This issue is dedicated in honour of Prof. K. Ramachandra who passed away in early 2011

On the Half Line: K. Ramachandra

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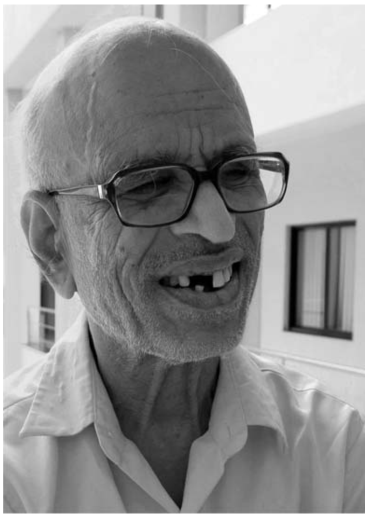


Abstract. A short biographical note on the life and works of K. Ramachandra, one of the leading mathematicians in the field of analytical number theory in the second half of the twentieth century.

1. Introduction

Kanakanahalli Ramachandra (1933–2011) was perhaps the real successor of Srinivasa Ramanujan in contemporary Indian mathematics. Ramachandra has made invaluable contributions to algebraic number theory, transcendental number theory and the theory of the Riemann zeta function. This article is a brief exposition on the life and works of Ramachandra. The title of this biographical note is motivated by the fact that Ramachandra was one of the few mathematicians who was still working on certain classical problems in number theory and many of his best results are theorems related to the values of the Riemann zeta function on the half line.

2. Early Life (1933–57)

Ramachandra was born on 18 August 1933 in the state of Mysore (now known as Karnataka) in southern India. His grandfather walked nearly a hundred and fifty kilometres to see the new born Ramachandra. Ramachandra hailed from a family with a modest background; his father passed away when Ramachandra was only 13. Ramachandra's mother managed his education by taking loan against their agricultural property. When Ramachandra was a student, he won a competition and was awarded a short biography of the legendary Indian mathematician Srinivasa Ramanujan. This was the book that ignited the interest for mathematics in Ramachandra.

Ramanujan's taxicab number $1729 = 9^3 + 10^3 = 1^3 + 12^3$ has become a part of mathematics folklore. During his college days, Ramachandra had a similar encounter with the number 3435. His college principal had a car with the number 3430 on the number plate. Ramachandra worked on the mathematical possibilities of this number and in the process he found that upon adding 5, the number 3435 is the only number with the unique property that when each digit was raised to a power equal to itself and the resulting numbers were added up, the sum equals the original number, i.e.

$$3^3 + 4^4 + 3^3 + 5^5 = 3435.$$

Ramachandra completed his graduation and post graduation from Central College, Bangalore. Due to family responsibilities, he had to look for a job at a young age and, just like Ramanujan, he also worked as a clerk. Ramachandra worked as a clerk at the Minerva Mills where Ramachandra's father had also worked. In spite of taking up a job quite remote from mathematics, Ramachandra studied number theory all by himself in his free time; especially the works of Ramanujan. Later, he worked as a lecturer in BMS College of Engineering. Ramachandra also served a very short stint of only six days as a teacher in the Indian Institute of science, Bangalore.

After the death of Ramanujan, Hardy wrote a series of lectures on the works of Ramanujan. These lectures were published as a book *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work.* This became one of the classic books on Ramanujan. When Ramachandra was a college student, he found a copy of this book in the public library of the state of Mysore and he studied the works of Ramanujan with great devotion. It was this book that inspired Ramachandra to become a mathematician and in particular, a number theorist. Unfortunately someone tore the cover of the book and took away the picture of Ramanujan. This made Ramachandra very disappointed and half a century later he still did not forget the irresponsible person who tore the particular page. Later Ramachandra got a copy of the book which he always kept with him as a personal favourite.

3. TIFR Bombay (1958–1995)

In 1958, Ramachandra secured a post in TIFR and it was here that he met K. Chandrashekharan who was one of the experts in the theory of the Riemann zeta function in India at that time. Ramachandra studied the theory of the Riemann zeta function under K. Chandrashekharan. Later Ramachandra would himself become one of the leading experts in the theory of the Riemann zeta function and make several invaluable contributions to the subject. For the next three decades until his retirement, Ramachandra remained at TIFR and established one of the most prestigious schools of analytic number theory in collaboration with his gifted students as well as leading number theorists all over the world.

Ramachandra believed that as a mathematician one not only has to contribute to the subject but also guide the next generation of mathematicians. He worked hard to perpetuate Number theory as an active research area and succeeded in inspiring the interested students to take up the subject. Ramachandra acted as the doctoral advisor for eight students; today some of his students are among the most renowned mathematicians in the field of analytic number theory.

