Ramanujan's route to roots of roots

Talk in IIT Madras on the occasion of

Ramanujan's $\sqrt{1+119\sqrt{1+120\sqrt{1+121\sqrt{\cdots}}}}$**-th birthday**

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Introduction

The fancy title points out to a subject which originated with Ramanujan or, at least, one to which he gave crucial impetus to. We can't help but feel a sense of bewilderment on encountering formulae such as

$$\sqrt[3]{\sqrt[3]{2}-1} = \sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{4}{9}};$$

$$\sqrt{\sqrt[3]{28}-\sqrt[3]{27}} = -\frac{1}{3}(-\sqrt[3]{98}+\sqrt[3]{28}+1);$$

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}} = \frac{1}{3}(-\sqrt[3]{25}+\sqrt[3]{20}+\sqrt[3]{2});$$

$$\sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{8\pi}{7}} = \sqrt[3]{\frac{5-3\sqrt{7}}{2}};$$

$$\sqrt[6]{7\sqrt[3]{20}-19} = \sqrt[3]{\frac{5}{3}} - \sqrt[3]{\frac{2}{3}};$$

$$\sqrt{8-\sqrt{8+\sqrt{8-\cdots}}} = 1+2\sqrt{3}\sin 20^\circ;$$

$$\sqrt{23-2\sqrt{23+2\sqrt{23+2\sqrt{23-\cdots}}}} = 1+4\sqrt{3}\sin 20^\circ;$$

$$\frac{e^{-2\pi/5}}{1+}\frac{e^{-2\pi}}{1+}\frac{e^{-4\pi}}{1+}\frac{e^{-6\pi}}{1+}\cdots = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}.$$

The last expressions for the so-called Rogers-Ramanujan continued fraction appeared in Ramanujan's first letter to Hardy. These formulae are among some problems posed by Ramanujan in the Journal of the Indian Mathematical Society. As was usual with Ramanujan, he had a general formula hidden in the background and, singled out striking special cases in these problem sections. Usually, a study of his notebooks revealed what general formula he had in mind. The first thing to notice is that radicals are multi-valued and the meaning of expressions where radicals appear has to be made clear. Specially, where there is a 'nesting' of radicals, the level of complexity increases exponentially with each radical sign and it is computationally important to have equivalent expressions with the least number of radical signs. One more point to note is that often an equality, once written down, is almost trivial to verify simply by taking appropriate powers and simplifying. Thus, the intriguing question is to find out how such a formula was discovered and how to determine other such identities. The appropriate language to analyse this problem is the language of Galois theory.

1 The identity in the title

We briefly discuss the title expression and go on to analyze the other expressions in the next sections. A beautiful, elementary discussion of the convergence and evaluation of these expressions appears in an article written by Shailesh Shirali in Resonance.
Ramanujan posed the problem of finding the values of :

$$\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}$$

and

$$\sqrt{6+2\sqrt{7+3\sqrt{8+\cdots}}}.$$

These are special cases of Ramanujan's theorem appearing as Entry 4 on page 108, chapter 12 of his second notebook. Leaving aside the questions of convergence of these infinite radicals, the values can easily be discovered. Indeed, (keeping in mind the title expression), it is better to look at the more general form of the first expression

$$f(x) = \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+\cdots}}}}$$

for $x > 0$. What is the meaning of this? One is looking to find functions $f : [0,\infty) \to [0,\infty)$ satisfying

$$f(x) = \sqrt{1+xf(x+1)}.$$

As $f(x) \geq 1$, we have

$$f(x) \leq \sqrt{(1+x)f(x+1)} \leq \sqrt{1+x}\ \sqrt{\sqrt{(2+x)f(x+2)}} \leq \cdots$$

Thus, $f(x) \leq \prod_{k=1}^{\infty}(k+x)^{\frac{1}{2^k}}$.
For $x \geq 1$, it is easy to bound the above infinite product above as

$$\prod_{k=1}^{\infty}(k+x)^{\frac{1}{2^k}} \leq \prod_{k=1}^{\infty}(2kx)^{\frac{1}{2^k}} = 2x\prod_{k=1}^{\infty}k^{\frac{1}{2^k}} < 2x\prod_{k=1}^{\infty}2^{\frac{k-1}{2^k}} \leq 4x.$$

In other words, for any $x \geq 0$, we have $f(x+1) < 4(x+1)$ and, therefore, $f(x) = \sqrt{1 + xf(x+1)} < \sqrt{1 + 4x(x+1)} = 2x+1$. A fortiori, $f(x) < 4x+1$ for any $x \geq 0$.
Playing the same game, if $f(x) \leq ax + 1$ for some $a > 0$ (and all $x \geq 0$), we get - on using $f(x+1) \leq a(x+1) + 1$ - that

$$f(x) \leq \sqrt{1 + (a+1)x + ax^2} \leq \sqrt{1 + (a+1)x + (\frac{a+1}{2}x)^2} \leq 1 + \frac{a+1}{2}x.$$

Hence, starting with $a = 4$, we have the inequality $f(x) \leq ax + 1$ recursively for $a = \frac{5}{2}, \frac{7}{4}, \frac{11}{8}$ etc. which is a sequence converging to 1. Thus, $f(x) \leq 1 + x$ for all $x \geq 0$. Similarly, using the fact that $f(x+1) \geq f(x)$, one has $f(x) \geq \sqrt{1 + xf(x)}$ which gives $f(x) \geq 1 + \frac{x}{2}$ for $x \geq 0$. The earlier trick of iteration gives us that if $a > 0$ satisfies $f(x) \geq 1 + ax$ for all $x \geq 0$, then $f(x) \geq 1 + \sqrt{a}x$. Thus, starting with $a = \frac{1}{2}$, we get $f(x) \geq 1 + ax$ for $a = \frac{1}{2^{1/2^k}}$ for all $k \geq 1$. The latter sequence converges to 1 and we therefore have a perfect sandwich (which probably Ramanujan survived on, in England !) to get $f(x) = 1 + x$ for all $x \geq 0$. Therefore,

