Beginnings of Polya Theory

Naren Manjunath

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1 Introduction

Polya theory is, unlike most of high- school combinatorics, not a bag of tricks that are situation- specific. It deals with questions where clearly understanding the set that is to be counted is the main difficulty.
An example of this is when we are trying to find the number of nationalities represented in a group of people: having counted the first Indian, we must ignore the other Indians as the counting proceeds.
In this case certain groups of people were related by nationality; however, the problems of Polya theory involve relations of symmetry operating among elements (people), and the crux of their solution is to understand how symmetry changes the relationships between these elements.
In what follows we will consider symmetry problems from elementary combinatorics, such as necklaces and chessboards, as well as solid geometry and chemistry, and we will hint at the number theoretic applications that abound in this field. The full scope of Polya theory is, of course, far greater, and is visible in graph theory and higher algebra as well.

Remark 1.1 We begin with a simplified chessboard example that, although easily solved by brute force, quickly generalises to problems that beg for more systematic methods of solution.

Example 1.2 Consider a 2×2 board whose squares are coloured either in red(r) or in black(b). We must find the number of different boards. Elementary counting suggests that since each square can have one of two colours, there are $2^4 = 16$ possible arrangements. A lot, however, depends on what we mean by 'different'. The consensus among most people is that two boards are equivalent if one can be obtained from the other by a rotation of some multiple of 90 anticlockwise. Assuming this, we find the following arrangements: a) 4 red squares b) 3 red, 1 black c) 2 red, 2 black (there are 2 possiblities here, where similar squares are on the same half or the same diagonal).

Thus the total number of non-equivalent boards is $2 \cdot (1+1) + 2 = 6$, since the two remaining cases are (a) and (b) with the colours interchanged. The reader will appreciate the intrinsic difficulty of the problem immediately upon replacing the ' 2×2 ' even with ' 3×3 ', and certainly with any greater number. Problems of a similar nature abound in elementary settings: The number of necklaces of n beads, with a choice of m colours, where necklaces with rotational symmetry are equivalent; the number of colourings of the vertices of a cube or tetrahedron in m colours, equivalent up to rotation; and several others that will be returned to.

2 Abstracting the problem

In each case mentioned above, there is an assignment of colours to squares, beads or vertices- a *function* from a domain to the set of colours. Elementary counting gives us the cardinality of the set of all such functions R^D, where D is the domain and R, the range. However, some functions are 'equivalent' (the elucidation of 'equivalence' being central to our study and done presently) and thus the real problem is to determine the number of *eqivalence classes* of functions on R^D.

So, when are two functions equivalent? Returning to our examples, we see that equivalence was defined by the property that one arrangement (function) could be obtained from another by an operation on the elements of the domain. What sort of operation? Clearly, any such operation *permutes* the elements among themselves, otherwise the power set is ill defined. Furthermore, as in our example, if we can speak of rotating a chessboard by multiples of 90, we must surely include rotations by zero (the identity operation) and negative multiples (inverses). We see that the set of operations possess a group structure; and, as our discussion will show, it is sufficiently general to consider that the operations form a permutation group.

We will formalize the concept of equivalence of functions. Let G be a permutation group acting on a set D. There exists a set R and the set R^D, the set of all functions from D to R.

Definition 2.1 *Two functions $f, g \in R^D$ are equivalent if there exists a permutation $\pi \in G$ such that for all $d \in D, f(d) = g(\pi(d))$. Also, for every $\pi \in G$, define $e_\pi : R^D \to R^D$ such that $e_\pi(f) = g$ iff f and g are equivalent.*

e_π is well defined. For, if for some i, $f(\pi^{-1}(i) = g(i)$, then setting $i = \pi(d)$, which is possible as π is a permutation, we get $f(d) = g(\pi(d))$.

Further, $e_(\pi)$ is injective. If $e_\pi(f) = e_\pi(h) = g$, then $f(d) = h(d)$ for all $d \in D$ and so $f = h$. This, together with the fact that the domain and range have equal cardinality, imply that e_π is also surjective and so is a permutation on R^D.

The equivalence class of f is called a *pattern*. Thus, the set of chessboards that form a pattern can be obtained from each other by rotation. Our objective is to enumerate the patterns on R^D.

3 A Partial Answer- Burnside's Lemma

The full machinery that we seek to employ is not always necessary, at least when we set ourselves the limited problem of counting the number of patterns without asking for any further information about their nature or their constituent functions.