At the invitation of Norwegian mathematician Atle Selberg, Ramachandra went to the Institute for Advanced Study in Princeton, USA, as a visiting professor and spent a period of six months. This was Ramachandra's first foreign trip and years later when Ramachandra constructed his house in Bangalore, he named it 'Selberg House' in honor of Atle Selberg. Over the course of his career, Ramachandra visited several countries and collaborated with some of the leading number theorists of the world and also invited many of the leading mathematicians to TIFR, including the legendary Paul Erdös. Erdös visited India in 1976 and stayed as a guest in Ramachandra's house. Ramachandra is one of the few mathematicians with Erdös Number 1. He published two joint papers with Erdös.

In 1978 Ramachandra founded the Hardy-Ramanujan Journal, which is considered among the most prestigious journals of number theory. It is one of the very few privately run mathematical journals in the world, funded entirely by Ramachandra and R. Balasubramanian who acted as its editors until Ramachandra passed away in 2011. The journal is published every year on 22nd December, on the occasion of the birthday of Srinivasa Ramanujan.

4. NIAS (1995–2011)

After retiring form TIFR, Ramachandra joined the National Institute of Advanced Studies, NIAS, Bangalore as a visiting faculty on the invitation of nuclear physicist and Founder Director of NIAS, Dr. Raja Ramanna. Ramachandra remained in NIAS and continued working on the theory of the Riemann zeta function until his death. During this period he was also associated with TIFR Bangalore as a visiting faculty. In 2003, a conference was held on the occasion of the seventieth birthday of Ramachandra in TIFR Bangalore. Several mathematicians from all over the world attended the conference and celebrated the event.

Ramachandra *left* (in the words of Erdös) on 17 January 2011. His health had broken down and he had been hospitalized for about two weeks. According to his wife Mrs. Saraswati Ramachandra: *The doctors had advised him to take complete rest and not work on mathematics. But as soon as he was discharged from the hospital, he started working on a problem that had been bothering him for the past six months.* He is survived by his wife and daughter.

5. Ramachandra's Mathematical Gods

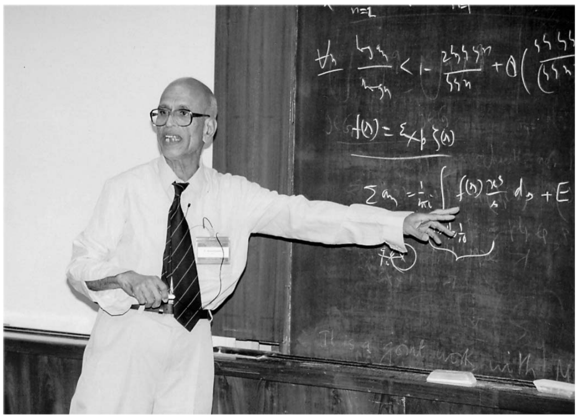
Although Ramachandra was born in a brahmin family and was a devout Hindu, he also had his own perception of Gods, his mathematical Gods. He used to address the great mathematicians as Gods and his reason was simple. "*Only a God can prove such a result*," he used to say. Decorating the walls of his room at NIAS, were the poster size pictures of G. H. Hardy, Srinivasa Ramanujan and I. M. Vinogradov. 'When he spoke about these mathematicians, he used to refer them as "*They are my Gods*." Sometimes when he spoke about a particular mathematical result he used to thank his mathematical God who was an expert in that field. "*With Siegel's blessings, I was able to prove some results*," Ramachandra said, referring to the German mathematician C. L. Siegel, as he spoke about his theorems in transcendental number theory.

Once when I asked Ramachandra whom he thought was the greatest mathematician, he immediately replied, *Ivan Matveyevich Vinogradov* had devised an ingenious method of evaluating exponential sums which reduced the error in the prime number theorem to

$$\pi(x) = Li(x) + O(x \exp(-c_1 (\ln x)^{3/5} (\ln \ln x)^{-1/5})).$$

In the eighty years since Vinogradov had published this result, no one was able to improve upon it. Therefore Ramachandra held Vinogradov in the highest regard. In second position he named Ramanujan and in third position he named Hardy. Ramachandra met Vinogradov twice when he was invited to attend conferences held in Russia on the occasions of the eightieth and the ninetieth birthdays of I. M. Vinogradov.