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}} = 3(!)$$

We leave the reader to ponder about the value of

$$\sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + \cdots}}}.$$

2 A theorem of Ramanujan

Ramanujan proved :
If m, n are arbitrary, then

$$\sqrt{m\sqrt[3]{4m - 8n} + n\sqrt[3]{4m + n}} =$$

$$\pm\frac{1}{3}(\sqrt[3]{(4m+n)^2} + \sqrt[3]{4(m-2n)(4m+n)} - \sqrt[3]{2(m-2n)^2}).$$

As mentioned above, this is easy to verify simply by squaring both sides ! However, it is neither clear how this formula was arrived at nor how general it is. Are there more general formulae? In fact, it turns out that Ramanujan

was absolutely on the dot here; the following result shows Ramanujan's result cannot be bettered :
Let $\alpha, \beta \in \mathbf{Q}^*$ *such that* α/β *is not a perfect cube in* $\mathbf{Q}$. *Then,* $\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$ *can be denested if and only if there are integers* m, n *such that* $\frac{\alpha}{\beta} = \frac{(4m-8n)m^3}{(4m+n)n^3}$.

For instance, it follows by this theorem that $\sqrt{\sqrt[3]{3} + \sqrt[3]{2}}$ cannot be denested.

What is meant by denesting ?
By the denesting of a nested radical one means rewriting it with fewer radical symbols. More formally, over any field K, nested radicals are defined as follows. Start with the elements in K - these are said to be nested radicals of depth zero. Use addition, subtraction, product, division and $\sqrt[n]{x}$ for $n \geq 2$ to form expressions (possibly not in K as we are taking n-th roots). With the expressions so formed, one can apply all the above procedures to form new expressions. Thus, a nested radical is an expression obtained from earlier-formed nested radicals by means of these procedures. One defines the depth dep $=$ dep $_K$ of a nested radical over K inductively by :
depth $(x) = 0$ for $x \in K$;
depth $(x \pm y) =$ depth $(x.y) =$ depth $(x/y) = \max$ (depth x, depth y);
depth $\sqrt[n]{x} = 1+$ depth x.

The usual convention used in fixing the values of radical expressions is as follows. An expression $\sqrt[3]{t}$ for a real number t will stand for the unique real cube root and, if s is a positive real number, $\sqrt{s}$ stands for the value which is the positive square root.
For example, the expression $\sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2}$ has value 1 !
Indeed, if t is the value (according to the agreed-upon convention above), then t is seen to be a (real) root of the polynomial X^3+3X-4. As $X^3+3X-4 = (X-1)(X^2+X+4)$, the only real root is 1.

3 Galois theory for denesting

To denest an expression such as $\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$, one needs to locate a field where the denesting exists. Usually, one will need to go to a field where enough roots of unity exist. Let K be any field of characteristic zero and let $\bar{K}$ be an algebraic closure. The definition of depth over a field K implies that there are subfields $K^{(d)}$ of $\bar{K}$ defined by $\text{depth}_K(x) = d$ for $x \in \bar{K}$ if and only if

$x \in K^{(d)} \setminus K^{(d-1)}$. Here, $K^{(d)}$ is generated by radicals over $K^{(d-1)}$. In fact, $K^{(d)} := \{x \in \bar{K} : x^n \in K^{(d-1)}\}$.
For example, $\sqrt[6]{7\sqrt[3]{20}-19} = \sqrt[3]{\frac{5}{3}} - \sqrt[3]{\frac{2}{3}}$ shows that the element on the left side which is in $\mathbf{Q}^{(2)}$ is actually contained in $\mathbf{Q}^{(1)}$ itself.
An element $x \in \bar{K}$ is a nested radical over K if and only if there exists a Galois extension L of K and a chain of intermediate fields

$$K \subset K_1 \subset \cdots \subset K_n = L$$

such that K_i is generated by radicals over K_{i-1} and $x \in L$.
Normally, if an element x is a nested radical over K, one obtains a chain as above successively generated by radicals such that $x \in L$ but L may not be automatically a Galois extension. For example, the left hand side above generates a non-Galois extension of $\mathbf{Q}$ and one needs to attach the 6-th roots of unity to get a Galois extension containing it.
So, why is it so important/useful to have a Galois extension ?
The fact of the matter is that Galois's famous theorem tells us that $x \in \bar{K}$ is a nested radical if and only if the Galois closure of $K(x)$ over K has a solvable Galois group. Thus, the extensions $K^{(d)}$, if they are Galois extensions of $K^{(d-1)}$, have an abelian Galois group and this theory is well-studied under the title of 'Kummer theory'. Therefore, one may adjoin enough roots of unity at the first step of the chain to get a chain of Galois extensions and may apply Kummer theory.
For instance,

$$\mathbf{Q} \subset \mathbf{Q}(\zeta_6) \subset \mathbf{Q}(\sqrt[3]{20}, \zeta_6) \subset \mathbf{Q}(\sqrt[6]{7\sqrt[3]{20}-19}, \zeta_6)$$

is a chain with each successive extension abelian.
Roughly speaking, Kummer theory can be summarized by the following :
Main Theorem of Kummer theory.
If K contains the n-th roots of unity, then abelian extensions L of K whose Galois groups have exponent n correspond bijectively to subgroups Ω of K^* containing $(K^*)^n$ via $L \mapsto K^* \cap (L^*)^n$ and its inverse map $\Omega \mapsto K(\Omega^{1/n})$.

