KRMO Solutions

KRMO-1995

1. Show that there are infinitely many positive integers A such that $2A$ is a square, $3A$ is a cube and $5A$ is a fifth power.

   Solution: First we observe that 2,3,5 divide A. So we may take $A = 2^{\alpha}3^{\beta}5^{\gamma}$. Considering $2A, 3A$ and $5A$, we observe that $\alpha+1, \beta, \gamma$ are divisible by 2; $\alpha, \beta+1, \gamma$ are divisible by 3; and $\alpha, \beta, \gamma+1$ are divisible by 5. We can choose $\alpha = 15 + 30n$; $\beta = 20 + 30n$; $\gamma = 24 + 30n$. As n varies over the set of natural numbers, we get an infinite set of numbers of required type. We may also take $A = 2^{15}3^{20}5^{24}n^{30}$ to get a different such infinite set.

2. Find all real numbers p, q such that the two roots of the equation $x^2 - px - 1 = 0$ and the two roots of the equation $x^2 - qx - 1 = 0$ form, in some order, the four terms of an arithmetic progression.

   Solution: First observe that one of the roots of each equation is negative and the other should be positive. Let a and b be the roots of $x^2 - px - 1 = 0$ and c and d be those of the equation $x^2 - qx - 1 = 0$. We may assume that $a < 0$ and $b > 0$; $c < 0$ and $d > 0$. We may further assume that $a < c$. The possible arithmetic progressions with these constraints are (1)a, c, b, d; (2)a, c, d, b. Let u be the common difference of either of the AP's. Clearly $u \neq 0$.

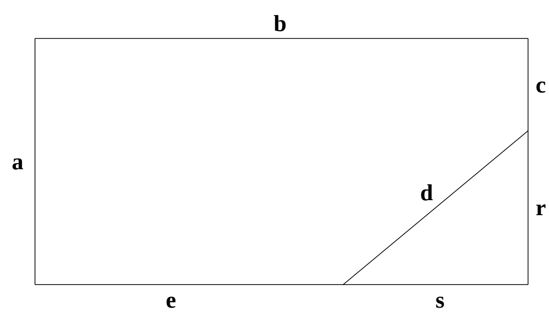
   (1) We have $c = a + u$, $b = a + 2u$, $d = a + 3u$. Using the relations $ab = -1$ and $cd = -1$, we get $u = -2a/3$. This gives $a(a - 4a/3) = -1$ and hence $a = -\sqrt{3}$. Using $a + b = p$ we get $p = -2/\sqrt{3}$. Similarly we obtain $q = 2/\sqrt{3}$.

   (2) Using $c = a+u, d = a+2u, b = a+3u$ and the relations between the roots and coefficients of a quadratic equation, we obtain $u = 0$. Thus this case does not arise.

   Hence the real values of p, q for which an AP is possible are $p = -2/\sqrt{3}, q = 2/\sqrt{3}$ and $p = 2/\sqrt{3}, q = -2/\sqrt{3}$.

3. From a rectangular piece of paper a triangular corner is cut off resulting in a pentagon. If the sides of the pentagon have lengths 10, 17, 18, 24 and 39 in *some order* find the sides of the rectangle and the sides of the triangle cut off.

   Solution: Let a, b, c, d, e denote the sides of the pentagon(see the adjoining figure). Then $d^2 = (a - c)^2 + (b - e)^2$. Since d is one among $\{10, 17, 18, 24, 39\}$, and d is a member of a Pythagorean triplet corresponding to the hypotenuse, we see that $d = 10, 17$ or $39 (10^2 = 8^2 + 6^2; 17^2 = 15^2 + 8^2; 39^2 = 36^2 + 15^2)$. Note that $d \neq 10$ since 8 is not a difference of two elements of $\{17, 18, 24, 39\}$. Similarly, we can rule out $d = 39$ because being the largest,



   it must be a side of the rectangle. The only possibility is $d = 17$. In this case we get $a = 18, b = 39, c = 10, e = 24, r = 8, s = 15$. Thus the rectangle has sides 39,18,39,18 and the triangle has sides 17,8,15.

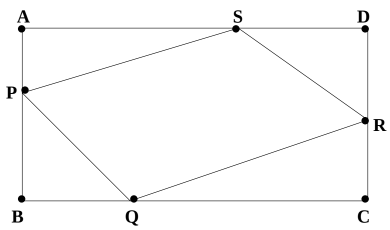
4. There are eight points in the plane, no three of them collinear. Find the maximum number of triangles formed out of these points such that no two triangles have more than one vertex in common.

Solution: First we observe that each vertex can be present in at most 3 triangles, for, having chosen a vertex, there remains 7 points from which 3 pairs are possible. If there are 9 or more triangles, these account for at least 27 vertices, repetitions allowed. In that case, one vertex has to occur in at least 4 triangles, a contradiction. Thus, there can be at most 8 triangles. The following example shows that 8 are possible; name the points as 1,2,3,4,5,6,7,8; the triangles are with vertices 123,145,167,248,368,578,256,347.

5. Suppose $ABCD$ is a rectangle and P, Q, R, S are points on the sides AB, BC, CD, DA respectively. Show that

$$PQ + QR + RS + SP > \sqrt{2}\, AC.$$

Solution:



We have

$$\begin{aligned}
&(PQ + QR + RS + SP)^2 \\
&\quad = PQ^2 + QR^2 + RS^2 + SP^2 + 2PQ \cdot QR \\
&\qquad + 2PQ \cdot RS + 2PQ \cdot SP \\
&\qquad + 2QR \cdot RS + 2QR \cdot SP + 2RS \cdot SP \\
&\quad > PB^2 + BQ^2 + QC^2 + CR^2 + RS^2 \\
&\qquad + DS^2 + SA^2 + AP^2 + 2BQ \cdot QC \\
&\qquad + 2PB \cdot PA + 2CR \cdot RD + 2SD \cdot SA \\
&\qquad (\text{since } PQ \cdot QR > BQ \cdot QC, \text{ etc.}) \\
&\quad = (PA + PB)^2 + (BQ + QC)^2 \\
&\qquad + (CR + RD)^2 + (DS + SA)^2 \\
&\quad = AB^2 + BC^2 + CD^2 + DA^2 \\
&\quad = AC^2 + BD^2 = 2AC^2.
\end{aligned}$$

Hence $PQ + QR + RS + SP > \sqrt{2}\, AC$.