Consider a group G acting on a set S . For every $g \in G$, let S^g denote the subset of S fixed by g. Also for every $s \in S$ define the *stabilizer* G_s of s, the subgroup of G (this is easily proved to be a subgroup) which fixes s, so that $gs = s$ for all $g \in G_s$. Now we find the cardinality of the set $(g, s) : gs = s$.

Fixing a g and then summing over G gives $\sum_{g\in G} |S^g|$. Alternatively, fixing an s and summing over S yields $\sum_{s\in S} |G_s|$. The latter sum can be simplified with the Orbit- Stabilizer Theorem. Indeed,

$$\begin{aligned}\sum_{s\in S} |G_s| &= \sum_{s\in S} \frac{|G|}{|O_s|} \\ &= |G| \sum_{s\in S} \frac{1}{|O_s|} \\ &= |G| \sum_{orbits O} \frac{|O_s|}{|O_s|} \\ &= |G| \cdot N,\end{aligned}$$

where N is the number of orbits of S and $|O_s|$ is the order of the orbit of s. This gives an expression for N:

Lemma 3.1 (Burnside's Lemma) *Given a group G acting on a set S, the number of orbits N due to this action is given by*

$$N = \frac{1}{|G|} \sum_{g\in G} |S^g|.$$

Remark 3.2 It is rather interesting to find that this is just as often often referred to as the *not- Burnside Lemma* in recognition of its prior discovery independently by Cauchy and Frobenius.

We immediately apply Burnside's Lemma to Example 1.2 letting G be the group C_4, the cyclic group of rotations by multiples of 90^0. We number the squares as quadrants. Letting S^θ denote the set of coloured boards (functions) which remain unchanged upon rotation by θ, we see that

a) Any board is fixed by rotation through $\theta = 0$. There are 16 such boards and so $|S^0 = 16$.

b) A 90^0 rotation sends 1 to 2, 2 to 3, 3 to 4, 4 to 1. If the board is unchanged, the colour of square 1 should be the same as that of 2, and similarly for all pairs. We thus have a monochromatic board, giving $|S^{90}| = 2$(red or black).

Symmetry gives $|S^{270}| = 2$ as well.
c) A 180^0 rotation is a reflection about the x- axis and sends 1 to 4, 2 to 3. Arguing as before, we get $|S^{180}| = 2^2 = 4$.

Burnside's Lemma now gives $N = \frac{1}{4}(16 + 2 + 2 + 4) = 6$, which we had obtained previously.

However, there are some clear limitations to the use of this lemma: computing fixed points of sets under group action is often tedious and does not allow us to solve problems such as the following:

Problem: Find the number of different cubes that can be obtained by colouring its vertices in 2, 3, m colours. Two cubes are considered equivalent under rotational symmetry.

4 A Better Understanding of Equivalent Functions; The Cycle Index

We need a better understanding of the properties of a function that is fixed by a permutation in G. A key insight is the following lemma, which restricts the values of the functions that can be fixed by a permutation $\pi \in G$.

We will use the fact that under action of π,we obtain a permutation of the elements of the domain which can be decomposed into cycles. If there are b_i cycles of length i, the permutation is said to have *cycle type* $(b_1, b_2, ..., b_n)$.

Consider one of the cycles of length j, C = $(a_1, a_2, .., a_j)$. We have the following lemma:

Lemma 4.1 *If π fixes a function f on R^D then f is constant for all $a_i \in C$.*

Proof: The above is equivalent to saying that $e_\pi(f) = f$ and so for all $d \in D, f(d) = f(\pi(d))$. But if d is in C, ie. $d = a_i$ for some i, then $f(a_i) = f(\pi(a_i)) = f(a_{i+1})$ and so f is constant within C.

In the language of example 1.2, if upon rotation by 90°, which generates a single cycle of length 4 among the four squares, if we get an unchanged board then for all squares in the cycle (basically, all 4 squares), the value of the function (the colour assigned to the square) must be constant for all elements of the cycle (squares).

Upon rotation by 180° we get two cycles, (14) and (23). So within each cycle, any assignment must be constant over the cycle. How this is to be used to count the patterns will be deferred until we have a scheme to represent assignments suitably.