Using Kummer theory, one may analyze all nested radicals x over $\mathbf{Q}$. In this talk, we do not go into the details of this general result proved by M.Honsbeek in 1999 but focus mainly on the result relevant to Ramanujan's theorem. We will use the notation $\overline{\mathbf{Q}}$ for the field of all algebraic numbers. In this field, every nonconstant polynomial over it has all its roots in it. The following consequence of the above main theorem of Kummer theory and will be a key to denesting radicals.

Proposition 1.
Let K denote a field extension of $\mathbf{Q}$ containing the n-th roots of unity. Suppose $a, b_1, b_2, \cdots, b_r \in \overline{\mathbf{Q}}^*$ are so that $a^n, b_1^n, \cdots, b_r^n \in K$.
Then, $a \in K(b_1, \cdots, b_r)$ if, and only if, there exist $b \in K^*$ and natural numbers $m_1, m_2, \cdots, m_r$ such that

$$a = b \prod_{i=1}^{r} b_i^{m_i}.$$

Proof.
The 'if' part is easily verified. Let us assume that $a \in L := K(b_1, \cdots, b_r)$. The subgroup Ω of L^* generated by the n-th powers of elements of (K^*) along with $b_1^n, \cdots, b_r^n$ satisfies $L = K(\Omega^{1/n})$ by Kummer theory. So, $a^n \in (L^*)^n \cap K^* = \Omega$. Thus, there exists $c \in K^*$ so that

$$a^n = c^n \prod_{i=1}^{r} b_i^{m_i n}.$$

Taking n-th roots on both sides and multiplying by a suitable n-th root of unity (remember they are in K), we get

$$a = b \prod_{i=1}^{r} b_i^{m_i}$$

for some $b \in K^*$. The proof is complete.

The following technical result from Galois theory which uses the above proposition is crucial in the denesting of $\sqrt{1 + \sqrt[3]{\beta/\alpha}}$ over $\mathbf{Q}$. Although this result is not difficult to prove, we shall not go into the details of the proof. We give a sketch for the cognoscenti; others may skip it.

Theorem 2.
Let c be a rational number which is not a perfect cube. Let $\delta \in \mathbf{Q}(\sqrt[3]{c})$ and let G denote the Galois group of the Galois-closure M of $\mathbf{Q}(\sqrt{\delta})$ over $\mathbf{Q}$. Then, the nested radical $\sqrt{\delta}$ can be denested over $\mathbf{Q}$ if, and only if, the second commutator group G'' of G is trivial. Further, these conditions are equivalent to the existence of $f \in \mathbf{Q}^*$ and some $e \in \mathbf{Q}(\delta)$ so that $\delta = fe^2$.

Sketch of Proof.
The essential part is to show that when G'' is trivial, then there are $f \in \mathbf{Q}^*$ and $e \in \mathbf{Q}(\delta)$ with $\delta = fe^2$.
We consider the field $K = \mathbf{Q}(\delta, \zeta_3)$, the smallest Galois extension of $\mathbf{Q}$ which contains δ. If δ_2, δ_3 are the other Galois-conjugates of δ in K, the **main claim** is that if $\delta_2\delta_3$ is not a square in K, then G'' is not trivial. To see this, suppose $\delta_2\delta_3$ (and hence its Galois-conjugates $\delta\delta_3, \delta\delta_2$) are non-squares as well. Then, $\sqrt{\delta\delta_2}$ cannot be contained in $K(\sqrt{\delta_2\delta_3})$ because of the above proposition. So, the extension $L = K(\sqrt{\delta\delta_2}, \sqrt{\delta_2\delta_3})$ has degree 4 over K and is contained in the Galois closure M of $\mathbf{Q}(\sqrt{\delta})$ over $\mathbf{Q}$. The Galois group Gal (L/K) is the abelian Klein 4-group V_4. Indeed, its nontrivial elements are $\rho_1, \rho_2, \rho_1\rho_2$ where :
ρ_1 fixes $\sqrt{\delta\delta_2}$ and sends $\sqrt{\delta_2\delta_3}$ and $\sqrt{\delta\delta_3}$ to their negatives;
ρ_2 fixes $\sqrt{\delta_2\delta_3}$ and sends $\sqrt{\delta\delta_2}$ and $\sqrt{\delta\delta_3}$ to their negatives.
Also, Gal $(K/\mathbf{Q})$ is the full permutation group on $\delta, \delta_2, \delta_3$. We also put δ_1 instead of δ for convenience.
Suppose, if possible, $G'' = \{1\}$. Now, the second commutator subgroup of Gal $(L/\mathbf{Q})$ is trivial as it is a subgroup of G''. In other words, the commutator subgroup of Gal $(L/\mathbf{Q})$ is abelian.
Consider the action of Gal$(K/\mathbf{Q})$ on Gal (L/K) defined as :