6. If a, b are natural numbers such that

$$\frac{a+1}{b} + \frac{b+1}{a}$$

is an integer prove that $\gcd(a, b) \le \sqrt{a+b}$.

Solution: Let d denote the **gcd** of a and b. We observe that d^2 divides ab. Since $(a^2 + b^2 + a + b)/ab$ is an integer, d^2 divides $a^2 + b^2 + a + b$. However d^2 divides both a^2 and b^2. Hence d^2 divides $a + b$. This forces $d^2 \le a + b$, i.e., $d \le \sqrt{a+b}$.

7. For positive real numbers a, b, c, d satisfying $a + b + c + d \le 1$ prove the following inequality:

$$\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c} \le \frac{1}{64abcd}.$$

Solution: The given inequality reduces to

$$a^2cd + b^2cd + c^2ab + d^2ab \le 1/64.$$

We observe that $a^2cd + b^2cd + c^2ab + d^2ab = (ac + bd)(ad + bc)$. Hence using the inequality $xy \le (x + y)^2/4$, we get

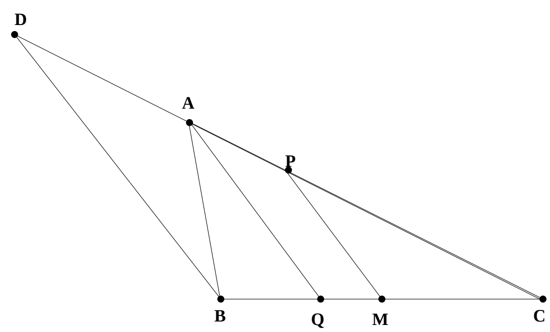
$$a^2cd + b^2cd + c^2ab + d^2ab \le \frac{(ac + bd + ad + bc)^2}{4}.$$

But $ac + bd + ad + bc = (a + b)(c + d)$. One more application of the same inequality gives $ac + bd + ad + bc \le (a + b + c + d)^2/4$. Combining both these inequalities and using the data that $a + b + c + d \le 1$, we get the required inequality.

KRMO-1996

1. In triangle ABC, M is the midpoint of BC. A line passing through M divides the perimeter of triangle ABC into two equal parts. Show that this line is parallel to the internal bisector of $\angle A$.

   Solution. Extend CA to D such that $AB = AD$. Draw AQ bisecting $\angle BAC$ and meeting BC in Q. Since ADB is an isosceles triangle, we have $\angle ADB = \angle ABD = A/2 = \angle BAQ$. Therefore AQ is parallel to DB. Also by data, $MC + CP = PA + AB + BM$. As $BM = MC$, and $AB = AD$, this becomes $CP = PA + AD = PD$. So P is the mid point of DC. Hence PM is also parallel to DB and the result follows.
   Alternatively, it can be proved that $CM/MQ = (b + c)/(b - c) = CP/PA$.



2. Find all positive integer solutions x and n of the equation

$$x^2 + 615 = 2^n.$$

   Solution. Note that $x^2 \equiv 1 \pmod 3$. Since 3 divides 615 we have $2^n \equiv 1 \pmod 3$. Hence n is even, say, $n = 2k$.The given equation can then be rewritten as $(2^k - x)(2^k + x) = 615 = 1{\cdot}615 = 3{\cdot}205 = 5{\cdot}123 = 15{\cdot}41$. Hence $(2^k - x, 2^k + x) = (1, 615), (3, 205), (5, 123), (15, 41)$. We get the possibilities : $(2^k, x) = (308, 307), (104, 101), (64, 59), (28, 13)$. Only the pair $(64, 59)$ gives a solution, namely $(k, x) = (6, 59)$, and hence $(n, x) = (12, 59)$.

3. (a) If p and q are two distinct positive integers, show that at least one of the equations

$$x^2 + px + q = 0 \quad \text{and} \quad x^2 + qx + p = 0 \qquad (*)$$

   has real roots.
   (b) If $p = 4n^2 + 1$ for some positive integer n, then find the number of q's ($\neq p$) for which *both* the equations in (*) have real roots.

   Solution. (a) Without loss of generality assume that $p > q$.
   If $q = 1$, then $p \ge 2$; so $p^2 - 4q = p^2 - 4 \ge 0$.
   If $q = 2$, then $p \ge 3$; so $p^2 - 4q = p^2 - 8 > 0$.
   If $q = 3$, then $p \ge 4$; so $p^2/4p > 4q$ and $p^2 - 4q > 0$.
   Thus in all cases $p^2 - 4q \ge 0$. Hence the first equation has real roots.
   (b) If both equations have real roots, then $p^2 \ge 4q$ and $q^2 \ge 4p$, i.e., $2\sqrt{p} \le q \le p^2/4$.
   If $p = 4n^2 + 1$, then we have

$$2\sqrt{4n^2 + 1} \le q \le \frac{1}{4}(16n^4 + 8n^2 + 1),$$

   which is the same as

$$4n + 1 \le q \le 4n^4 + 2n^2.$$

   The number of such integral q is $4n^4 + 2n^2 - 4n$ If we exclude the case $q = p = 4n^2 + 1$, which does fall in the above range, we get the desired number as $4n^4 + 2n^2 - 4n - 1$.