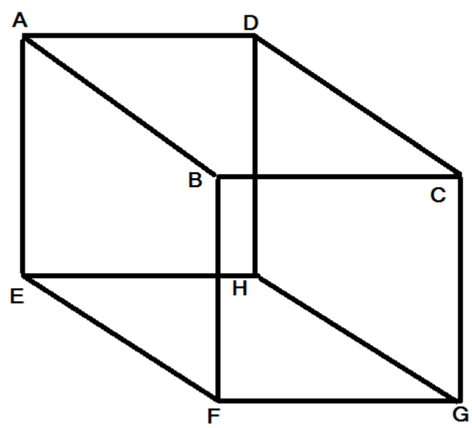
Moreover, in order to use our knowlegde of Lemma 4.1 we will need some sort of bookkeeping device that stores information about the cycles generated separately by all the elements of G. The 'device' most suitable for computation is a polynomial whose each term describes one element of G, and so we define the following:

Definition 4.2 (Cycle Index) For each $\alpha \in G$ with $|G| = n$ define a *monomial* $x_1^{b_1} x_2^{b_2} ... x_n^{b_n}$ where the x_i are arbitrary variables and b_i is the number of cycles of length i in α. Then the *cycle index* of G in its action on set S is defined as

$$P_G(x_1, x_2, ..., x_n) = \frac{1}{|G|} \sum_{\alpha \in G} (monomial(\alpha)).$$

Since calculating the cycle index of our groups is fundamental to what is to come, it is worthwhile to perform the calculation for some well- known objects.

Example 4.3 Now let G be the group of rotations (rotational symmetries) of a cube $ABCDEFGH$. The total number of symmetries, $|G|$, is 24. Indeed, a vertex can be rotated to any other face under the action of G, so its orbit has order 8. Also, its stabiliser is the group of rotations by multiples of 120^0 about the axis passing through the vertex and the centre of the cube, and has order 3. The Orbit-Stabilizer Theorem now gives $|G| = 8 * 3 = 24$. The rotations in G can be classified as follows:



a) Rotations fixing vertices: A rotation fixing, say, A and G can either send $E \to B \to D \to E$ (120° rotation) or can send $E \to D \to B \to E$ (240° rotation). We recognize that these two rotations, along with the identity, constitute the stabilizer of A and G, mentioned previously. Thus, for each such rotation, there are two 1-cycles and two 3-cycles, to which we assoociate the monomial $x_1^2 x_3^2$. There are two rotations per pair of opposite vertices and 4 such pairs in the cube, giving a factor of 8 to the above monomial.

b) Rotations fixing edges: Suppose we rotate the cube by 180° about the axis formed by joining the midpoints of AB and GH. Then, we have the cycles (AB), (GH), (DF), (CE) and the associated monomial x_2^4. There are 6 axes, one for each pair of opposite edges.

c)Rotations fixing faces: Here the axis is through the centre of a face and perpendicular to it. For rotations by 90° or 270°, the vertices that bound these faces are permuted among themselves in two 4-cycles, ie. $2 \cdot x_4^2$. For a 180°- rotation, there are four 2- cycles formed by vertices on the same face, diagonal to each other. We can choose the axis in 3 ways.
This gives us the cycle index of G:

$$P_G(x_1, x_2, .., x_n) = \frac{x_1^8 + 8x_1^2x_3^2 + 9x_2^4 + 6x_4^2}{24}$$

where the first term is the identity permutation monomial.

Example 4.4 The purpose of this example is to show that in addition to the degree, even the coefficients of the cycle index can change depending on which element of the cube we choose as our set. Let us perform the same calculation for the faces of the cube.

a) As above, for a 120° rotation the cycles are (A)(G)(EBD)(FCH). So $ABCD \to ADHE \to AEFB \to ABCD$ and similarly for the other three faces. This gives the term $8x_3^2$.

b) Using the cycle decomposition for vertices, we have $ABCD \to BAEF, ADHE \to BFGC, DCGH \to FEHG$ with the associated term $6x_2^3$.

c) For 90° and 270° rotations, two faces are fixed and the others go around in a 4-cycle. For 180°, the 4-cycle breaks into two 2-cycles. The contributionto the cycle index reads $6x_1^2x_4 + 3x_1^2x_2^2$.
Thus, [$P_G(x_1, x_2, ..., x_n) = \dfrac{x_1^6 + 8x_3^2 + 6x_2^3 + 6x_1^2x_4 + 3x_1^2x_2^2}{24}$.]