$$(\sigma, \tau) \mapsto \sigma_L \tau \sigma_L^{-1}$$

where, for $\sigma \in$Gal $(K/\mathbf{Q})$, the element $\sigma_L \in$Gal $(L/\mathbf{Q})$ which restricts to K as σ.
The following computation shows that the commutator subgroup of Gal $(L/\mathbf{Q})$ cannot be abelian.
If π :Gal $(L/\mathbf{Q}) \to$Gal$(K/\mathbf{Q})$ is the restriction map, look at any lifts a, b, c of $(12), (13), (23)$ respectively. For any $d \in$Gal (L/K), the commutator $ada^{-1}d^{-1}$ is defined independently of the choice of the lift a since Gal (L/K) is abelian. An easy computation gives :

$$a\rho_2 a^{-1}\rho_2^{-1} = \rho_1$$

$$b\rho_1 b^{-1}\rho_1^{-1} = \rho_1\rho_2$$

$$c(\rho_1\rho_2)c^{-1}(\rho_1\rho_2)^{-1} = \rho_2.$$

Therefore, the whole of Gal (L/K) is contained in the commutator subgroup of Gal $(L/\mathbf{Q})$. Now $(123) = (13)(23)(13)(23)$ implies that $d = bcb^{-1}c^{-1}$

which is in the commutator subgroup of Gal $(L/\mathbf{Q})$ is a lift of (123). Thus, $dgd^{-1}g^{-1} = Id$ for any $g \in$Gal (L/K) as Gal (L/K) is contained in the commutator subgroup of Gal $(L/\mathbf{Q})$ (an abelian group). But note that $d\rho_1 d^{-1}$ fixes $\sqrt{\delta_2\delta_3}$ and hence, cannot be equal to ρ_1. Thus, we have a contradiction to the assumption that $G'' = \{1\}$ while $\delta_2\delta_3$ is a nonsquare in K; the claim follows.
Now, assume that G'' is trivial. We would like to use the claim proved above to show that there are $f \in \mathbf{Q}^*$ and $e \in \mathbf{Q}(\delta)$ with $\delta = fe^2$.
Start with some $\eta \in K$ with $\delta_2\delta_3 = \eta^2$. We would like to show that $\eta \in \mathbf{Q}(\delta)$. This will prove our assertion, for then,

$$\delta = \frac{\delta_1\delta_2\delta_3}{\delta_2\delta_3} = \frac{\delta_1\delta_2\delta_3}{\eta^2} = fe^2$$

where $f = \delta_1\delta_2\delta_3 \in \mathbf{Q}$ and $e = \eta^{-1} \in \mathbf{Q}(\delta)$.
Suppose $\eta \notin \mathbf{Q}(\delta)$. Since the product $\delta_2\delta_3 \in \mathbf{Q}(\delta)$, on applying the above proposition to $K = \mathbf{Q}(\delta, \zeta_3) = \mathbf{Q}(\delta, \sqrt{-3})$, we get $\sqrt{\delta_2\delta_3} = \sqrt{-3}\theta$ for some $\theta \in \mathbf{Q}(\delta)$; that is,

$$\eta^2 = \delta_2\delta_3 = -3\theta^2.$$

Taking norms over $\mathbf{Q}$, we get $N(\eta) = (-3)^3 N(\theta)^2$ which is a contradiction since $(-3)^3$ is not a square in $\mathbf{Q}$. Therefore, η indeed belongs to $\mathbf{Q}(\delta)$ and we are done.

4 Existence of Denesting

In this section, we determine conditions under which elements e, f as in theorem 2 exist. For any non-zero α, β in $\mathbf{Q}$, the polynomial

$$F_{\beta/\alpha}(t) = t^4 + 4t^3 + 8\frac{\beta}{\alpha}t - 4\frac{\beta}{\alpha}$$

plays a role in determining the denestability of the nested radical $\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$ over $\mathbf{Q}$.

Lemma 3.
Let $\alpha, \beta \in \mathbf{Q}^*$ such that α/β is not a perfect cube in $\mathbf{Q}$. Then, $\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$ can be denested if and only if the polynomial $F_{\beta/\alpha}$ has a root in $\mathbf{Q}$.
Proof.

Now $\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$ can be denested if and only if $\sqrt{1+\sqrt[3]{\beta/\alpha}}$ can be denested. By the theorem, this happens if and only if there exists $f, x, y, z \in \mathbf{Q}$ with

$$1+\sqrt[3]{\beta/\alpha} = f(x+y\sqrt[3]{\beta/\alpha}+z\sqrt[3]{\beta^2/\alpha^2})^2 \cdots\cdots \diamondsuit$$

Assume that denesting can be done. The elements $1, \sqrt[3]{\beta/\alpha}, \sqrt[3]{\beta^2/\alpha^2}$ are linearly independent over $\mathbf{Q}$. Thus, we may compare like powers of $\sqrt[3]{\beta^2/\alpha^2}$ in $\diamondsuit$ to get

$$1/f = x^2 + \frac{2yz\beta}{\alpha}$$
$$0 = y^2 + 2xz$$
$$1/f = \frac{\beta z^2}{\alpha} + 2xy$$

After a simple calculation, it is easy to see that $z \neq 0$ and that y/z is a root of $F_{\beta/\alpha}$.
Conversely, suppose $F_{\beta/\alpha}$ has a rational root s. Then, working backwards, a denesting is given as :

$$\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}} = \pm\frac{1}{\sqrt{f}}\left(-\frac{s^2\sqrt[3]{\alpha^2}}{2}+s\sqrt[3]{\alpha\beta}+\sqrt[3]{\beta^2}\right)$$

where $f = \beta - s^3\alpha$. The proof is complete.