6. Ramachandra as a Teacher



Ramachandra has mentored a few of the brightest mathematical minds today including R. Balasubramanian and T. N. Shorey. He was extremely generous to his students and he often credited his students in his papers even for very minor contributions. Ramachandra was proud of the achievements of his students and often spoke about their works. *Balu and Shorey have brought a great name to me*, he said, referring to R. Balasubramanian and T. N. Shorey. It is often said in lighter vein that Ramachandra's greatest contribution to the theory of transcendental numbers is

T. N. Shorey because Shorey has proved some extraordinary results in this field. Ramachandra often said that his student R. Balasubramanian was a better mathematician. During the days when R. Balasubramanian was his doctoral student at TIFR Bombay, Ramachandra often introduced R. Balasubramanian as: *This is R. Balasubramanian. We are teacher and student but at different times the roles will be reversed.*

7. Mathematical Works

During his career as a mathematician, Ramachandra published over hundred and fifty papers which included several important works in the fields of algebraic number theory, transcendental number theory and the theory of the Riemann zeta function. Ramachandra was among the pioneers in evaluating the fractional moments of the Riemann zeta function. He was also one of the first mathematicians to consider the gap between numbers with large prime factors. Several key areas of analytic number theory that Ramachandra had pioneered, continue to be active areas of research even today.

In his early years as a number theorist, Ramachandra worked in the field of algebraic number theory. His first paper was: Some applications of Kronecker's limit formula, Ann. of Math (2), 80 (1964), 104–148. The reviewer M. Eichler remarked: *This paper contains some remarkable new results on the construction of the ray class field of an imaginary quadratic number field*. Ramachandra completed his Phd under the supervision of K. G. Ramanathan at TIFR Bombay (now known as Mumbai) in 1965.

When the seminal work of Alan Baker appeared in the 1960's, Ramachandra and his students, especially T. N. Shorey, took up transcendental number theory and made remarkable contributions to both the theory and its applications to problems of classical number theory. A detailed exposition of Ramachandra's contribution can be found in Michel Waldschmidt's paper 'On Ramachandra's Contributions to Transcendental Number Theory.'

In 1974, Ramachandra turned his attention to hard core classical analytic number theory, especially the theory of Riemann zeta function and general Dirichlet series. His contributions to the theory of the Riemann zeta function is best summarized in the words of British mathematician and a Fellow of the Royal Society, Roger Heath-Brown:

As soon as I entered research, 30 years ago, yours became a familiar name; and your infuence has remained with me ever since. Time permits me to mention in detail only one strand of your work – but it is one that clearly demonstrates how important your research has been. A little over 20 years back you proved the first results on fractional moments of the Riemann Zeta-function. At first I could not believe they were correct!! Since then however the ideas have been extended in a number of ways. They have lead of course to a range of important new results about the Zeta-function and other Dirichlet series. But just as significantly the ideas have led to new conjectures on the moments of the Riemann Zeta-function. These conjectures provide the first successful test for the application of Random Matrix Theory in this area. Nowadays this is a growing area which has contributed much to our understanding of zeta-functions. And it can all be traced back to your work in the late 1970's.

8. Selected Theorems – The Little Flowers

Ramachandra used to dedicate his best results to his mathematical Gods in papers whose title began with *Little flowers to* .... For instance when he visited Russia on the occasion of the ninetieth birthday of I. M. Vinogradov, the paper that he presented in the conference was titled *Little flowers to I. M. Vinogradov*. We shall present a few flowers from Ramachandra's garden that roughly cover his genre of work. For more details on Ramachandra's work, I would request the reader to refer to the volumes of the Hardy-Ramanujan journal which are available online at `http://www.imsc.res.in`

Theorem 8.1 (K. Ramachandra). *Let λ be any constant satisfying $1/2 < \lambda < 1$ and l a non-negative integer constant. Put $H = T^{\lambda}$. Then we have*

$$(\ln T)^{1/4+l} \ll \frac{1}{H}\int_T^{T+H} \left|\frac{d^l}{dt^l}\zeta(1/2+it)dt\right| \ll (\ln T)^{1/4+l}.$$

Theorem 8.2. (R. Balasubramanian and K. Ramachandra). *Let t be a fixed transcendental number and $x \geq 1, y \geq 1$ be integers. Let n be any integer such that $x \leq n < x+y$ for which 2^{t^n} defined as $\exp(t^n \ln 2)$ is algebraic. The number of such integers is $< (2y)^{1/2} + O(y^{1/4})$.*

Theorem 8.3. (M. Jutila, K. Ramachandra and T. N. Shorey). *Let $k > 2$ and $n_1 = n_1(k), n_2 = n_2(k), \ldots$ be the sequence of all positive integers which have at least one prime factor $> k$. Put $f(k) = \max(n_{i+1} - n_i)$ the maximum being taken over all $i > 1$. Then*

$$f(k) \ll \frac{k}{\ln k}\left(\frac{\ln\ln\ln k}{\ln\ln k}\right).$$

Theorem 8.4 (K. Ramachandra). *For all sufficiently large m, between m^2 and $(m+1)^2$, there is an n and a prime p dividing n such that $p > n^{1/2+1/11}$.*