Example 4.5 We shall try to find the cycle index of the cyclic group C_n of rotations acting on a regular n- gon- a situation encountered in the necklace problem outlined along with Example 1.2. First we observe that this group is isomorphic to the additive group $\mathbb{Z}_n$ of integers modulo n, acting on the set $[n] = (1, 2, ..., n)$.
Next, we see that the difference between consecutive numbers in a cycle is the same by definition, and for this difference d, the length of the cycle is the least number k such that $kd \equiv 0 \pmod{n}$. For a fixed d, this k is unique and so the length of all cycles is the same. Thus, if there is a cycle of length k, then there are n/k cycles, partitioning $\mathbb{Z}_n$. Of course $k|n$ by this argument.

Now, we must find the number of values of d that act on $[n]$ in this manner, for a fixed k. For the above congruence to hold clearly $d = \frac{n}{k} \cdot t$ for some $t \leq k$. Notice that with this value of d, the number $k' = \frac{k}{(t,k)}$ is also a solution to the congruence, because the product $k'd$ is still a multiple of n. Since k is the least solution for fixed d we must have $(t,k) = 1$ and since $t < k$ there are exactly $\phi(k)$solutions, ie. $\phi(k)$ permutations with cycle length k.
These contribute the term $\phi(k)x_k^{n/k}$ to the cycle index. Thus, we finally have our expression:

$$P_G(x_1, x_2, .., x_n) = \frac{\sum_{k|n} \phi(k)x_k^{n/k}}{n}.$$

Corollary Since the sum of coefficients in the cycle index is by definition 1, we obtain $\sum_{k|n} \phi(k) = n$.

5 What do we really want now? Weight for it . . .

To summarize what we have done so far, we have restated the problem of assigning properties to sets as a problem of finding the number of patterns, or classes of functions within a power set, some of which are equivalent under the action of a permutation group. The number of patterns is closely related to the number of functions that are fixed by the permutations of the group. Burnside's Lemma is the clearest example of this. Furthermore, if a permutation fixes a function we have proved it must be constant over its individual cycles. All the information we could possibly need about cycles is contained in the cycle index, which is a lot easier to compute than the fixed points in Burnside's Lemma.
So we have the tools we need to find the number of patterns in a group. But with the amount of machinery we have at our disposal, a little more effort can actually allow us to solve a much bigger question: What is the exact nature of the patterns? How many of the squares in Example 1.2 have equal numbers of red and black? How many cubes exist where exactly three vertices are coloured red, and the rest blue?
The new idea we present is motivated by the need to distinguish between different elements of the range R. We define for each $r \in R$ a *weight function* $w : R \to \mathbb{R}$. Also, we define the following:

Definition 5.1 The weight of a function $f : D \to R$ is given by

$$w(f) = \prod_{d \in D} f(d).$$

Also, the total weight of R is $w(R) = \sum_{r \in R} w(r)$.
For example, if there are six mathematicians to each of whom one problem from a selection of hard problems is assigned, if three are given the Goldbach Conjecture, two the P/NP problem and one the Riemann Hypothesis, the weight assignment (using an obvious convention) for this function would be g^3p^2r.
Our definition of weights has the following very desirable property:

Proposition 5.2 *Equivalent functions have equal weight.* Proof: Suppose $e_\pi(f) = g$. Then for all $d \in D$, $f(d) = g(\pi(d))$ and thus

$$\begin{aligned} w(f) &= \prod_{d \in D} f(d) \\ &= \prod_{d \in D} f(\pi^{-1}d) \\ &= \prod_{d \in D} g(d) \\ &= w(g) \end{aligned}$$

This allows us to make the following definitions:

Definitions 5.3 For each pattern F, define $w(F) = w(f)$ for some $f \in F$. By Proposition 5.2, this function is well-defined.
Now define the *pattern inventory*, which is given by

$$P.I. = \sum_F w(F)$$

where the sum covers all patterns in R^D.
At last we have clearly defined what we were looking for: The pattern inventory contains by means of its weighted terms all the information about the action of the group that we could possibly need. The number of patterns, which we calculate when we have considered all patterns to be equally important, is indeed obtained by setting all weights equal to unity. Knowledge of the pattern inventory is, in principle at least, the solution to all the problems we have set ourselves hitherto.
To see this, we consider the following example:

Example 5.4 We are to place marbles into a container with three holes arranged in the form of an equilateral triangle. There are 6 marbles in total. We must list all possible arrangements, assuming that the dihedral group D_3 acts on the triangle.
Here, if we assign weight a_i to a group of i marbles in the same hole, the possible arrangements are given in the pattern inventory

$$P.I. = a_6 + a_5a_1 + a_4a_1^2 + a_4a_2 + a_3^2 + a_3a_2a_1 + a_2^3.$$

We leave it to the reader to verify that the pattern inventory for Example 1.2, assuming that red and black are weighted with r and b respectively, is

$$P.I. = r^4 + r^3 b + 2r^2 b^2 + rb^3 + b^4.$$

This example reminds us of something very important:

Remark *Different patterns may have the same weight.*
We now have everything we need to find the pattern inventory purely by examining group actions on the given domain. There is a final link, the relation between a function constant over given subsets of the domain and its weight, which is given by the following lemma:

Lemma 5.5 For a function $f : D \to R$ let $D = \bigcup_i D_i$ such that f is constant over any given D_i. Then the total weight of all such functions f is

$$W = \prod_i (\sum_{r \in R} w(r)^{|D_i|})$$

Proof: Any such function has the weight $w_1^{|D_1|} w_2^{|D_2|} ... w_i^{|D_i|}$... and so belongs to the expression W. Conversely, any term in W corresponds to the unique function f with $f(d) = w_i$for all $d \in D_i$.
Thus, if our six mathematicians (see def. 5.1) asked for a reallocation of problems but demanded that their teammates remain unchanged, even if two teams got the same problem, the possible weights of functions in this case would be terms from the product [$(g^3 + p^3 + r^3)(g^2 + p^2 + r^2)(g + p + r)$]
where it is ensured that any assignment remains constant over a given team, as was required.

6 Polya's Fundamental theorem on Enumeration

Drumroll, please...

Theorem 6.1 (Polya's fundamental enumeration theorem) The pattern inventory of a set of functions R^D acted upon by a permutation group G is obtained by replacing in the cycle index of the group, the variable x_i with the sum $\sum_{r \in R} w(r)^i$. That is,

$$P.I. = P_G(\sum_{r \in R} w(r), \sum_{r \in R} w(r)^2, ..., \sum_{r \in R} w(r)^n).$$

Proof: Some patterns have equal weight W_i. Let there be m_i such patterns. The set of functions of weight W_i is a union of disjoint subsets of $R^D, T = \bigcup_{i=1}^{m_i} F_i$.
Now G acts on T producing m_i orbits, so Burnside's Lemma gives

$$m_i = \frac{1}{|G|} \sum_{\pi \in G} |S^\pi|$$

where S^π is the subset of T fixed by π.
The pattern inventory is therefore given by

$$\begin{aligned} P.I. &= \sum_i m_i W_i \\ &= \frac{1}{|G|} \sum_{\pi \in G} \sum_i |S^\pi W_i| \\ &= \frac{1}{|G|} \sum_{\pi \in G} |(W_f ix(\pi)|. \end{aligned}$$

where $W_f ix(\pi)$ is the total weight of all functions fixed by π.
Choose a $\pi \in G$ with cycle type $(b_1, b_2, ..., b_n)$. Any function in S^π, we have seen, is constant over the cycles of π, which together partition D. Thus, Lemma 5.5 gives

$$P.I. = \frac{1}{|G|} \sum_{\pi \in G} \prod_j (\sum_{r \in R} w(r)^j)^{b_j}.$$

This is because we collect the similar expressions for the b_j cycles of equal length j. Comparing this with the definition

$$P_G(x_1, x_2, .., x_n) = \frac{1}{|G|} \sum_{\pi \in G} \prod_j x_j^{b_j}$$

the theorem is proved.

Corollary- The number of patterns is given by $P_G(|R|, |R|, ..., |R|)$.