Examples 4.
For $\alpha = 5, \beta = -4$ we get $s = -2$ to be the rational root of $F_{-4/5}(t) = t^4 + 4t^3 - \frac{32}{5}t + \frac{16}{5} = 0$. Thus, $f = -4 + 40 = 36$ and we have

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}} = \frac{1}{6}(-2\sqrt[3]{25} - 2\sqrt[3]{-20} + \sqrt[3]{16})$$

$$= \frac{1}{3}(-\sqrt[3]{25} + \sqrt[3]{20} + \sqrt[3]{2}).$$

Similarly, for $\alpha = 28, \beta = 27$, we have $s = -3$ and $f = 27^2$ and we get

$$\sqrt{\sqrt[3]{28}-\sqrt[3]{27}} = -\frac{1}{27}\left(-\frac{9}{2}\sqrt[3]{28^2} - 3\sqrt[3]{(-27)(28)} + \sqrt[3]{27^2}\right)$$

$$= -\frac{1}{3}(-\sqrt[3]{98} + \sqrt[3]{28} + 1).$$

5 Connection with Ramanujan's denesting

We saw that denesting of $\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$ involved the rational root of a certain related polynomial. The connection with Ramanujan's denesting comes while trying to characterize the α, β for which the polynomial $F_{\beta/\alpha}$ has a root in $\mathbf{Q}$. This is easy to see as follows :

Lemma 5.
Let $\alpha, \beta \in \mathbf{Q}^*$ where the ratio is not a cube. Then $\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$ can be denested over $\mathbf{Q}$ if, and only if, $F_{\beta/\alpha}$ has a root s in $\mathbf{Q}$ which is if, and only if, there are integers m, n so that

$$\frac{\alpha}{\beta} = \frac{(4m-8n)m^3}{(4m+n)n^3}.$$

Proof.
Of course, we need to prove only the second 'if and only if' and, even there, it suffices to prove the 'only if' part as the other implication is obvious. Now $s^4+4s^3+8s\beta/\alpha-4\beta/\alpha=0$ implies (on taking $s=n/m$) that

$$\frac{\beta}{\alpha} = \frac{s^3(s+4)}{4-8s} = \frac{(4m+n)n^3}{(4m-8n)m^3}.$$

Remarks.

This is not quite the same as Ramanujan's denesting formula although the ratio β/α is the same. Ramanujan's theorem is :

$$\sqrt{m\sqrt[3]{4m-8n}+n\sqrt[3]{4m+n}} =$$

$$\pm\frac{1}{3}(\sqrt[3]{(4m+n)^2}+\sqrt[3]{4(m-2n)(4m+n)}-\sqrt[3]{2(m-2n)^2}).$$

If we apply this to denest $\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$ where $\frac{\beta}{\alpha}=\frac{n^3(4m+n)}{m^3(4m-8n)}$, we would get

$$\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}} = \frac{1}{\sqrt{m}}\sqrt[6]{\frac{\alpha}{4m-8n}}\sqrt{m\sqrt[3]{4m-8n}+n\sqrt[3]{4m+n}}.$$

This formula looks quite awkward and the natural question arises as to whether it is true that for integers α, β there exist integers m, n with $\alpha =$

$m^3(4m - 8n)$ and $\beta = n^3(4m + n)$.
It turns out that this is not always the case. For example, if $\alpha = -4, \beta = 5$, the integers $m = n = 1$ work whereas for the other choice $\alpha = 5, \beta = -4$, there are no such integers (!) Thus, when we are denesting $\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$, we are actually denesting $\sqrt{1 + \sqrt[3]{\beta/\alpha}}$ and it is better to use the method we discussed.
The asymmetry between m and n in Ramanujan's formula can be explained as follows. If one changes m to $m' = -\frac{n}{\sqrt{2}}$ and n to $n' = m\sqrt{2}$, it turns out that

$$4(m - 2n)m^3 = (4m' + n')n'^3$$

$$(4m + n)n^3 = 4(m' - 2n')m'^3.$$

Thus, we have the same denesting !

6 Denesting nested square roots

As mentioned in the beginning, using Galois theory, one may investigate the denesting of any nested radical in principle. However, it is not clear how to do in general. When the nested radical consists only of nested square roots, some nice results can be proved. We merely state two of them due to Borodin, Fagin, Hopcroft and Tompa :

Theorem 6.
Let K have characteristic 0 and $a, b, r \in K$ with $\sqrt{r} \notin K$. Then,
(a) $\sqrt{a + b\sqrt{r}} \in K(\sqrt{r}, \sqrt{a_1}, \cdots, \sqrt{a_n})$ *for some* $a_i \in K \Leftrightarrow \sqrt{a^2 - b^2 r} \in K$;
(b) $\sqrt{a + b\sqrt{r}} \in K(\sqrt[4]{r}, \sqrt{a_1}, \cdots, \sqrt{a_n})$ *for some* $a_i \in K \Leftrightarrow$ *either* $\sqrt{a^2 - b^2 r}$ *or* $\sqrt{r(-a^2 + b^2 r)}$ *is in* K.

Theorem 7.
Let $K \subset \mathbf{R}$ and $a, b, r \in K$ with $\sqrt{r} \notin K$. Let $a_1, \cdots, a_n \in K$ be positive and let $r_1, \cdots, r_n \geq 1$. If $\sqrt{a + b\sqrt{r}} \in K(\sqrt[r_1]{a_1}, \cdots, \sqrt[r_n]{a_n})$, then $\sqrt{a + b\sqrt{r}}$ is already in $K(\sqrt[4]{r}, \sqrt{b_1}, \cdots, \sqrt{b_n})$ for some $b_1, \cdots, b_n \in K$.