Theorem 8.5. (Ramachandra, Shorey and Tijdeman). *There exists an absolute constant $c_2 > 0$ such that for $n \geq 3$ and $g = g(n) = \left[c_2\left(\frac{\ln n}{\ln\ln n}\right)^3\right]$, it is possible to choose distinct primes $p_1, p_2, \ldots, p_g$ such that $p_i|(n+1)$ for $1 \leq i \leq g$.*

9. Acknowledgement

I am indebted to Mrs. Saraswati Ramachandra, wife of K. Ramachandra, for sharing the details of the last few days of Ramachandra's life. She was kind enough to give me a small photograph of Ramachandra that she had preserved. I would also like to thank C. S. Aravinda, TIFR Bangalore, and Kishor Bhat, NIAS Bangalore, for providing me with valuable information and encouraging me to write this biography. I wish to thank S. L. Ravi Shankar for providing some of the photographs in this article. I would also like to thank Ankita Jain and Roshani Nair, GLIM Chennai, for reviewing this article and giving valuable suggestions.

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Prof. K. Ramachandra: Reminiscences of his Friends

M. Pavaman Murthy

A Few Reminiscences of K. Ramachandra in his Early Years at TIFR

I joined TIFR in 1958 along with Ramachandra, R. R. Simha and Vasanthi Rao. Ramachandra stood out in this group of four because of his remarkable dedication to mathematics and his knowledge of advanced number theory. He had made up his mind even before joining TIFR to work in number theory. A few months after joining the institute, Ramachandra was studying research papers in number theory when the rest of us in the group were struggling with topics exposed in the "baby seminars." The four of us were good friends. In recent times, whenever I visited TIFR from Chicago, Ramachandra used to enter in to my office and inquire about my family and tell me about his work and his students' work. He always spoke of his students with pride.

Here are some of my reminiscences of Ramachandra in his early years at TIFR.

In the first or second year after joining the institute, one early morning in TIFR hostel in the Old Yacht Club, I still remember witnessing a heated discussion between Ramachandra and Raghavan Narasimhan comparing the contributions of Hilbert and Kummer to number theory. Probably that was the first time that I came to know that there was a famous number theorist called Kummer.

Here is a glimpse of Ramachandra's sense of humor. In our early TIFR years, the institute was located in the Old Yacht Club building adjacent to Gateway of India. Several of the members of the School of Mathematics along with us new comers worked in a large hall with many tables. One day Ramachandra came to my table and showed me a theorem of Siegel and said "See Siegel proves this theorem in ten pages. I have proved the same in five pages." Indeed Siegel's proof covered ten pages. What Ramachandra had done was to copy verbatim Siegel's proof in his note book in five pages (with very small handwriting)!

Sometime after joining the institute, we four new comers were told that there would not be any oral examinations at the end of our first year, contrary to the usual practice. Instead we were to give some talks on selected topics. Our progress in the first year was to be judged by those talks which were to be held in October 1958 after the summer vacation. I had gone to my hometown Hyderabad, hoping to prepare for my talks. Within a few days after my arrival in Hyderabad I received a letter from Ramachandra who was still at the institute. Ramachandra had written advising me to prepare the talks well as he had heard that KC (Prof. K. Chandrasekharan) was to attend our talks and might ask questions in those talks. It was out of kindness and concern about me that Ramachandra had written that letter. Little did he realize that he had spoiled my vacation! I will miss Ramachandra.

M. Pavaman Murthy

Michel Waldschmidt

K. Ramachandra: Some Reminiscences

I was thrilled when I received a letter from Ramachandra around 1974, who invited me to spend some months at the Tata Institute of Bombay and give a course on transcendental numbers. When I was young it was my dream to visit India, and I did not expect that I would have such an opportunity. I knew very well his paper [1] *Contributions to the theory of transcendental numbers* published in Acta Arithmetica in 1968: this was the main reference of my thesis, submitted in 1972. I was able to pursue his work in several directions, including algebraic groups. My first attempt to prove a new result was motivated by one of his problems which is now called the four exponentials conjecture, which had been proposed independently by S. Lang and Th. Schneider. This has been the problem on which I have spent most of my efforts during all my mathematical life, and it is still open. I believed a couple of times that I had a solution, especially in 1970; it turned out that there was a gap in my argument, but that I could nevertheless get something new: instead of solving the four exponentials conjecture, which is the first open problem proposed by Th. Schneider in his book, I could solve the 8th of these problem, on the transcendence of one at least of the two numbers e^e and e^{e^2}. As a matter of fact, the same solution was found at the same time and independently by W. D. Brownawell. For this result, we shared the Distinguished Award of the Hardy-Ramanujan Society which was attributed to both of us by Ramachandra in 1986.

