_0(x)\theta(x)^{-1}$ is a map from $\mathrm{Gal}(L/K)$ to Aut $G_0(L)$ satisfying the one-cocycle identity

$$\beta(xy) = \beta(x)^{\theta(x)}\beta(y).$$

Let us recall the useful construction of twisting more formally. If $c \in Z^1(K, Aut(G))$, an one-cocycle, then we have the twisted group G_c to be that K-form of G such that $G_c(K_{sep}) = G(K_{sep})$ and Γ acts on $G_c(K_{sep})$ as

$$\gamma * g = c(\gamma)(\gamma.g).$$

Any K-form of G is of the form G_c for some c as above. So, we have :
There is a bijection between the set of isomorphism classes of connected semisimple K-groups having the root datum Ψ and the set $H^1(K, Aut(G))$.
This follows from the lemma we proved in the section on diagram automorphisms and from the oft-used bijection between the K-forms of a variety V and first cohomology set of $Aut(V)$.
One can twist $Aut(G)$ itself also and obtain the K-group $Aut(G)_c$ by twisting $Aut(G)$ by an one-cocycle $c \in Z^1(K.Aut(G))$, where the group acts on itself by inner conjugation. The action of Γ is then

$$\gamma * \sigma = c(\gamma)(\gamma\sigma)c(\gamma)^{-1}.$$

We similarly have $Inn(G)_c$. One can see easily by first principles that :
The group of K-automorphisms (respectively, inner K-automorphisms) of G_c is isomorphic to $Aut(G)_c(K)$ (respectively, $Inn(G)_c(K)$).
We discuss one last ingredient to the existence theorem to be proved in the next section. Let G be a connected semisimple K-split group as above and, for an one-cocycle c in $Z^1(K, Aut(G))$, we look at the K-form G_c given by the twisting procedure above. Let S be a maximal K-split torus in G_c and $T \supset S$ a maximal K-torus in G_c. Let R be the root system of (G, T) and choose a positive system R^+ (as in 15.5.1) characterized by the property that α is positive if and only if $\pi(\alpha)$ is in the positive K-root system where π is the map induced by the restriction map $X^*(T) \to X^*(S)$. If D is the simple roots determined by this R^+, and $B \supset T$, the corresponding Borel subgroup,

we have consequently an indexed root datum $\Psi = (X, D, X^\vee, D^\vee, D_0, \tau)$. Now, the parabolic subgroup $P_{D_0} \supset B$ is a minimal K-parabolic subgroup of G_c (recall from 15.4.7 that minimal K-parabolics are generated by the centralizer of a maximal K-split torus and the root groups $U_{(\alpha)}$ corresponding to $\alpha \in R^+$). Let now T_0 be a maximal torus of G which is$K$-split (remember G is K-split) and let $B_0 \supset T_0$ be a Borel over K. Hence, there exists some $g \in G(K_{sep})$ with $B_0 = gBg^{-1}T_0 = gTg^{-1}$. So $P_0 = gP_{D_0}g^{-1}$ is defined over K. If we replace c by the cocycle $\gamma \mapsto g^{-1}c(\gamma)(\gamma g)$, which represents the same element in $H^1(K, Aut(G))$, we have $T = T_0, P_0 = P_{D_0}$. Thus, T, P_{D_0} are both subgroups of G and H although the K-structures are different. Thus, the Levi subgroup of P_{D_0} determined by (that is, containing) T is also defined over K for both K-structures as P_{D_0} is defined over K for both structures. We have the K-subgroup $R(L) = Z(L)^0 = \bigcap_{\alpha \in D_0} Ker(\alpha)$ of G. Consider the map τ on Γ; by its definition, its image is contained in the subgroup A_0 (recall A_0 consists of diagram automorphisms acting trivially on $Z(L)^0$). Also, we saw in 15.5.3, $\tau(\Gamma)$ leaves both D_0 and $D - D_0$ invariant. Since A_0 contains automorphisms which are all defined over K, we may view τ as a homomorphism from Γ to $Aut(G)(K)$. Further, we may view τ as an one-cocycle in $Aut_{K_{sep}}(L)$. This gives rise to a twisted K-group L_τ. Similarly, we have a twisted K-group G_τ. Now, for our semisimple K-split group G, the group $Aut(G)$ has the closed K-subgroup $Aut_{D_0}(G)$ of automorphisms which stabilize P_{D_0} and its Levi L. The group $Inn_{D_0}(G)$ is similarly defined and it can be seen that $Inn_{D_0}(G)$ is the image of L in $Inn(G)$ and $Aut_{D_0}(G)$ is the semidirect product of $Inn(G)$ and the stabilizer of D_0 in A_0. Thus, we also have a map $z \mapsto \bar{z}$;