These theorems also provide an algorithm for denesting. We do not go into it and proceed to the last section where we discuss certain values of the Rogers-Ramanujan continued fractions as nested radicals.

7 Rogers-Ramanujan continued fraction as nested radicals

The Rogers-Ramanujan continued fraction is the function

$$R(z) = \frac{q^{1/5}}{1+}\frac{q}{1+}\frac{q^2}{1+}\cdots$$

defined on the upper half plane (where $q = e^{2i\pi z}$). It is a holomorphic function and is also given by the product

$$R(z) = q^{1/5}\prod_{n=1}^{\infty}(1-q^n)^{(\frac{n}{5})}$$

where $\frac{n}{5}$ here denotes the Legendre symbol. We had mentioned at the beginning Ramanujan's beautiful formula which he wrote in his first letter to Hardy :

$$\frac{e^{-2\pi/5}}{1+}\frac{e^{-2\pi}}{1+}\frac{e^{-4\pi}}{1+}\frac{e^{-6\pi}}{1+}\cdots = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}.$$

In other words, as $z = i$ gives $q = e^{-2\pi}$, we have

$$R(i) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}.$$

Actually, this follows from a reciprocity theorem for $R(z)$ that Ramanujan wrote in his second letter to Hardy and which was proved by Watson. There are several other such identities written by Ramanujan like :

$$R(e^{i\sqrt{5}}) = \frac{\sqrt{5}}{1+\sqrt[5]{5^{3/4}\sqrt{(\frac{\sqrt{5}-1}{2})^5}-1}} - \frac{\sqrt{5}+1}{2}.$$

Each of these and many others can be proved using eta-function identities which were known to Ramanujan. Two such are :

$$\frac{1}{R(z)} - R(z) - 1 = \frac{\eta(z/5)}{\eta(5z)}\cdots\cdots\cdots(\heartsuit)$$

$$\frac{1}{R(z)^5} - R(z)^5 - 11 = (\frac{\eta(z)}{\eta(5z)})^6\cdots\cdots\cdots(\spadesuit)$$

Here, the Dedekind eta function $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ where $q = e^{2i\pi z}$ as before. Note that $\eta(z)$ is essentially the generating function for the sequence $p(n)$ of partitions of a natural number n. Further, $\eta(z)^{24}$ is the discriminant function (sometimes called Ramanujan's cusp form) $\Delta(z) = q\prod_{n=1}^{\infty}(1-x^n)^{24}$ which is the unique cusp form (upto scalars) of weight 12 and its Fourier expansion $\Delta(z) = (2\pi)^{12}\sum_{n=1}^{\infty}\tau(n)e^{2i\pi nz}$ has Fourier coefficients $\tau(n)$, the Ramanujan tau function. Thus, this is familiar territory to Ramanujan. But, keeping in mind our focus on nested radicals, we shelve a discussion of proofs of these - there are proofs due to Berndt et al. in the spirit of Ramanujan and other ones by K.G.Ramanathan using tools like Kronecker limit formula which were unfamiliar to Ramanujan. For readers interested in these, there is a Memoir by Andrews, Berndt, Jacobsen and Lamphere in Memoirs of the AMS 477 (1992). We go on to briefly discuss in the modern spirit how values of $R(z)$ at imaginary quadratic algebraic integers z on the upper half-plane can be expressed as nested radicals. This will involve class field theory, modular functions and the theory of complex multiplication. In order to be comprehensible to a sufficiently wide audience, we merely outline this important but rather technical topic.

What is the key point of this procedure ?
The discussion below can be informally summed up as follows. The values of 'modular functions' (functions like $R(z)$) at an imaginary quadratic algebraic integer τ on the upper half-plane generate a Galois extension over $\mathbf{Q}(\tau)$ with the Galois group being abelian. This is really the key point. For, if we have any abelian extension L/K, one has a standard procedure whereby one could express any element of L as a nested radical over a field L' with degree $[L' : K] < [L : K]$. Recursively, this leads to a nested radical expression over K. Let us make this a bit more formal now.

Modular functions come in.
A meromorphic function on the extended complex upper half-plane $\mathcal{H} \cup \mathbf{P}^1(\mathbf{Q})$ which is invariant under the natural action of $\Gamma(N) := Ker(SL_2(\mathbf{Z}) \to SL_2(\mathbf{Z}/N\mathbf{Z}))$ is known as a *modular function of level* N. One has :
The Rogers-Ramanujan continued fraction $R(z)$ *is modular, of level* 5.
Now, a modular function of level N (since it is invariant under the transformation $z \mapsto z + N$) has a Fourier expansion in the variable $e^{2i\pi z/N} = q^{1/N}$. Those modular functions of level N whose Fourier coefficients lie in $\mathbf{Q}(\zeta_N)$ form a field F_N. In fact, in the language of algebraic geometry, F_N is the

function field of a curve known as the modular curve $X(N)$ (essentially, the curve corresponding to the Riemann surface obtained by compactifying the quotient of the upper half-plane by the discrete subgroup $\Gamma(N)$). Moreover, F_N is a Galois extension of F_1 with Galois group $GL_2(\mathbf{Z}/N\mathbf{Z})/\{\pm I\}$. The classical modular j-function generates F_1 over $\mathbf{Q}$ and, hence, induces an isomorphism of $X(1)$ with $\mathbf{P}^1(\mathbf{Q})$. Let us briefly define it.
For τ on the upper half-plane, consider the lattice $\mathbf{Z} + \mathbf{Z}\tau$ and the functions

$$g_2(\tau) = 60 \sum_{m,n}{}' \frac{1}{(m+n\tau)^4} \left(= \frac{(2\pi)^4}{12} \left(1 + \sum_{n=1}^{\infty} \sigma_3(n) e^{2\pi i n \tau} \right) \right)$$

$$g_3(\tau) = 140 \sum_{m,n}{}' \frac{1}{(m+n\tau)^6} \left(= \frac{(2\pi)^6}{12} \left(1 + \sum_{n=1}^{\infty} \sigma_5(n) e^{2\pi i n \tau} \right) \right).$$