When I received the invitation of Ramachandra, I decided to accept it and to go to India with my wife. However, shortly afterwards, she became pregnant, so I postponed this visit and came alone, after the birth of my son Alexis in May 1976. I visited TIFR from the end of October to end of December 1976. It was not easy for me to leave my young son for such a long time at his early age. My stay in India has been an unforgettable experience for me. I loved it immediately, even if it took me some time before I could adjust to the food.

My lectures were on *transcendental numbers and group varieties*. Since I was going to deliver lectures on that topic at Collge de France (*cours Peccot*) a few months later, I used this opportunity to polish my presentation. The precise topic was a development of Ramachandra's work with applications to commutative algebraic groups. These notes were going to be published in Astrisque in 1979. I did not know that Ramachandra had shifted his interest from transcendental number theory to the Riemann zeta function two year earlier (so I had no influence on this shift!); and, most of all, I did not know that he was disliking commutative groups as much as he loved numbers. To mix both was not the best thing to do to please him, but I was innocent. My TIFR course was supposed to be published by the Tata Institute, a research student of Ramachandra was supposed to write it down. I left him the notes (it was not that easy at that time to make xerox copies), but the course was never written down, my notes got lost and I had to reconstruct them from scratch.

This was my first experience of spending some time in a non–French speaking country, and my English was quite poor. To spend two months like this was very efficient from this point of view, and since Ramachandra was among the people with whom I spoke often, I made progress during this stay to understand him better. Later, it happend quite a few times that I was with an English speaking mathematician, from UK or USA, and I served as a translator, repeating with my French accent what Ramachandra said with his Indian accent, and it was helpful for the concerned colleague!

I met Ramachandra again in 1979 in Kingston at Queen's University where we participated in a conference on recent developments in number theory, organized by P. Ribenboim, where I was with my family. I also met him later, in July 1987, again in Canada, during an International number theory conference held at Universit Laval organized by Jean–Marie De Koninck and Claude Levesque.

My second trip to India was in 1985. A conference in honor of Bambah was scheduled in Chandigarh. My trip was supported by an agreement between the two Academy of Sciences of India and France; I already had my ticket when the conference was postponed for security reasons. Nevertheless I was allowed by the two academies to maintain the project and I first visited Bombay. In Bombay I was invited by Ramachandra, at his apartment and in the evening at the Tanjore restaurant of the Taj Mahal. I also went to Madras and visited *Matscience* (which became later *IMSc – Institute of Mathematical Sciences*), and on my way back I visited Delhi.

I came back in 1987 for the centenary of Ramanujan, and I could participate to a conference in Annamalai University (next to Chidambaram in Tamil Nadu), and this was the opportunity for my first trip to Kumbakonam. I visited the home town of Ramanujan three more times later (December of 2007, 2008 and 2009) when I was in the selection committee of the Ramanujan SASTRA Prize.

This visit in 1987 gave me the opportunity to organize my next visit, one year later, with my family. I had been invited by Alf van der Poorten to spend two months (July and August 1987) in Australia. The advantage of my trip to the southern hemisphere was that it was giving us the opportunity to visit India on the way back. This visit to India for my family was threatened at the last minute (at the airport of Sydney, just before boarding for Bombay) for a question of visa, but fortunately we could make it. With my wife Anne, my son Alexis who was just 12, and my daughter Hélǹe who was 10, we arrived in Bombay on August 20, 1988, we went to Madras on 22, we visited Pondicherry from 23 to 25, and came back to France on 28. This was a tight schedule, but this has been an unforgetable experience, one of the high points in my life as well as in the life of my children (my daughter Hélǹe came back on her own to India in 2000 and in 2001). In Bombay we stayed at the Tata Institute. We were invited by Ramachandra, who took us to the Prince of Wales Museum and was our guide. My children were to remember that he had a pink shirt during that visit: this is not common for us that a man would wear a shirt of that color.

After that I was to come back on a regular basis to India, on the average more than once a year, and I met Ramachandra very often. I was there for the two major conferences which were organized for his birthdays, the sixtieth in July 1993 (organized by R. Balasubramanian in Madras) and the seventieth in 2003 (organized by K. Srinivasa in Bangalore). I wrote a survey based on Ramachandra's paper [1]. This survey was completed after the first conference and published after the second one [2]. In December 2003, during this conference, I visited Ramachandra's office at NIAS. It was almost empty. Only a picture of G. H. Hardy and a picture of S. Ramanujan were on the wall, to whom he was deeply devoted. He told me that once a year, he had to leave his office which was used by other people for a few days, so he could not keep anything personal there.

