Macaulay Expansion

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Abstract. Given natural numbers n and r, the "greedy" algorithm enables us to obtain an expansion of the integer n as a sum of binomial coefficients in the form $\binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \cdots + \binom{a_1}{1}$. We give an alternate interpretation of this expansion, which also proves its uniqueness in an interesting manner.

The 1996 Iranian mathematical olympiad competition contained the following problem. *For natural numbers n and r, there is a unique expansion*

$$n = \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \cdots + \binom{a_1}{1}$$

with each a_i an integer and $a_r > a_{r-1} > \cdots > a_1 \geq 0$.

The existence is fairly easy to prove using the "greedy" algorithm. This expansion is sometimes known as the Macaulay expansion. However, the following alternate interpretation does not seem to be well known; it gives uniqueness in an interesting manner. In what follows, the following well-known convention is used: the binomial coefficient $\binom{n}{r}$ is equated to 0 if $n < r$.

For each natural number r, denote by S_r the set of all r-digit numbers in some base b whose digits are in strictly decreasing order of size. Evidently, S_r is nonempty if and only if $b \geq r$; in this case, S_r has $\binom{b}{r}$ elements. Let us now write the elements of S_r in increasing order.

For instance, in base 10, the first few of the 120 members of S_3 are:

$$(2, 1, 0), (3, 1, 0), (3, 2, 0), (3, 2, 1), (4, 1, 0), (4, 2, 0), (4, 2, 1), (4, 3, 0), \ldots.$$

We will prove the following.

Theorem. *Given any positive integer n, and any base b such that $\binom{b}{r} > n$, the $(n+1)$-th member of S_r is $(a_r, \ldots, a_2, a_1)$, where $n = \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \cdots + \binom{a_1}{1}$. In particular, for each n, the Diophantine equation $\binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \cdots + \binom{a_1}{1} = n$ has a unique solution in positive integers $a_r > a_{r-1} > \cdots > a_1 \geq 0$.*

Here are a couple of examples to illustrate the theorem.

(i) Let $r = 3$ and $n = 12$. We may take any base b so that $\binom{b}{3} > 12$. For example, $b = 6$ is allowed because $\binom{6}{3} = 20$. Among the 20 members in S_3, the 13th member is $(5, 2, 1)$. Note that

$$\binom{5}{3} + \binom{2}{2} + \binom{1}{1} = 12.$$

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(ii) Let $r = 3, n = 74$. We may take $b = 10$ as $\binom{10}{3} = 120$. The 75th member of S_3 is $(8, 6, 3)$. Note that

$$\binom{8}{3} + \binom{6}{2} + \binom{3}{2} = 74.$$

Proof of theorem. First of all, we notice that the number of members in S_r that have first digit $< m$ equals $\binom{m}{r}$; this is because we are choosing r numbers from $\{0, 1, \ldots, m-1\}$ and arranging them in decreasing order. Now, suppose the $(n+1)$th member of S_r is

$$(a_r, a_{r-1}, \ldots, a_1).$$

The number of members of S_r with first digit $< a_r$ is $\binom{a_r}{r}$. The number of members of S_r, whose first digit is a_r and which occur before the above member, is the number of members of S_{r-1} occurring prior to $(a_{r-1}, \ldots, a_1)$. Inductively, it is clear that this equals

$$\binom{a_{r-1}}{r-1} + \cdots + \binom{a_2}{2} + \binom{a_1}{1}.$$

Therefore, the number of members of S_r occurring prior to the $(n+1)$th member above (which must be n) is

$$\binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \cdots + \binom{a_1}{1}.$$

This proves our result. ■

Remark. We may proceed in a slightly different direction, if we do not use the first observation in the proof. For any k, we can obtain by induction that the number of elements in S_k *starting with* some a is $\binom{a}{k-1}$. Indeed, to prove this by induction, we use the identity

$$\binom{n}{r} = \sum_{m=1}^{n-1} \binom{m}{r-1},$$

which is itself seen by induction on n.

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