7 Applications

Example 7.1 We return to the problem of vertex colouring of a cube in two colours, red and blue. The pattern inventory, by Polya's theorem, is

$$\frac{(r+b)^8 + 8(r+b)^2(r^3+b^3)^2 + 9(r^2+b^2)^4 + 6(r^4+b^4)^2}{24}$$

a) The number of patterns

$$
\begin{aligned}
&= \frac{2^8 + 8.2^2.2^2 + 9.2^4 + 6.2^2}{24} \\
&= 23.
\end{aligned}
$$

b) The number of patterns of 4 vertices each coloured red and blue = coeffficient of r^4b^4 in the P.I.

$$
\begin{aligned}
&= \frac{\binom{8}{4} + 8 \cdot 2 \cdot 2 + 9 \cdot 6 + 6 \cdot 2}{24} \\
&= 7.
\end{aligned}
$$

Example 7.2 How many necklaces of n beads can be made in m colours, if the group acting on them is the cyclic group C_n? We saw that the cycle index was given by

$$P_G(x_1, x_2, .., x_n) = \sum_{k|n} \phi(k) x_k^{n/k}$$

and thus the number of necklaces, by Polya's theorem, is

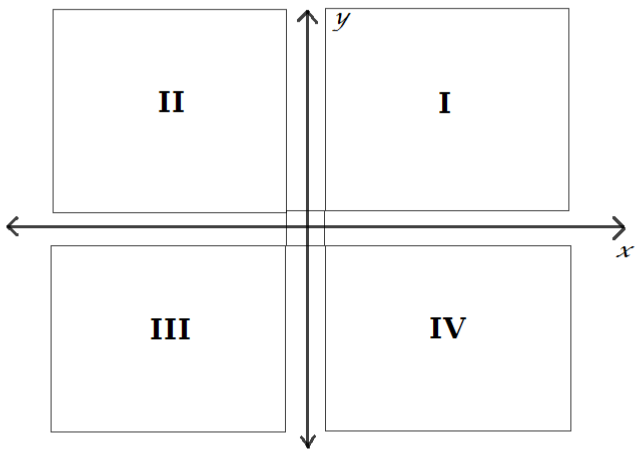
$$N = \sum_{k|n} \phi(k) mn/k.$$

Remark: There is an elegant solution to the above problem by means of the Mobius Inversion Formula. The necklace must, as we showed, have cycle length $k|n$, and if the d repetitions due to permutations within each cycle are considered we have

$$m^n = \sum_{k|n} kM(k)$$

where $M(k)$ is the number of patterns with cycle length k. It can be proved that $M(k)$ is multiplicative and thus the Inversion Formula can be applied, giving us the same answer after a lengthy computation which we will avoid here.

Example 7.3 We consider the action of the dihedral group D_4 on an $n \times n$ square chessboard where n is even. For the purpose of calculation we introduce a 'dummy' square at the origin and allow each unit square to have side 1 so that a lattice point is at their centre. The sides of the square have length 1. We first obtain the cycle index of the board.



a) Rotations by 90° or 270°: Each 1×1 square belongs to a subgroup of four squares under these motions and so for every four squares a 4-cycle is generated thus giving a monomial of $2 \times x_4^{n^2/4}$. The 180° rotation gives $x_2^{n^2/2}$.
b) Here we assume that the motion of any unit square corresponds to the motion of the lattice point within the square. Consider the reflection about the x-axis followed by rotation through 90° or 270°. To analyse this we use the representation of points on $\mathbb{R}^{\not\vDash}$ as a 2-D coloumn vector and recall that the motions of D_n can be represented using orthogonal matrices A with $A^2 = I$. Thus any square will be part of a 2- cycle or less. It is now sufficient to find the number of fixed points.
Rotation by 90° and reflection about the x- axis have the associated matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; and their product is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which operates on the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ to give $\begin{pmatrix} y \\ x \end{pmatrix}$. Thus, under this operation, only the n squares along one diagonal are stabilized and the rest belong to 2- cycles. The monomial is the same for the cas of reflection and then 270° rotation, so we finally get the term $2 \cdot x_1^n x_2^{(n^2-n)/2}$. The remaining reflection is just reflection about the y- axis with the term $x_2^{n^2/2}$.
Our cycle index, then, is

$$\frac{x_1^{n^2} + 2x_4^{n^2/4} + 3x_2^{n^2/2} + 2x_1^n x_2^{(n^2-n)/2}}{8}$$

and replacing the variables with 2 gives the answer.
The reader is invited to perform the similar calculation for odd n and see if a closed form can be obtained without separating the two cases.