$$Z^1(K, (Inn_{D_0}G)_\tau) \to Z^1(K, (Inn(L))_\tau).$$

Lemma.
(i) *G_τ, L_τ are quasi-split K-groups.*
(ii) *There exists an one-cocycle $z \in Z^1(K, (Inn_{D_0}G)_\tau)$ such that $G_c = (G_\tau)_z$.*
(iii) *Any connected reductive K-group G is an inner K-form of a quasi-split K-group - inner form of a group H means an element of $H^1(K, Inn(G))$.*
(iv) *The anisotropic kernel of G_c is K-isomorphic to $(L_\tau)_{\bar{z}}$.*
Proof.
Now, clearly B is a Borel of G_τ defined over K and hence G_τ is quasi-split over K. Similarly, L_τ is also quasi-split over K.
As the Γ-actions on D both for G and G_c are the same, it follows that G_c is

an inner K-form of G_τ. By that (lone) lemma on diagram automorphisms, we get $z \in Z^1(K, (Inn(G))_\tau)$ with $G_c = (G_\tau)_z$. To show that it actually comes from an one-cocycle with values in $(Inn_{D_0}(G))_\tau$ (and remembering that $Inn_{D_0}(G)$ is identified with the image of L in $Inn(G)$), we proceed as follows. If $z(\gamma) = Int(g_\gamma)$, then g_γ's centralize $S(K_{sep})$ as S is a K-split torus in G_c. Since $L =$ centralizer of S, it follows that $g_\gamma \in L$ for all γ. The proof is complete.

Existence theorem in general

The condition for a given admissible triple (Ψ, H, C) to come from a connected reductive K-group (for some S, T) will be discussed. As usual, the argument reduces to the triples which would correspond to connected semisimple groups. Thus, one leaves out the central torus C and considers a pair (Ψ, H) where Ψ is an abstract based root datum and H is an anisotropic K-group.

Let G be a connected semisimple K-group which is K-split. Suppose $\Psi_0 = (X, D, X^\vee, D^\vee)$ is the based root datum of G. Let Ψ be an abstract based root datum whose underlying based root datum is Ψ_0 and the other entries are D_0 and τ; that is, $\Psi = (X, D, X^\vee, D^\vee, D_0, \tau)$. If P_{D_0} denotes the standard parabolic subgroup of G determined by D_0, and L is its Levi subgroup which contains T, then L is K-split. If H is an anisotropic K-form of L, let us try to see when (Ψ, H) comes from a K-group. There are quasi-split K-groups G_τ, L_τ which are K-forms of G, L and such that G, L are inner forms. There is a homomorphism from $(Inn_{D_0}G)_\tau$ to $(Inn(L))_\tau$. There is a corresponding map $\theta : H^1(K, (Inn_{D_0}G)_\tau) \to H^1(K, (Inn(L))_\tau)$. Then, we have :

Representability criterion:
Let $c \in Z^1(K, (Inn(L))_\tau)$ be such that $H = (L_\tau)_c$. Then (Ψ, H) is representable if and only if the class of c in $H^1(K, (Inn(L))_\tau)$ belongs to Image(θ).
Proof.
If (Ψ, H) is representable, then (ii) of the previous lemma implies that the cohomology class of c is in the image of θ. Conversely, if this property holds, we consider the twisted K-group $(G_\tau)_c$ which group contains a K-split torus S whose centralizer L_1 is isomorphic to H over K. Thus, this group represents (Ψ, H). Moreover, S is the maximal K-split torus of $Z(L_1)^0$ and must,

thus, be a maximal K-split torus in $(G_\tau)_c$. So, L_1 is the anisotropic kernel of $(G_\tau)_c$ and the indexed root datum is as specified.