4. Let X be a finite set containing n elements. Find the number of all ordered pairs (A, B) of subsets of X such that neither A is contained in B nor B is contained in A.

Solution. For any given subset A of X containing r elements, every subset B of X that has the property that neither A is contained in B nor B is contained in A can be obtained by taking the union of a nonempty subset of $X \setminus A$ and a proper subset of A. The number of such B is therefore $(2^{n-r} - 1)(2^r - 1), 1 \le r \le n-1$. Hence the number of pairs (A, B) is given by

$$\begin{aligned} \sum_{r=1}^{n} \binom{n}{r} (2^{n-r} - 1)(2^r - 1) &= \sum_{r=0}^{n} \binom{n}{r} (2^{n-r} - 1)(2^r - 1) \\ &= \sum_{r=0}^{n} \binom{n}{r} (2^n - 2^r - 2^{n-r} + 1) \\ &= 2^n(1+1)^n - (1+2)^n - (2+1)^n + (1+1)^n \\ &= 4^n - 3^n - 3^n + 2^n = 4^n - 2.3^n + 2^n. \end{aligned}$$

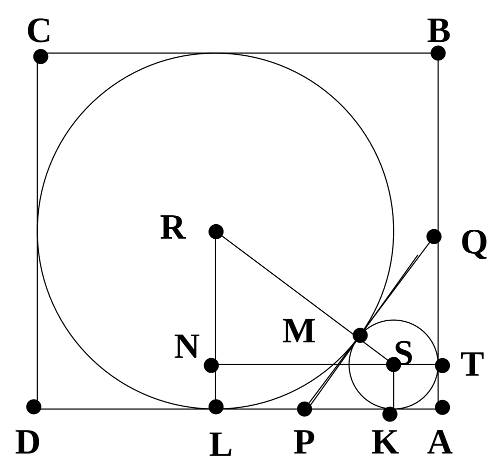
OR

Let $S = \{(A, B) | A \subset B\}$ and $T = \{(A, B) | B \subset A\}$. we have $|S| = \sum_{r=0}^{n} \binom{n}{r} 2^{n-r}$ (take an r-element subset A of X and add a subset (possibly empty) of $X \setminus A$ to get B).

$|S| = (1+2)^n = 3^n$. Similarly $|T| = 3^n$. Also $|S \cap T| = |\{(A, A) : A \subset X\}| = 2^n$. Hence $|S \cup T| = |S| + |T| - |S \cap T| = 3^n + 3^n - 2^n = 2.3^n - 2^n$. The number of all possible ordered pairs (A, B) is $2^n.2^n = 4^n$. Hence the required number is $4^n - 2.3^n + 2^n$, as we want the cardinality of the complement of $(S \cup T)$.

5. Two circles whose radii are in the ratio $4:1$ touch each other externally at M and lie inside a rectangle $ABCD$ such that the larger circle touches sides AD, BC and CD, and the smaller circle touches the sides AB and AD. The common tangent at M to the circles meet sides AD and AB at P and Q. Find the ratios AP/PD and AQ/QB.

Solution.



Let S and R be the centres of the smaller and larger circles respectively and K and L be the points of contact of these circles with AD. Join SK and $RL$and draw TN through S parallel to AD to meet AB and RL in T and N respectively. Let the radii of the circles be $4r$ and r. We have AB = diameter of the bigger circle $= 8r$; $SR = SM + MR = r + 4r = 5r; LN = SK = TA = r; RN = RL - NL = 4r - r = 3r$. So $SN = \sqrt{SR^2 - RN^2} = 4r = KL$. But $PK = PM = PL$; so each is equal to $2r$. Hence $AP = AK + KP = r + 2r = 3r$, and $PD = PL + LD = 2r + 4r = 6r$. So $AP/PD = 3r/6r = 1/2$.
From right triangle AQP, in which $AQ = AT + QT = r + QT$, $PQ = PM + MQ = 2r + MQ = 2r + QT$, $AP = AK + KP = r + 2r = 3r$, we have $PQ^2 = AP^2 + AQ^2$; i.e., $(2r + QT)^2 = (3r)^2 + (r + QT)^2$, giving $QT = 3r$. Therefore $AQ = AT + TQ = r + 3r = 4r$; so $QB = 4r$ as well. Hence $AQ/QB = 1$.

6. (a) Find all positive integers n for which 11 divides $n^2 + 3n + 5$.
(b) Show that for no positive integer n, 121 divides $n^2 + 3n + 5$.

Solution. (a) Considering $n = 11r + k$, for $k = 0, 1, 2, \ldots, 10$, we see that only for $k = 4$, 11 divides $n^2 + 3n + 5$. Hence the set of integers n for which the given condition is satisfied is $\{11r + 4 \,:\, r \in \mathbf{N} \cup \{0\}\}$.

(b) If $n = 11r + 4$, then $n^2 + 3n + 5 = 121r^2 + 121r + 33$, which is clearly not divisible by 121. Hence the result.

7. If $a^2 = 7b + 51$ and $b^2 = 7a + 51$ where a and b are real numbers, find the product ab.

Solution. From the given equations by subtraction we have $a^2 - b^2 = 7(b - a)$. That is, $(a - b)(a + b + 7) = 0$. So either $a = b$ or $a + b = -7$. If $a = b$, then $a^2 - 7a - 51 = 0$, which gives $a = (7 \pm \sqrt{253})/2$ and hence $ab = a^2 = 7a + 51 = (151 \pm 7\sqrt{253})/2$.

If $a \neq b$, then $b = -7 - a$, and so $a^2 = -49 - 7a + 51 = 2 - 7a$; $ab = a(-7 - a) = -a^2 - 7a = -2 + 7a - 7a = -2$.

OR

If $a = b$, proceed as before. So assume $a \neq b$. We have $a^2 = 7(a + b) - 7a + 51$, i.e.,

$$a^2 + 7a - [7(a + b) + 51] = 0.$$

Similarly, $b^2 + 7b - [7(a + b) + 51] = 0$.