I have a specially fond and very moving memory of our meetings in Bangalore in early 2005 (end of January – beginning of February). I was the representative of CIMPA for the school on *Security of computer systems and networks*, organized at the Indian Institute of Science by K. Gopinath. This was only four months after my daughter passed away, and Ramachandra found the right words to speak with me. He also gave me his personal reminiscences. He told me the difficulties he had during his own life. He spoke of his brother and his nephews. He mentioned that his father passed away while he was only 13, that he needed to take care of his family, and that he was fully dedicated to mathematics – this much I already knew! He told me how difficult it has been for him to take certain decisions, like that of moving from one place to another. And, of course, we shared our concerns as fathers who care about their daughters. This is certainly one reason why I had the feeling to be so close to him, and I did my best to meet him as often as possible. I stayed in India two months in December 2009 – January 2010. I was ready to go from Chennai to Bangalore in December, 2009 to visit Ramachandra, but Kishor Bhat, who was taking care of the arrangement, told me that the daughter of Ramachandra had to go to the hospital and he suggested me to postpone my visit, which I did. So this visit took place in January 2010, and this was to be our last meeting. At that time he gave me some money for P. Philippon, to whom he attributed the Hardy–Ramanujan award. When I told this to Philippon he was grateful and suggested that the money go to an orphanage, which I could do immediately thanks to Prem Prakash in Chennai.

I came back to India for ICM2010 in Hyderabad in August 2010 and for a satellite conference just after in Chennai, I was in transit in Bangalore on the way, but my schedule was too tight and I did not visit him that time – I missed an opportunity.

It has been a great privilege for me to know Ramachandra. I never met anyone else who would be so dedicated to mathematics. I also knew him on a more personal basis. I admire him, he was truly exceptional. I miss him.

Michel Waldschmidt, May 9, 2011

This text is an abstract of a Colloquium talk given at TIFR CAM Bangalore on April 26, 2011, at the invitation of C. S. Aravinda. The author is thankful to C. S. Aravinda for this invitation, to K. Sandeep who took care of the organisation, to Kishor Bhat and K. Srinivasa who made possible a visit the same day to Mrs. Ramachandra.

The colloquium talk included also a mathematical discussion concerning Ramachandra's contributions to transcendental number theory. The pdf file of the talk is available on the web site of the author. The main reference is

[1] K. Ramachandra, Contributions to the theory of transcendental numbers (I), *Acta Arith.* **14**, 65–72 (1968); (II), id., 73–88.

A description of this work is given in

[2] M. Waldschmidt, On Ramachandra's contributions to transcendental number theory; Ramanujan Mathematical Society, Lecture Notes Series Number 2, *The Riemann Zeta function and related themes: papers in honour of Prof. K. Ramachandra*, Proceedings of International Conference held at National Institute of Advanced Studies, Bangalore 13–15 December, 2003 Ed. R. Balasubramanian, K. Srinivasa (2006), 155–179.

K. Soundararajan

It is a privilege to have known as great a man and mathematician as Prof. Ramachandra. I first met him in 1989 when I was in high school and learning mathematics from Prof. Balasubramanian. Balu had mentioned my interests to Ramachandra, and I was astonished to receive shortly afterwards an invitation from Ramachandra to spend a couple of weeks at TIFR. I learnt a great deal from him during those two weeks, especially on the subjects close to his heart – the distribution of prime numbers and the behavior of the zeta function. Equally, I was struck by his warmth and friendliness and his childlike love and enthusiasm for mathematics. The distance between our years or accomplishments was completely absent in our interactions. To borrow a phrase from "My Fair Lady," Ramachandra treated all flower-girls as duchesses. Over the years, I was fortunate to have many more interactions with him, and I have benefited immensely from his encouragement, advice, and generosity in sharing ideas. And of course, his many beautiful papers in number theory have been a source of inspiration for me and many others.

In 1990, Aleksandar Ivic gave a series of lectures in TIFR on mean-values of the zeta-function. Ramachandra kindly invited me to attend this series, and I spent a very happy month in Bombay learning from these lectures and many conversations with Balu and Ramachandra. I was nearing then the end of my high school years, and was thinking about where to pursue my undergraduate education. I sought advice from Ramachandra on this, and he enthusiastically recommended my going to the University of Michigan to work with Hugh Montgomery. I did so, and my next occasion to meet Ramachandra was on the happy occasion of his sixtieth birthday in 1993 when a celebratory conference was organized at the Institute for Mathematical Sciences, Chennai. I was honored to speak at this conference, and Balu and I wrote a paper *On a conjecture of R. L. Graham* which we dedicated to him.