Determining the K-root system

Let us see how to determine the relative Dynkin diagram of G over K from the Tits index. We assume G is absolutely (almost) simple for simplicity. The general semisimple case can be deduced from the case for absolutely simple groups by a combination of operations of the three types : central isogenies, restriction of scalars and direct products. Let us denote by Φ the absolute system of roots and Σ, a basis. Then, the vertices of ${}_K\Sigma$ are in bijection with circled orbits $O(u)$ of vertices u. We need to determine when an element of ${}_K\Sigma$ is multipliable and how two vertices are connected. Let u' be an element of $O(u)$ and consider the complement in Σ of the union of all circled orbits different from $O(u)$. Let $\Sigma_{u'}$ be the connected component of u' in that complement. Denote $O'(u) = O(u) \cap \Sigma_{u'}$ and $c'(u) =$ sum of the coefficients of the element of $O'(u)$ in the highest root of $\Sigma_{u'}$. Note that $c'(u) = 1$ or 2; it is 2 iff u is multipliable. To determine when two roots are connected, let u, v be two distinct vertices of ${}_K\Sigma$. Consider the complement in Σ of the union of all circled orbits other than $O(u), O(v)$. Let Σ_{uv} be a connected component of this complement which intersects both the orbit of u and of v (if such a component exists). In case such a component does not exist, then u and v will not be connected in ${}_K\Sigma$. Suppose it exists. Let $c(u), c(v)$ denote the sums of the coefficients of highest root of Σ_{uv} occurring in $O(u), O(v)$ respectively. If $c(u) = c(v) = 2$, assume that u is multipliable; then, there is a double bond between u and v in ${}_K\Sigma$ with the arrow going from v to u. If $c(u) = c(v) = 1$, then u, v are joined in ${}_K\Sigma$ by a single bond. If $c(u) = 1, c(v) = 2$, there is a double bond with the arrow going from u to v. Finally, if $c(u) = 2, c(v) = 3$ then there is a triple bond with arrow going from u to v.

Example of Tits index :
Consider a quadratic Galois extension L/K (with $\mathrm{Gal}(L/K) =< \sigma >$) and a L/K-Hermitian form (V, f) of Witt index r and dimension $2m+1$. Assume K is infinite. Consider the K-group $G = SU(f) := \{g \in SL(V) :^t g^\sigma hg = h\}$ where h represents f with respect to some chosen basis. In other words, $SU(f)$ is the fixed point subgroup of $SL(V)$ under the involution

$$g \mapsto h^{-1}({}^t g^\sigma)^{-1}h.$$

To determine the Tits index of G, we note that Dynkin diagram ${}_L\Sigma$ of G over L is the diagram of type A_{2m}. The maximal parabolic L-subgroups are stabilizers of proper nonzero subspaces of V. If we number the vertices of ${}_L\Sigma$ from 1 to $2m$ such that the i-th vertex represents the stabilizers of i-dimensional subspaces. Now, the involution σ transforms subspaces of V into their orthogonal complements (w.r.t. f) and thus, acts on ${}_L\Sigma$ by exchanging i and $2m - i + 1$. Thus, the pairs $\{(i, 2m - i + 1)\}$ are the orbits in ${}_L\Sigma$ under $\mathrm{Gal}(L/K)$. Arbitrary L-parabolics are the stabilizers of flags in V; therefore, the K-parabolics are the stabilizers of self-orthogonal flags. This is a minimal K-parabolic iff the flag is maximal self-orthogonal. Let F be one such maximal self-orthogonal flag and let $X \in F$ have dimension $\leq m$ and maximal w.r.t. to these properties. Then, by this maximality, the flag F' of all subspaces contained in X and belonging to F forms a complete flag inside X. By hypothesis, dim $X = r$ and thus, by the last statement, the dimensions of the elements of F which constitute the minimal K-parabolic Stab F are the integers $1, 2, \cdots, r, 2m - r + 1, 2m - r + 2, \cdots, 2m$. In other words, the Witt index of G looks as follows :