Therefore a and b are the roots of the quadratic equation $x^2 + 7x - [7(a + b) + 51] = 0$.

Hence $a + b = -7$ and so $ab = -[7(a + b) + 51] = -[-49 + 51] = -2$.

KRMO-1997

1. A trapezoid has perpendicular diagonals and altitude 10. Find the area of the trapezoid if one diagonal has length 13.

Solution: Because triangles ABP and CDP are similar,

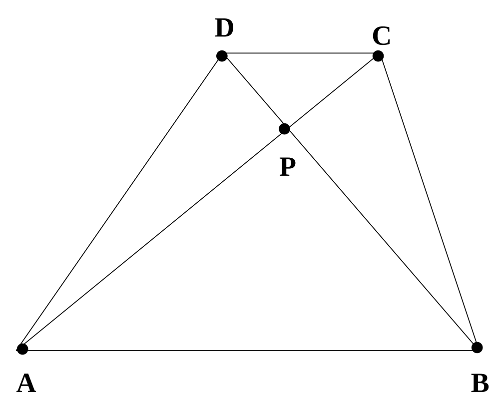
$$\frac{x}{13 - x} = \frac{y}{z}. \qquad (1)$$

Because triangles ABP and BDX are similar,

$$\frac{x}{DX} = \frac{y}{10}.$$

But $DX = \sqrt{69}$. Therefore

$$\frac{x}{\sqrt{69}} = \frac{y}{10}. \qquad (2)$$



By (1) $z = (13 - x)y/x$ and by (2) $y/x = 10/\sqrt{69}$. Therefore, required area is

$$\frac{1}{2}BD \cdot AC = \frac{1}{2}13(y + z) = \frac{13}{2}\left(y + \frac{(13 - x)y}{x}\right) = \frac{13}{2} \cdot \frac{13y}{x} = \frac{845}{\sqrt{69}}.$$

2. Find the greatest common divisor of all **even** 6-digit numbers obtained by using each of the digits 1,2,3,4,5,6 exactly once.

Solution: Let the greatest common divisor of all the numbers under consideration be d. Since all the numbers are even, 2 divides d. Since the sum of digits of any such number is

$1+2+3+4+5+6 = 21$, each such number is divisible by 3 but not by 9. Thus 3 divides d and 9 does not divide d. Considering the numbers 123546 and 123564 having difference 18, we conclude that d divides 18. Since 2 and 3 divide d while 9 does not , we conclude that $d = 6$.

3. If a and b are two positive real numbers such that $a + b = 1$, prove that

$$\frac{1}{3} \leq \frac{a^2}{a+1} + \frac{b^2}{b+1} < \frac{1}{2}.$$

Solution: Since $a + b = 1$,

$$\frac{a^2}{a+1} + \frac{b^2}{b+1} = \frac{1-ab}{2+ab}.$$

Because $ab > 0$, $1 - ab < 1$ and $2 + ab > 2$. Therefore

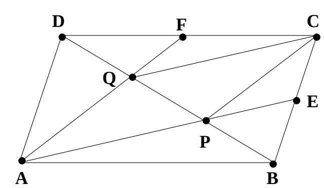
$$\frac{1-ab}{2+ab} < \frac{1}{2}.$$

Again, because $(a+b)/2 \geq \sqrt{ab}$, $1/4 \geq ab$. Therefore $1 - ab \geq 3/4$ and $2 + ab \leq 9/4$. Therefore

$$\frac{1-ab}{2+ab} \geq \frac{1}{3}.$$

4. In a quadrilateral $ABCD$, two points P and Q are chosen on diagonal BD such that $BP = PQ = QD$. Suppose AP meets BC in E and AQ meets CD in F. If $BE = EC$ and $CF = FD$, show that $ABCD$ is a parallelogram.

Solution:



In triangle BCQ, we have $BE = EC$ and $BP = PQ$. Hence PE is parallel to QC. Thus AE and QC are parallel. Similarly AQ and PC are parallel. Therefore $APCQ$ is a parallelogram. Hence AC and PQ bisect each other. Therefore AC and BD bisect each other. Thus $ABCD$ is a parallelogram.

5. If a, b, x and y are real numbers such that

$$x^{1997} - ax - b = 0, \quad y^{50} - by - a = 0 \quad \text{and } xy = 1,$$

prove that $(a-1)^2 = b^2$.

Solution: Multiply the first equation by y and use the second equation to get

$$x^{1996} = a + by = y^{50}.$$

Therefore

$$x^{1996+50} = (xy)^{50} = 1.$$

This gives $x = \pm 1$. Substituting this in the first equation, we obtain $b = \pm(1-a)$. Squaring this, we get $b^2 = (1-a)^2$.

6. How many increasing 3-term geometric progressions can be obtained from the sequence $1, 2, 2^2, 2^3, \dots, 2^n$?

[e.g., $\{2^2, 2^5, 2^8\}$ is a 3-term geometric progression for $n \geq 8$.]

Solution: Let us start counting 3-term GP's with common ratios $2, 2^2, 2^3, \dots$.

The 3-term GP's with common ratio 2 are

$$1, 2, 2^2; 2, 2^2, 2^3; \cdots ; 2^{n-2}, 2^{n-1}, 2^n.$$

They are $(n-1)$ in number. The 3-term GP's with common ratio 2^2 are

$$1, 2^2, 2^4; 2, 2^3, 2^5; \cdots ; 2^{n-4}, 2^{n-2}, 2^n.$$

They are $(n-3)$ in number. Similarly we see that The 3-term GP's with common ratio 2^3 are $(n-5)$ in number and so on. Thus the number of 3-term GP's which can be formed from the sequence $1, 2, 2^2, 2^3, \dots, 2^n$ is equal to

$$S = (n-1) + (n-3) + (n-5) + \cdots .$$

Here the last term is 2 or 1 according as n is odd or even. If n is odd, then

$$S = (n-1) + (n-3) + (n-5) + \cdots + 2 = 2\left(1 + 2 + 3 + \cdots + \frac{n-1}{2}\right) = \frac{n^2-1}{4}.$$

If n is even, then

$$S = (n-1) + (n-3) + (n-5) + \cdots + 1 = \frac{n^2}{4}.$$

Hence the required number is $(n^2-1)/4$ or $n^2/4$ according as n is odd or even.