8 Applications to Chemistry

The fact that molecules have higher probabilities of assuming symmetrical configurations makes it possible to obtain valuable insights via group theory. All the problems we discuss are related to the counting of isomers.
The first example is from my own chemistry class, and one can see that it can turn messy without group theoretic ideas.

Example 8.1 A *coordination complex* is a compound formed when a positively charged transition metal atom attracts negatively charged *ligands* in a solution. The number of ligands that coordinate with it determine the geometry of the complex.
Thus, six ligands form an octahedral complex written as $M(abcdef)$ where M is the metal and the rest are ligands, some of which may be equal to each other. We are to find the number of isomers where a, b, c, d, e, f are all distinct.
To solve this, we note that the octahedron is a dual of the cube in the sense that the vertices of the octahedron correspond to the faces of the cube, and vice versa. Therefore, the cycle index of the group of rotations actng on the vertices is exactly the same as that of the same group acting on the faces of the cube. and this was found in Example 5.4 to be

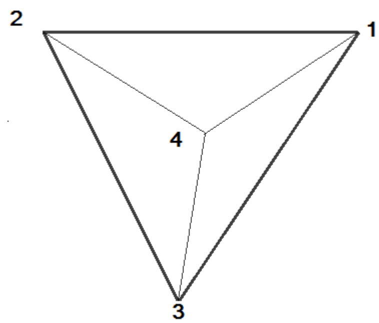
$$\frac{x_1^6 + 8x_3^2 + 6x_2^3 + 6x_1^2x_4 + 3x_1^2x_2^2}{24}.$$

By Polya's theorem, the number of diferent cubes obtained is the coefficient of $abcdef$ in

$$\begin{aligned}
&\frac{(a+b+c+d+e+f)^6}{24} \\
&= \binom{6}{1,1,1,1,1,1} \\
&= 30.
\end{aligned}$$

Note that this procedure counts both geometrical and optical isomers.

Example 8.2 There is a great deal of scope for the application of Polya Theory in the study of organic molecules. Carbon in its unsaturated form, is the centre of a tetrahedral molecule idealised as shown.



There are two rotations that fix the substituent 4, through 120° and 240°, permuting the other 3 substituents in a 3- cycle. Thus, these rotations contribute a term $4 \cdot 2 \cdot x_1 x_3$ to the cycle index. Next, we fix the axis on the line joining the midpoints of opposite sides, say 14 and 23. 180° rotations about this axis give us the cycles $(14), (23)$. There are 3 such axes, giving us the term $3 \cdot x_2^2$. Including the identity, we obtain the cycle index

$$\frac{x_1^4 + 8x_1x_3 + 3x_2^2}{12}.$$

a) Existence of the enantiomeric form: Suppose the substituents p, q, r, s are all different. We want to find the number of molecules of the type $Cpqrs$. This is the coefficient og $pqrs$ in the pattern inventory, which is $\dfrac{\binom{4}{1,1,1,1}}{12} = 2$. The occurrence of two patterns (non- superimposable molecules) when the substituents are different is called *chirality* and the molecules form an*enamtiomeric pair.*
These molecules form a mirror- image. Indeed, by fixing two substituents, we can interchange the other two, i.e. reflect them about the mirror formed by the plane of the fixed substituents. This also tells us why the substituents need to be different: otherwise, we can choose the mirror so that one molecule is reflected into itself.
b) A chemist studying the hydrogen content of alkyl halides with one carbon must know how many compounds are possible for a given number of hydrogen substituents. This asks for the pattern inventory with H weighted with h and the halogens Cl, Br, I weighted with 1 each. The pattern inventory is obtained by substituting for x_i the quantity $h^i + 3$ in the cycle index, and carrying out the required computation we obtain

$$P.I. = h^4 + 3h^3 + 6h^2 + 11h + 5$$

which solves the problem.

This theory can be extended and generalised, in ways we shall not look into here; for example, what if a group acts on the elements of the range as well? A generalisation to Polya's theorem in this regard was provided by Nikolaas de Bruijn, and there are various extensions that I hope to be able to acquaint myself with in future.

I am grateful to Professor B. Sury of the Indian Statistical Institute, Bangalore, for guiding me and suggesting that I read some articles, including his own, related to Burnside's Lemma and related aspects of group theory, which became the motivation for this study.

9 References

1) *Combinatorial Techniques*, Sharad S. Sane (This was my main source for this topic)
2) *Algebra*, M. Artin (I learnt the basic group theory required from this book)