[Note that $p'(z)^2 = 4p(z)^3 - g_2(\tau)p(z) - g_3(\tau)$ where the Weierstrass p-function on $\mathbf{Z} + \mathbf{Z}\tau$ is given by $p(z) = \frac{1}{z^2} + \sum_w (\frac{1}{(z-w)^2} - \frac{1}{w^2})$.]
It can be shown that $\Delta(\tau) \stackrel{d}{=} g_2(\tau)^3 - 27g_3(\tau)^2 \neq 0$. The elliptic modular function $j : h \to \mathbf{C}$ is defined by

$$j(\tau) = 12^3 \cdot \frac{g_2(\tau)^3}{\Delta(\tau)}.$$

Similar to the generation of F_1 by the j-function is the fact that the field F_5 also happens to be generated over $\mathbf{Q}(\zeta_5)$ by a single function (in other words, $X(5)$ also has genus 0). An interesting classical fact is :
The function $R(z)$ generates F_5 over $\mathbf{Q}(\zeta_5)$.
The idea used in the proof of this is the following. The j-function can be expressed in terms of the eta function and on using the two identities (♡) and (♠), it follows that

$$j = \frac{(1 + 228R^5 + 494R^{10} - 228R^{15} + R^{20})^3}{(-R + 11R^6 + R^{11}}.$$

From this, one can derive also explicitly the minimal polynomial of R^5 over $\mathbf{Q}(j)$, a polynomial of degree 12. Thus, $\mathbf{Q}(R)$ has degree 60 over $\mathbf{Q}(j)$. In fact, this minimal polynomial is a polynomial in j with integer coefficients. In particular, if τ is an imaginary quadratic algebraic integer on the upper half-plane, $R(\tau)$ is an algebraic integer as this is true for j.
If τ is an imaginary quadratic algebraic integer on the upper half-plane,

one can form the $\mathbf{Z}$-lattice $\mathcal{O}_\tau$ with basis $1, \tau$. This is an order (a subring containing a $\mathbf{Q}$-basis) in the imaginary quadratic field $\mathbf{Q}(\tau)$. When $\mathcal{O}_\tau$ is a maximal order, the 'class field' H of $\mathcal{O}_\tau$ is the 'Hilbert class field' of $\mathbf{Q}(\tau)$ - a finite extension of $\mathbf{Q}(\tau)$ in which each prime of $\mathbf{Q}(\tau)$ is unramified and principal. More generally, for any N, the field H_N generated by the values $f(\tau)$ of modular functions f of level N is nothing but the 'ray class field' of conductor N over $\mathbf{Q}(\tau)$ when $\mathcal{O}_\tau$ is a maximal order. We have :
The first main theorem of complex multiplication :
H_N is an abelian extension of $\mathbf{Q}(\tau)$.
The Hilbert class field H is H_1 in this notation. That is, $H_1 = \mathbf{Q}(\tau)(j(\tau))$ is Galois over $\mathbf{Q}(\tau)$ with the Galois group isomorphic to the class group of $\mathcal{O}_\tau$. In general, for any N, one may explicitly write down the action of $\mathrm{Gal}(H_n/\mathbf{Q}(\tau))$ on $f(\tau)$ for any modular function f of level N. One may use that and the fact mentioned earlier that $R(\tau)$ is an algebraic integer to prove :
If τ is an imaginary quadratic algebraic integer on the upper half-plane, and $\mathcal{O}_\tau$ is the order in $\mathbf{Q}(\tau)$ with $\mathbf{Z}$-basis $1, \tau$, the ray class field H_5 of conductor 5 over $\mathbf{Q}(\tau)$ is generated by $R(\tau)$.
Let us begin with this data to describe an algorithm to express elements of H_N as nested radicals over $\mathbf{Q}(\tau)$.

Procedure for expressing as nested radicals
Let L/K be any abelian extension and let $w \in L$. Choose an element $\sigma \in Gal(L/K)$ of order $n > 1$. Take $L' = L^\sigma(\zeta_n)$ where L^σ denotes the fixed field under $< \sigma >$. Now, L' is an abelian extension of K and

$$[L' : K] \leq \phi(n)[L^\sigma : K] < n[L^\sigma : K] = [L : L^\sigma][L^\sigma : K] = [L : K].$$

Look at the Lagrange resolvents $h_i = \sum_{k=1}^{n} \zeta_n^{ik} w^{(\sigma^k)}; 0 \leq i < n$ of w with respect to σ. Thus, we have

$$w = \frac{h_0 + h_1 + \cdots + h_{n-1}}{n}.$$

Since $h_0 = tr_{L/L^\sigma}(w) \in L'$, it is fixed by every element of $Gal(L(\zeta_n)/L')$. Furthermore, any element $\rho \in Gal(L(\zeta_n)/L')$ acts by some power of σ on L and as identity on ζ_n. Therefore, for $i = 1, 2, \cdots, n-1$;

$$h_i^\rho = \zeta_n^{-ia} h_i$$

where ρ is acting as σ^a on L.
Hence, we get $h_1^n h_2^n \cdots h_{n-1}^n \in L'$. Thinking of h_i as $\sqrt[m]{h_i^m}$, the expression

$$w = \frac{h_0 + h_1 + \cdots + h_{n-1}}{n}$$

is actually an expression as a nested radical over L'. Next, one can apply this procedure to $h_0, h_1^n, h_2^n, \cdots, h_{n-1}^n$ in L' etc.