7. Suppose x, y, z are three integers which are in arithmetic progression. If x is of the form $8n+4$ where n is an integer and each of y, z is expressible as a sum of squares of two integers, show that $\gcd(x, y, z)$ cannot be odd.

Solution : Suppose $\gcd(x, y, z)$ is odd. Then, since x is even, y must be odd and z must be even. Let $z = z_1^2 + z_2^2$ where z_1 and z_2 are integers. Then either z_1 and z_2 are both even or both odd.

Case 1. Let z_1 and z_2 be both even.

Then z must be of the form $4k$ where k is an integer. Since x also is divisible by 4, $y = (x+z)/2$ cannot be odd.

Case 2. Let z_1 and z_2 be both odd.

Then z must be of the form $8k+2$ where k is an integer. Since x is of the form $8n+4$, this forces y to be of the form $4l+3$ where l is an integer. As every square is of the form $4m$ or $4m+1$, y cannot be expressed as a sum of two squares.

KRMO-1998

1. Let P be an interior point of an equilateral triangle ABC such that $AP^2 = BP^2 + CP^2$. Prove that $\angle BPC = 150°$.

Solution :

Draw CQ such that $\angle PCQ = 60°$ and $CP = CQ$ (as shown in the fig.) Then : ΔPCQ is equilateral and therefore, $PQ = PC$. Also, in triangles APC and BQC; $AC = BC$; $PC = QC$ and $\angle ACP = 60° - \angle PCB = \angle BCQ$ The triangles are congruent. Therefore, $AP = BQ$. Substituting these in $AP^2 = BP^2 + CP^2$, we obtain $BQ^2 = BP^2 + PQ^2$, which implies $\angle BPQ = 90°$. Therefore we obtain

$$\angle BPC = \angle BPQ + \angle QPC = 90° + 60° = 150°.$$

2. Does there exist a positive integer N such that the number formed by the last two digits of the sum $1 + 2 + 3 \ldots + N$ is 98 ?

   Solution: Suppose such an integer N exists. Then we must have $1+2+3+\ldots+N = 100k+98$, where k is a nonnegative integer. This gives $N(N+1) = 200k+196$; i.e. $(2N+1)^2 = 800k+785$. But, in the above equation there is a perfect square in the L.H.S. and a number which can never be a perfect square in the R.H.S. ; a perfect square ending with 5 can have only 2 in the 10's place. The number $800k + 785$ ends with 5 and has 8 in the 10's place.(Alternately, one can solve the quadratic equation in N and look at the condition for existence of integer roots.) So there cannot exist a positive integer N such that the sum $1 + 2 + 3 \ldots + N$ ends with 98.

3. Find all real numbers a which satisfy the equation.

$$\sqrt{a - \frac{1}{a}} + \sqrt{1 - \frac{1}{a}} = a$$

   Solution : The given equation can be written in the form

$$\frac{1}{a} = \frac{1}{\sqrt{a - \frac{1}{a}} + \sqrt{1 - \frac{1}{a}}}.$$

   On rationalising the expression and rearranging, we obtain

$$\frac{a-1}{a} = \sqrt{a - \frac{1}{a}} - \sqrt{1 - \frac{1}{a}}.$$

   Taking $x = \sqrt{a - \frac{1}{a}}$, we obtain

$$\frac{a-1}{a} + a = 2x$$

   and this simplifies to $x^2 + 1 = 2x$. We get $x = 1$. Solving $a - \frac{1}{a} = 1$, we obtain

$$a = \frac{1 \pm \sqrt{5}}{2}.$$

   Suppose $a = \frac{1-\sqrt{5}}{2}$. We can write the given equation in the form

$$\sqrt{\frac{a-1}{a}} = a - 1$$

   since a satisfies the relation $a^2 - a - 1 = 0$. But then lhs is positive and rhs is negative. This incompatibility shows that $a = \frac{1-\sqrt{5}}{2}$ is not possible. We conclude that the only value that a takes is

$$a = \frac{1 + \sqrt{5}}{2}.$$

4. Let A denote the set of all numbers between 1 and 700 which are divisible by 3 and let B denote the set of all numbers between 1 and 300 which are divisible by 7. Find the number of all ordered pairs (a, b) such that $a \in A, b \in B, a \neq b$ and $a + b$ is even.

Solution : First, note that A has 233 elements of which 116 are even and 117 are odd, B has 42 elements of which 21 are even and 21 are odd and $A \cap B$ has 14 elements.
Therefore, required number is:

$$\begin{aligned}
n &= \#\{(a,b) \ : \ a \in A, b \in B, a+b \text{ is even } \} - \#\{(a,b) \ : \ a \in A, b \in B, a = b\} \\
&= \$\{(a,b) \ : \ a \in A, b \in B, a \text{ is even, } \ b \text{ is even } \} + \#\{(a,b) \ : \ a \in A, b \in B, a \text{ odd, } b \text{ odd } \} \\
&\quad -\#\{(a,b) \ : \ a \in A, b \in B, a = b\} \\
&= 116 \times 21 + 117 \times 21 - 14 = 4879.
\end{aligned}$$

5. Let $ABCD$ be a rectangle, if P and Q are points respectively on AD and DC such that the areas of the triangles BAP, PDQ and QCB are all equal, find the ratios $\frac{DP}{PA}$ and $\frac{DQ}{QC}$.

