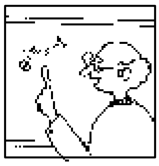
Classroom



In this section of* Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

On How to Arrange Infinitely many Marriages!

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The well-known 'marriage problem' asks whether it is possible to fix marriages for a set of n girls given that each set of r girls is acquainted with at least r boys. Here, it is understood that each girl is to be married to one of her acquaintances and that no polygamy is allowed. It should be clarified as to what the word 'at least' that occurs in the formulation means. This means that, for every $r \geq 1$, and, corresponding to any group of r girls, there is a group of r boys each of whom is acquainted with one girl from the group. The answer to the above question is 'yes' as can be shown by applying induction on n (for instance, a proof on these lines can be found in [1], p.72).

Given infinitely many girls (!), every r of whom are acquainted with a finite number $\geq r$ of boys for every r, can each girl get married to one of her acquaintances?

Going a step further, let us discuss the possibility of infinite marriage-fixing. That is to say, given infinitely many girls (!), every r of whom are acquainted with a finite number $\geq r$ of boys for every r, can each girl get married to one of her acquaintances? A moment's thought tells us that it is not all that obvious to answer. There seems to be no simple-minded resolution of this

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dilemma. However, the answer does turn out to be 'yes' and we can prove this using some basic notions of general topology. As a matter of fact, one needs a version of an axiom of set theory known as the axiom of choice. Axiom of choice is a statement which on the first glance, looks obvious. Far from it! Let us try to discuss it in simple language. First, look at any two non-empty sets X_1 and X_2. One can consider the Cartesian product $X_1 \times X_2$ of these two sets. This is the set of all tuples (x_1, x_2) with $x_1 \in X_1, x_2 \in X_2$. In other words, any point of $X_1 \times X_2$ can be described as a simultaneous choice of an element each from X_1 and from X_2. Evidently, this definition can be carried over to finitely many sets. How can one define the Cartesian product of infinitely many non-empty sets, say, sets X_α parametrized by $\alpha \in \Gamma$ for some set Γ? A point of this must mean a simultaneous choice of an element from each X_α. Intuitively, this may seem obviously possible. However, one cannot prove it from the basic axioms of set theory. That such a 'choice' is possible is the statement referred to as the axiom of choice. Axiom of choice is independent of the basic axioms of set theory. It is equivalent to a statement for topological spaces known as Tychonov's theorem whose statement on first glance looks to be stronger than the axiom of choice. As a matter of fact, we shall require only the countable version of this for our discussion of the generalisation of the marriage problem.

Tychonov's theorem: *Let $\{X_\alpha\}_{\alpha\in\Gamma}$ be compact topological spaces. Then, the Cartesian product $X = \prod_{\alpha\in\Gamma} X_\alpha$ has the structure of a compact topological space.*

Recall that a topological space X is said to be Hausdorff if any two points have disjoint open neighbourhoods and such a space is called compact if it has the following property: any system of closed subsets $\{X_\alpha\}_{\alpha\in\Lambda}$ with the property that $\bigcap_{\alpha\in\Lambda} X_\alpha = \Phi$ must satisfy $\bigcap_{i=1}^{n} X_{\alpha_i} = \Phi$ for some $\alpha_1, \ldots, \alpha_n \in \Lambda$ (equivalently, finite intersection property for closed sets implies non-empty intersection). Let us use Tychonov's theorem to solve the infinite version of the marriage problem.

Axiom of choice appears under many guises. A widely-used version is known as Zorn's lemma after Max Zorn.

For each girl g, consider the set $B(g)$ of boys acquainted with her, as a topological space under the discrete topology. Being a finite set, this is a compact, Hausdorff space. Therefore, the Cartesian product B of all $\{B(g);\ g \text{ girl}\}$ is compact, by Tychonov's theorem. If $\{g_1, \cdots, g_n\}$ is a set of n girls, look at the subset $B(g_1, \cdots, g_n)$ of B consisting of all the tuples $\{b(g) : g \text{girl}\}$ in B such that $b(g_1), \ldots, b(g_n)$ are n different boys. By the solution to the finite marriage problem, $B(g_1, \cdots, g_n)$ is a non-empty, closed set. In other words, the class of all such sets F has the finite intersection property and, therefore, has non-empty intersection. An element (i.e., a tuple) in this intersection evidently produces partners for all girls!

Thus, this is a resolution by Zorn's (method) of our dilemma!

Although this solves the problem we have posed, one might still wonder whether the problem has an affirmative solution with the weaker assumption that some girl may possibly be acquainted with infinitely many boys. It turns out that this is false. For discussions on this as well as for a discussion of Tychonov's theorem, [2] is an excellent reference.

Suggested Reading

[1] B Sury, *Resonance*, Vol.3, No.2, 1998.

[2] L Mirsky, *Transversal theory*, Academic Press, New York-London, 1971.

Space and time are not conditions in which we live, they are modes in which we think

Albert Einstein