Ramachandra took great pride and pleasure in the accomplishments of his students – and I count myself as a student/grandstudent of his – and a nice result by one of his students gave him even more joy than his own great theorems. Through the 70's and 80's Ramachandra wrote a series of path-breaking papers on the zeta function, making great progress on understanding the moments and extreme values of the zeta function. Shortly after his sixtieth birthday conference, I was reading one of these beautiful papers (*J. London Math. Soc.*, 1975) on the fourth moment of the zeta-function, and this directly inspired me to work out new lower bounds for moments of the zeta-function. I sent the paper and related work to Ramachandra who was absolutely delighted. This reaction from one of the pioneers in the field was a source of great encouragement to me. A little later, Ramachandra himself wrote another paper on fractional moments of the zeta-function, and at the risk of seeming immodest I am very proud to say that he dedicated this paper to my 23rd birthday! No one except Ramachandra would think of dedicating a paper to someone's 23rd birthday, and to him this would have seemed perfectly natural!

After Ramachandra retired from TIFR and moved to Bangalore I had a couple of occasions to meet him. In 1997 he invited me to NIAS to give a couple of lectures and I spent a happy week with him. I next saw him at the 2003 meeting of the American Math Society in Bangalore. As always he was full of ideas and spoke excitedly of the many problems he was thinking about. The AMS meeting followed a conference in honor of Ramachandra's 70th birthday, which unfortunately I missed, but Granville and I wrote a paper in his honor on one of his favorite topics (extreme values of the zeta function). It did not occur to me that this would be the last time I saw him. We of course kept in touch over the next several years: I would always receive new year cards, and the next issue of the Hardy-Ramanujan Journal, and he always forgave my tardy responses. I thought I would see him at the 2010 ICM, or at Balu's 60th birthday conference, but that was not to be. I miss him greatly.

Prabhakar Vaidya

I, like the rest of us at NIAS, have not had sufficient time to reflect on the magnitude of the great personal loss that I have suffered. I am going to mention one immediate thing that springs to my mind.

I will miss visiting his office. His office was physically quite close to mine. Yet, the few steps I often took to visit his office were nothing short of magical. Once you entered his door, I was transformed into a magical world. There were these two towering portraits of Hardy and Ramanujan. I would look at them in reverence and then my eyes would wander to his scribbling on the whiteboard. He would notice where they had rested and his face would be lit. "This one is by Soundararajan. He has improved the result of Montgomery, which was in turn an improvement of the famous result of Vinogradov . . . ."

(This was his world. Numbers, theorems, Hardy, Ramanujan. Michigan was mostly Montgomery and when he announced at our faculty meeting that Vinogradov had passed away, he could not stop his tears.)

He would notice my confusion as I try to grasp this new result on the board and say in a kind voice, "oh don't worry about these constants, they don't matter . . . ." I keep asking myself, "$\log\log\log$ of x? . . . " How on earth does anyone think of this? He would start explaining and I would look at his face and say to myself, "Can God have a kinder face?"

No matter how many times I visited his office, the result was the same. I was a much purer version of me than the person who entered. I was back in my childhood, worshipping Ramanujan, dreaming that one day I will prove Fermat's last theorem "I would be lost in my books, in numbers, in dreams . . . ."

That boy is now mostly gone. And yet, in his room, he was back. The innocence, the purity "Prof. Ramachandra," radiated it to us. The glow would last for while, even after I used to leave his room. I knew that this was a very special Darshana. Yes, my rest of the day would sail quite smoothly now.

Was I really this lucky to have known this extraordinary genius, this man whose devotion to Ramanujan was greater than Hanuman's for Rama, this utterly, extraordinarily humble man, this living saint, who chose to say hello to me?

Prof. K. Ramachandra: Reminiscences of his Students

A. Sankaranarayanan

One of the Indian Legends Whom I Know Intimately

I am doubly fortunate to have had Prof. Ramachandra as my Phd thesis advisor. He has been not only an excellent researcher but also a good teacher. I had the opportunity to learn many intricate techniques in Number Theory from him and I am extremely happy about it. We had indeed collaborated on several research works and some of our results have still been unimproved for several years.

Prof. Ramachandra has contributed in various branches of Number Theory.

1. Algebraic Number Theory: On Kronecker's limit formula.
2. Transcendental Number Theory: On Baker's theory on linear forms and logarithms, transcendental measures of certain irrational numbers and so on.
3. Theory of Riemann zeta-function and L-functions: His contributions in this area are immense. I quote a few here namely, Omega theorems, lower and upper bound estimates

on various questions, zero-density estimates of certain L-functions, Mean-value theorems on certain vertical lines of certain L-functions and so on.

4. Elementary Number Theory: Various questions involving summatory functions of several interesting arithmetical functions, On Vinogradov's three primes theorem and so on.