Solution: Let

$$\frac{DP}{PA} = k, \frac{DQ}{QC} = l; PA = x, QC = y.$$

Using the given data, we obtain :

$$\frac{1}{2}(ly + y)(x) = \frac{1}{2}(kx)(ly) = \frac{1}{2}y(kx + x).$$

This simplifies to $l + 1 = kl = k + 1$. We obtain $l = k$ and hence $l + 1 = l^2$. Solving for l, we get

$$l = k = \frac{1 + \sqrt{5}}{2}.$$

(The value $\frac{1-\sqrt{5}}{2}$, being negative, is rejected.)

6. If $x_1, x_2 \ldots x_n$ are n distinct positive integers $(n > 1)$, show that there does not exist a positive integer y satisfying $x_1^{x_1} + x_2^{x_2} + \ldots + x_n^{x_n} = y^y$

Solution: Suppose that there exists a positive integer y

such that

$$x_1^{x_1} + x_2^{x_2} + \ldots + x_n^{x_n} = y^y.$$

Then, $x_j < y$ for $j = 1, 2 \ldots, n$. Let $x = \max\{x_j, j = 1, 2, \ldots n\}$. Then $x < y$ and using this bound we obtain, $x^x < y^x$. This leads to

$$x_j^{x_j} < y^x, j = 1, 2, \ldots, n.$$

Summing over j, we get,

$$x_1^{x_1} + x_2^{x_2} + \ldots + x_n^{x_n} < n.y^x.$$

But, because each $x_j \geq 1$ and x_j are distinct, $x \geq n$ and hence $y > n$. Thus we have

$$x_1^{x_1} + x_2^{x_2} + \ldots + x_n^{x_n} < y.\ y^x = y^{x+1} \leq y^y,$$

a contradiction. This proves that there does not exist a positive integer y such that

$$x_1^{x_1} + x_2^{x_2} + \ldots + x_n^{x_n} = y^y.$$

7. At each of the eight corners of a cube write +1 or -1 arbitrarily. Then, on each of the six faces of the cube write the product of the numbers written at the four corners of that face. Add all the fourteen numbers so written down. Is it possible to arrange the numbers +1 and -1 at the corners initially so that this final sum is zero ?

Solution :

Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ and x_8 be the numbers written at the corners. Then, the final sum is given by

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_1x_2x_3x_4 + x_5x_6x_7x_8 + x_1x_4x_5x_8$$

$$+x_2x_3x_6x_7 + x_1x_2x_5x_6 + x_3x_4x_7x_8 \ldots \qquad (\star)$$

Because there are fourteen terms in the above sum and each of the terms is +1 or -1, the sum will be zero only if some seven terms are +1 each and the remaining seven terms are -1 each.

But, the product of the fourteen terms is

$$(x_1x_2x_3x_4x_5x_6x_7x_8)(x_1x_2x_3x_4)(x_5x_6x_7x_8)(x_1x_4x_5x_8)(x_2x_3x_6x_7)(x_1x_2x_5x_6)(x_3x_4x_7x_8)$$

$$= (x_1x_2x_3x_4x_5x_6x_7x_8)^4 = (\pm 1)^4 = +1.$$

Therefore, it is impossible to have an odd number of -1 s in the sum $(\star)$.

We conclude that the desired arrangement is not possible.

KRMO-1999

1. Let AB be a chord of a circle and CD be the diameter of the circle perpendicular to AB. Let M be any point on the line passing through C and D, distinct from c and D. Let AM meet the circle again at K. If KD meets BC at N, prove that $\angle NMC = 90°$.

Solution: Suppose M is on the minor arc side of AB. (see Fig.1) Since $arc\ AD = arc\ BD$, we see that $\angle AKD = \angle BCD$. (K, C lie on the minor arc). This shows that M, N, K, C are con-cyclic. Hence $\angle CKD + \angle CMN = 180°$. Since CD is a diameter, $\angle CKD = 90°$. This gives$\angle CMN = 90°$.

Suppose M is on the major arc side of AB. Then K lies on the major $arc\ AB$. Again we get $\angle AKD = \angle BCD$. But $\angle AKD + \angle AKN = 180^\circ$. This gives $\angle NCM + \angle MKN = 180^\circ$. We conclude that M, K, N, C are con-cyclic. Since $\angle CKD = 90^\circ$, we also have $\angle CKN = 90^\circ$. Hence $\angle CMN = \angle CKN = 90^\circ$.

2. Find all prime numbers p such that there are integers x, y satisfying

$$p + 1 = 2x^2 \text{ and } p^2 + 1 = 2y^2.$$

Solution: We may assume that both x, y to be positive. We observe that p is odd. Taking the difference, we get

$$p^2 - p = 2(y^2 - x^2).$$

Write this in the form $p(p-1) = 2(y-x)(y+x)$. Since p is odd p cannot divide 2. If p divides $y - x$, then $p \le y - x$. But then $p - 1 \ge 2y + 2x$ is not possible. Hence p should divide $y + x$. This gives $p \le y + x$ and $p - 1 \ge 2(y - x)$. Eliminating y, we obtain $p + 1 \le 4x$. Since $p + 1 = 2x^2$, we get $2x^2 \le 4x$ and hence $x \le 2$. Taking $x = 1$, we get $p = 1$, which is not a prime. If $x = 2$, we get $p = 7$. Thus $p = 7$ is the only prime satisfying the given condition.

3. Find all cubic polynomials $p(x)$ such that $(x-1)^2$ is a factor of $p(x) + 2$ and $(x+1)^2$ is a factor of $p(x) - 2$.