Returning to our situation, when τ is an imaginary quadratic algebraic integer, we wish to look at the value $R(\tau)$. So, we will work with the ray class field H_5 over $\mathbf{Q}(\tau)$. Recall the identity

$$\frac{1}{R(z)} - R(z) - 1 = \frac{\eta(z/5)}{\eta(5z)} \cdots\cdots\cdots (\heartsuit)$$

It turns out that the elements $\frac{1}{R(\tau)}$ and $-R(\tau)$ of H_5 are Galois-conjugate over the field $\mathbf{Q}(\tau)(w(\tau))$ where we have written $w(\tau)$ for $\frac{\eta(\tau/5)}{\eta(5\tau)}$. Therefore, we have :
H_5 is generated over $\mathbf{Q}(\tau)$ by $w(\tau)$ and ζ_5.
Now $w(\tau)$ is an algebraic integer as both $\frac{1}{R(\tau)}$ and $-R(\tau)$ are so. Usually, one works with $w(\tau)$ instead of with $R(\tau)$ because the former has half the number of conjugates and also, it is defined in terms of the eta function which can be evaluated by software packages very accurately. The point is that the software packages exploit the fact that $SL_2(\mathbf{Z})$-transformations carry z to a new z with large imaginary part and so the Fourier expansion of $\eta(z)$ will converge quite rapidly. Therefore, to obtain expressions for $R(\tau)$, one works with $w(\tau)$ and uses $(\heartsuit)$.

What τ to choose ?
Analyzing the product expression

$$R(z) = q^{1/5} \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{n}{5}\right)}$$

one observes that $R(z)$ is real when $Re(z) = 5k/2$ for some integer k. Thus, keeping in mind that we need algebraic integers τ with the corresponding order being maximal, we look at :

$$\tau_n = \sqrt{-n} \quad if \quad -n \not\equiv 1 \ \ mod \ \ 4;$$

$$\tau_n = \frac{5+\sqrt{-n}}{2} \quad if \quad -n \equiv 1 \ \ mod \ \ 4.$$

Observe that $\tau_1 = i$. In general, it is quite easy to show that $\sqrt{5} \in \mathbf{Q}(\tau_n)$ and that $\frac{w(\tau_n)}{\sqrt{5}}$ is an algebraic integer. So, one can generate H_5 over $\mathbf{Q}(\tau_n)$ by $\tilde{w}(\tau_n) = \frac{w(\tau_n)}{\sqrt{5}}$ (instead of $w(\tau_n)$) along with ζ_5. Applying the procedure of finding nested radicals in an abelian extension, one can show (using software packages) :

$$\tilde{w}(\tau_1) = \tilde{w}(i) = 1.$$

Unwinding this for $w(i)$ and then for $R(i)$, one gets

$$R(i) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2} \ !$$

Similarly, one may obtain formulae like

$$R(\frac{5+i}{2}) = -\sqrt{\frac{5-\sqrt{5}}{2}} + \frac{\sqrt{5}-1}{2}$$

Last take on j-function

A romantic story goes that Ramanujan wrote somewhere that $e^{\pi\sqrt{163}}$ is 'almost' an integer ! Whether he did actually write it or not, he could have known the following interesting fact.

Applying the first theorem of complex multiplication to $j(\tau)$ for τ imaginary quadratic, it follows that $j(\tau)$ is an algebraic integer of degree = class number of $\mathbf{Q}(\tau)$ i.e, $\exists$ integers $a_0, \ldots, a_{h-1}$ such that $j(\tau)^h + a_{h-1}j(\tau)^{h-1} + \ldots + a_0 = 0$. Now, there are only finitely many imaginary quadratic fields $\mathbf{Q}(\tau) = K$ which have class number 1. The largest D such that $\mathbf{Q}(\sqrt{-D})$ has class number 1 is 163. Since $163 \equiv 3(4)$, the ring of integers is $\mathbf{Z} + \mathbf{Z}(\frac{-1+i\sqrt{163}}{2})$. Thus $j(\frac{-1+i\sqrt{163}}{2}) \in \mathbf{Z}$.

The Fourier expansion of j has integer coefficients and looks like $j(\tau) = \frac{1}{q} + 744 + \sum_{n\geq 1} c_n q^n$ with $c_n \in \mathbf{Z}$ and

$$q = e^{2\pi i(\frac{\pm 1+i\sqrt{163}}{2})} = -e^{-\pi\sqrt{163}}.$$

Thus $-e^{\pi\sqrt{163}} + 744 - 196884\ e^{-\pi\sqrt{163}} + 21493760\ e^{-2\pi\sqrt{163}} + \ldots = j(\tau) \in \mathbf{Z}$. In other words,

$$e^{\pi\sqrt{163}} - integer = 196884\ e^{-\pi\sqrt{163}} + 21493760\ e^{-2\pi\sqrt{163}} \ldots \approx 0!$$

To end, we should agree that Ramanujan **radically** changed the mathematical landscape !