Uncountably generated ideals of functions

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Undergraduates usually think that the study of continuous functions and the study of abstract algebra are divorced from each other. More often than not, they find it very surprising that concepts like rings and ideals could be applied to function spaces as well! Some applications in algebra texts concern the ring $C[0,1]$ of real-valued continuous functions on $[0,1]$; however, these texts restrict themselves to a few standard exercises although more could be accomplished with almost the same amount of labor. For instance, the exercises in ([1], p.388),([2], p.259) and ([3], p.140) ask for a proof that maximal ideals in $C[0,1]$ are not finitely generated. The fact that these maximal ideals are not *countably* generated does not seem to be as well-known as it should be although the proof is not harder! We will prove this, and then use it to produce some non-prime ideals in $C(0,1)$ which cannot be countably generated as well. Without further ado, let us begin:

Maximal Ideals in C[0,1]

Let $M_c = \{f \in C[0,1] : f(c) = 0\}$ where $c \in [0,1]$. Contrary to what we want to prove, assume that I_c is generated by a countable set $(f_1, f_2, \cdots)$. By re-scaling, we may assume that $|f_n(x)| \leq 1$ for all x and for all n. Consider the function

$$f(x) := \sum_{n=1}^{\infty} \sqrt{|f_n(x)|}/2^n.$$

By uniform convergence, f is continuous. Clearly, $f \in I_c$. By assumption $f = \sum_{i=1}^{r} g_i f_i$. Let M be an upper bound for $|g_i|$ for all $i \leq r$ and all x in $[0,1]$. Then,

$$|f(x)| \leq M \sum_{i=1}^{r} |f_i(x)|.$$

Now, by continuity, there is a neighborhood U of c such that

$$\sqrt{|f_i(x)|} < \frac{1}{2^i M}$$

for all $x \in U$ and all $i \leq r$. We may also assume that for each $x \in U$, $f_i(x) \neq 0$ for some $i \leq r$. Thus, for each $x \in U$, we get some i such that

$$|f_i(x)| < \frac{\sqrt{|f_i(x)|}}{2^i M}.$$

Hence,

$$|f(x)| \leq M \sum_{i=1}^{r} |f_i(x)| < \sum_{i=1}^{r} \frac{\sqrt{|f_i(x)|}}{2^i} \leq |f(x)|$$

which is a contradiction. This proves that I_c is not countably generated.

Prime ideals in $C[0,1]$

In any commutative ring R with identity 1, recall the basic method (essentially, the only method) of constructing prime ideals. One considers a multiplicatively closed subset S of R containing 1. In other words, S contains 1 and satisfies the property that $st \in S$ whenever $s, t \in S$. If P is any ideal of R which is maximal with respect to the property that P does not intersect S, then P must be a prime ideal. This is so, because if $ab \in P, a \notin P, b \notin P$, then the ideals $P + (a)$ and $P + (b)$ must intersect S, by the maximality property. Thus, if

$$s = p_1 + ar_1 \in S \cap (P + (a))$$

and

$$t = p_2 + br_2 \in S \cap (P + (b)),$$

then $st = p_1(p_2 + br_2) + p_2 ar_1 + abr_1 r_2 \in S \cap P$, which contradicts the choice of P.

Note also that, such an ideal P which is maximal with the property that it does not intersect S, can be chosen by using the set-theoretic property known as Zorn's lemma.

In $C[0,1]$, one could consider S to be the set of all polynomial functions on $[0,1]$ whose top coefficient is 1. Then, a prime ideal P with the property that $P \cap S = \emptyset$ is not maximal; indeed, if M_c is a maximal ideal containing P, then the polynomial $X - c$ is in $S \cap M_c$ and cannot, therefore, belong to P. It is not clear if this can be countably generated. In general, the following question is natural :

Are there any finitely generated prime ideals in $C[0,1]$?

A non-prime ideal in C(0,1)

There is a nice way to use the previous result to produce an ideal in the ring of continuous functions on the noncompact interval $(0,1)$, which is neither prime nor countably generated. Let

$$I = \{f \in C(0,1) : f(1/n) = 0 \text{ for all but finitely many } n\}.$$

Note that I is indeed an ideal but is not a prime ideal. For instance, if we consider some $f \in C(0,1)$ which vanishes at all $1/2n$ and does not vanish at any $1/(2n+1)$ and a function $g \in C(0,1)$ which vanishes at all $1/(2n+1)$ and does not vanish at any $1/2n$, then neither f nor g are in I whereas $fg \in I$.

We will show that I cannot be countably generated. To begin with, fix disjoint closed intervals

$$K_n := [\frac{1}{n+1} + \epsilon_n, \frac{1}{n}]$$

for all n.

Suppose, contrary to what we want to prove, that I is generated by a countable set $\{f_1, f_2, \cdots\}$ in $C(0,1)$. For each n, we look at the ring $C(K_n)$ and its maximal ideal $I_{1/n}$ consisting of those functions which vanish at $1/n$. For each n, $I_{1/n}$, as we have shown, is not countably generated. Now, the restrictions $f_i|_{K_n}$ which happen to vanish at $1/n$, form a countable subset in $I_{1/n}$. For each n, pick an element $\phi_n \in I_{1/n}$ which is *not* in the ideal generated by the restrictions $\{f_i|_{K_n} : i \geq 1, f_i(1/n) = 0\}$. As ϕ_n are defined on disjoint intervals, they have a continuous extension ϕ to the whole of $(0,1)$. So, now we have $\phi \in C(0,1)$ with $\phi|_{K_n} = \phi_n$. Note that this is possible because the limit point 0 is not in the set $(0,1)$. Since $\phi_n(1/n) = 0$ for every n, $\phi(1/n) = 0$ for each n; that is, $\phi \in I = (\{f_1, f_2, \cdots, \})$. Therefore, by assumption, we may write $\phi = \sum_{i=1}^{r} g_i f_i$, for suitable $g_i \in C(0,1)$, and some r. As $f_1, \cdots, f_r$ vanish at all but finitely many $1/n$, there is a common N (indeed, infinitely many) so that $f_i(1/N) = 0$ for $i = 1, \cdots, r$. Therefore $\phi(1/N) = 0$. However, the fact that

$$\phi_N = \phi|_{K_N} = \sum_{i=1}^{r} g_i|_{K_N} f_i|_{K_N}$$

contradicts the choice of ϕ_N in $I_{1/n}$. Therefore, I cannot be countably generated.

References

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[2] D.S.Dummit & R.M.Foote, *Abstract algebra*, 2nd ed., John Wiley & Sons, Inc., New York (Asian Edn.), Singapore, 2003.

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