Solution: If $(x - \alpha)$ divides a polynomial $q(x)$ then $q(\alpha) = 0$. Let $p(x) = ax^3 + bx^2 + cx + d$. Since $(x-1)$ divides $p(x) + 2$, we get

$$a + b + c + d + 2 = 0.$$

Hence $d = -a - b - c - 2$ and

$$\begin{aligned} p(x) + 2 &= a(x^3 - 1) + b(x^2 - 1) + c(x - 1) \\ &= (x-1)\{a(x^2 + x + 1) + b(x+1) + c\}. \end{aligned}$$

Since $(x-1)^2$ divides $p(x) + 2$, we conclude that $(x-1)$ divides $a(x^2 + x + 1) + b(x+1) + c$. This implies that $3a + 2b + c = 0$. Similarly, using the information that $(x+1)^2$ divides $p(x) - 2$, we get two more relations: $-a + b - c + d - 2 = 0$; $3a - 2b + c = 0$. Solving these for a, b, c, d, we obtain $b = d = 0$, and $a = 1, c = -3$. Thus there is only one polynomial satisfying the given condition: $p(x) = x^3 - 3x$.

4. Let M be a product of five distinct prime numbers. Find the number of pairs (m, n) of positive integers, such that m divides n and n divides M.

Solution: Let $M = p_1p_2p_3p_4p_5$ be the given number. Let us consider p_1. In how many ways can p_1 be a factor (or non-factor) of m?

5. Consider the two squares lying inside a triangle ABC with $\angle A = 90^\circ$ with their vertices on the sides of ABC: one square having its sides parallel to AB and AC, the other, having two sides parallel to the hypotenuse. Determine which of these two squares has greater area.

Solution: Let the side of the square with sides parallel to AB be x and the side of the other square be y. Let $AKLM$ be the square in the first case and $PQRS$ be the square in the other case (K, L, M on AB, BC, CA; p, q on BC, R on CA and S on AB).

Using the similarity of CML and CAB we get

$$\frac{x}{c} = \frac{b-x}{b},$$

and hence $x = bc/(b+c)$. Similarly using the similarity of ARS and ACB, we get

$$\frac{y}{a} = \frac{AS}{AB} = \frac{c - BS}{c}.$$

But BPS is similar to BAC and hence $BS/a = y/b$ giving $BS = ay/b$. This shows that $y = abc/(a^2 + bc)$. Let us now compute $x - y$:

$$x - y = \frac{bc}{b+c} - \frac{abc}{a^2+bc} = \frac{bc(a-b)(a-c)}{(b+c)(a^2+bc)}.$$

Since $a > b$ and $a > c$, we conclude that $x > y$. Thus the square with sides parallel to AB is larger than the other square.

6. Given three positive real numbers a, b, c with $abc = 1$ such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < a + b + c$$

show that exactly one of the three numbers is greater than 1.

Solution: We write the given inequality, using $abc = 1$, in the form

$$bc + ca + ab < a + b + c.$$

This is equivalent to

$$(1-a)(1-b)(1-c) < 0.$$

There are two possibilities: two of $1-a, 1-b, 1-c$ are positive and the remaining one is negative, or all the three are negative. The second case can be ruled out, for, in that case we see that $a < 1, b < 1, c < 1$ forcing $abc < 1$. Hence the first case prevails and we conclude that exactly one of the a, b, c is greater than 1.

7. Find all positive integers a, b satisfying

$$10a^2 - 10b^2 = 99 - 21ab.$$

Solution: We can write the given equation in the form

$$(2a+5b)(5a-2b) = 99.$$

But $99 = 1{\cdot}99 = 3{\cdot}33 = 9{\cdot}11$. Thus there are 6 cases to be considered: $(2a+5b) = 1, (5a-2b) = 99$; $(2a+5b) = 3, (5a-2b) = 33$; $(2a+5b) = 9, (5a-2b) = 11$; $(2a+5b) = 99, (5a-2b) = 1$; $(2a+5b) = 33, (5a-2b) = 3$; $(2a+5b) = 11, (5a-2b) = 9$. Solving these we get integer solution only for $(2a+5b) = 99, (5a-2b) = 1$, giving $a = 7, b = 17$.

1. Let ABC be a non-isosceles triangle in which the altitude AD through A falls inside the triangle. Let M be the mid-point of BC. If $\angle BAD = \angle CAM$, prove that $\angle A = 90^\circ$.

   Solution 1: Let AD extended meet the circumcircle Γ in K and AM extended meet it in L. (See Fig.)

   Since $\angle BAK = \angle CAL$, we obtain $\angle CBL = \angle CAL = \angle BAK = \angle BLK$. It follows that BC is parallel to KL. But AD is perpendicular to BC. We thus conclude that AK is perpendicular to KL. This implies that AL is a diameter of Γ. Hence the circumcentre O of Γ lies on AL. On the other hand O must also lie on the perpendicular bisector of BC, i.e., on the perpendicular through M. This forces that $M = O$. Thus BC is a diameter of Γ and $\angle BAC = 90^\circ$.

   Solution 2: We observe that $\angle ABC = \angle AKC$, $\angle CKL = \angle CAL = \angle BAK$. Since $\angle BAK + \angle ABC = 90^\circ$, it follows that $\angle AKL = \angle AKC + \angle CKL = 90^\circ$. Hence AL is a diameter. We proceed as in solution 1.

   Solution 3: Draw a line through B perpendicular to AB to meet AM produced in P. Then we get $\angle MBP = 90^\circ - \angle ABM = \angle BAD = \angle MAC$. We conclude that P, B, A, C are concyclic. Since $\angle ABP = 90^\circ$, AP is a diameter. Thus M lies on a diameter. Rest as in solution 1.

2. Given that a, b, c, d are natural numbers such that $a^5 = b^4, c^3 = d^2$ and $a - c = 17$, find a, b, c, d.

   Solution: Since $a^5 = b^4$, we have $a = x^4$ and $b = x^5$ for some integer x. [For if p^k is the largest power of a prime p which divides $m = a^5 = b^4$, then $k|5$ and $k|4$. This implies that $k|20$ and hence $k = 20n$ for some n. This in turn shows that p^{4n} is the largest power of p which divides a and p^{5n} is the largest power of p which divides b.] Similarly $c^3 = d^2$ gives $c = y^2$ and $d = y^3$ for some integer y. Using $a - c = 17$, we obtain $x^4 - y^2 = 17$ which shows that $x^2 + y = 17$ and $x^2 - y = 1$. From these we get $2x^2 = 18$ or $x = 3$. (We can rule out $x = -3$.) It follows that $y = 8$. Thus $a = 3^4 = 81$, $b = 3^5 = 243$, $c = 8^2 = 64$ and $d = 8^3 = 512$.

3. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

   $$\frac{a^3}{a^2 + b^2} + \frac{b^3}{b^2 + c^2} + \frac{c^3}{c^2 + a^2} \geq \frac{1}{2}.$$

   Solution: We observe that

   $$\frac{a^3}{a^2 + b^2} = a - \frac{ab^2}{a^2 + b^2} \geq a - \frac{ab^2}{2ab} = a - \frac{b}{2},$$

since $a^2+b^2 \geq 2ab$. Thus we obtain

$$\begin{aligned}\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{c^2+a^2} &\geq a-\frac{b}{2}+b-\frac{c}{2}+c-\frac{a}{2}\\ &= \frac{a+b+c}{2}=\frac{1}{2}.\end{aligned}$$

4. In a 4×4 array of 16 numbers, the sum of the numbers in each row, each column and each of the diagonals is 1. Find the sum of the numbers in the 4 corners of the array.

Solution: Let us denote by x the sum of four numbers in the corners of the array; y the sum of 4 numbers in the centre of the array; and z the sum of remaining 8 numbers. By adding the first row, the last row, the first column and the last column we obtain $2x+z=4$. Similarly, adding second row, third row, second column and third column we obtain $2y+z=4$. Finally adding two diagonals, we get $x+y=2$. Solving these we see that $x=1$, $y=1$ and $z=2$. Thus the sum of the numbers in the 4 corners of the array is 1.

5. In triangle ABC, lines KL and MN are drawn parallel to BC meeting AB in K, M and AC in L, N. Suppose the altitude through A in triangle ABC meets KL, MN, BC in P, Q, R respectively. If

$$[AKL]=[MBCN]=\frac{1}{3}[ABC],$$

find the ratio AP/QR. (Here $[X]$ denotes the area of the figure X.)

Solution: We have using similarity(see Fig.),

$$\frac{AP}{AR}=\frac{AK}{AB}, \quad \frac{KL}{BC}=\frac{AK}{AB}.$$

These give

$$\frac{[AKL]}{[ABC]}=\frac{AP\cdot KL}{AR\cdot BC}=\left(\frac{AK}{AB}\right)^2.$$

We obtain $\left(\frac{AK}{AB}\right)^2=\frac{1}{3}$ giving $\left(\frac{AK}{AB}\right)=\frac{1}{\sqrt{3}}$. similarly we obtain $\left(\frac{AK}{AM}\right)=\frac{1}{\sqrt{2}}$. Thus we get

$$\frac{AP}{AK}=\frac{1}{\sqrt{3}}, \quad \frac{AP}{AQ}=\frac{1}{\sqrt{2}}, \quad \text{and} \quad \frac{AP}{QR}=\frac{1}{\sqrt{3}-\sqrt{2}}=\sqrt{3}+\sqrt{2}.$$

6. Find all natural numbers n with the property that if $n=(a_1a_2\cdots a_k)_{10}$ is the decimal representation of n then there are digits a, b, c such that

$$(a_1a_2\cdots a_kabc)_{10}+\frac{n}{2}=1+2+\cdots+n.$$

Solution: Let us write $(abc)_{10} = x$. The given relation reads as

$$1000n + x + \frac{n}{2} = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

This reduces to

$$2000n + 2x = n^2.$$

It follows that n divides $2x$. If $2x \neq 0$, then $n \leq 2x$. Dividing by n, we get

$$n = 2000 + \frac{2x}{n}.$$

If $2x > 0$, then we see that $n > 2000$. But we know that $x \leq 1000$. Thus if $2x \neq 0$ then we get $n \leq 2x \leq 2000$. Combining these we conclude that $2x = 0$. It follows that $n^2 = 2000n$ or $n = 2000$. It is easy to verify that $n = 2000$ indeed is a solution to the problem.

7. Solve the system for real numbers x, y, z:

$$\begin{aligned} x^2 + 2yz &= 13, \\ y^2 + 2zx &= 10, \\ z^2 + 2xy &= 13. \end{aligned}$$

Solution: Adding these 3 relations we obtain

$$(x+y+z)^2 = 36,$$

giving $x + y + z = 6$ or $x + y + z = -6$. Subtracting the third relation from the first, we get

$$(x-z)(x+z-2y) = 0.$$

Thus there are two possibilities: $x = z$ and $x + z - 2y = 0$. Suppose $x = z$. Then there are only two equations:

$$x^2 + 2xy = 13, \quad y^2 + 2x^2 = 10.$$

These two imply that $(x-y)^2 = -3$, which is impossible for real x, y. Thus we are forced to the relation $x + z - 2y = 0$.

If $x + y + z = 6$, then we get $2y = x + z = 6 - y$ and hence $y = 2$. Using the second equation we get $2xz = 10 - y^2 = 6$ and $x + z = 2y = 4$. Solving, we obtain $x = 3$, $z = 1$ or $x = 1$, $z = 3$. Similarly $x + y + z = -6$ gives $y = -2$; $x = -3$, $z = -1$ or $x = -1$, $z = -3$. Thus we obtain four solutions

$$(x, y, z) = (3, 2, 1), (1, 2, 3), (-3, -2, -1), (-1, -2, -3).$$

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