Personally, he was a nice gentleman. He was my mentor, teacher and above all, he is an excellent caretaker. I recall here my student days. I used to stay in Room No: 414 at the Brahmagupta Hostel. Generally I used to work until late night. On several occasions, he had visited my room even after midnight to discuss mathematical problems and poured several nice ideas.

He is one of the most successful mathematicians who had a lot of students. Prof. T. N. Shorey, Prof. S. Srinivasan, Prof. R. Balasubramanian, Prof. M. J. Narlikar, Prof. V. V. Rane, myself, Prof. K. Srinivasa and Mr. Kishore Bhat (I learn that he has submitted his thesis to Mysore University recently) are all his Phd students.

Truly speaking, a very few of his students had certain privileges with him and definitely I am one among the slot. On special days of every year, he had invited me to his home for having lunch or dinner with him and his family. My relationship with Prof. Ramachandra is beyond a student and teacher relationship. I have been treated all along as one of his family members. He was more than a father figure to me and I truly owe him a lot to my existence.

One of his nice qualities is that whenever any student of him contributes some good mathematical results, he used to appreciate openly. I have heard from him such appreciations for Profs. Shorey and Balasubramanian of their nice mathematical contributions. I too am fortunate to get such appreciations from him on a few occasions. I feel that it is the right moment to record such an experience of mine from him. I recall here the year 1993–1994. During this period, I was working on the zeros of quadratic zeta-functions on the critical line over short intervals. I wrote a manuscript (first draft) which slightly improves the length of the short interval (of a result of Bruce Berndt (for the imaginary quadratic field case) and of a result of K. Chandrasekharan and Raghavan Narasimhan (for the real quadratic field case)) so that a zero of quadratic zeta-functions exists in such a shorter interval. I went with the manuscript to his office in the morning around 10 O'Clock on the next day, told him about the result what I got and requested him to check the manuscript. He was a bit hesitant and half hearted to check my manuscript. The reason for his reluctance was that during a tea-table chat, Prof. K. Chandrasekharan told Prof. K. Ramachandra that it would be very difficult to improve their result. I politely requested Prof. Ramachandra to check my manuscript and find a flaw and then let it go to the dust bin After my lunch at the cafeteria (my family was away on that day), I went home and peacefully slept in the afternoon. It was around 4 pm on that day, on hearing my house door calling bell, I opened the door, to my surprise, Prof. Ramachadra was standing outside my door with full of joy (I could see from his face). I understood from him that he sat with my manuscript after I left his office and checked it thoroughly all through the day. He congratulated me for this nice result and started telling about this result to all our faculty members the next day. He is such a nice person and this incident still runs in my mind as one of my green memoirs. I should also mention here that later developments show that there are certain improvements on the imaginary quadratic field case whereas in the real quadratic field case my result still stands as such till today.

He started on his own the Journal "Hardy-Ramanujan Journal" (devoted to primes, Diophantine equations, transcendental numbers and other questions on $1, 2, 3, 4, \ldots$) as Prof. R. Balasubramanian and Prof. K. Ramachandra as editors and successfully published it for the past 33 years. I hope that this journal will be continued.

Definitely he is very generous. A very good Indian number theorist who died at an early age left behind his spouse with four children. For those children's education, he used to send money every year. I know this very well because I myself have sent money on behalf of him sometimes.

When I received a phone call from his brother-in-law in the early morning 2.00 O'Clock on 18th January 2011 about his sad demise, I was naturally quite shocked and it took me almost 2 hours to recover myself. The only solace to my heart and my mind is that I was fortunate to be with him at the Ramaiah Hospital in Bangalore for at least five days in December 2010.

Prof. Ramachandra is one of the best mathematicians of our country and his passing away is a very big loss generally to our Nation and in particular to all Number Theorists worldwide. I really do not find any words to console his family at this critical moment. Though he is no more physically, I am sure that he will be living with us and he will be remembered for his excellent contributions to the subject Number Theory in the years to come.

I pray to the Almighty that his soul rests in peace.

A. Sankaranarayanan, School of Mathematics,
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Mangala Narlikar

When we came to Mumbai at the end of 1972 with two small daughters, we got to live in the TIFR housing colony as my husband was a professor in the department of Physics there. More than six years before that, I was first a research student and then research associate in the School of Mathematics there. In my spare time, I started going to the same place to see if I can work in Mathematics again. My interest was in Analysis and related subjects and I started attending some lectures along with new students. Prof. K. Ramachandra whom I had seen but not interacted with before, was lecturing on Analytic Number Theory and I liked those lectures. When Prof. Ramachandra found that I wished to study that branch of mathematics, he gave all the encouragement. Due to my household responsibilities, sometimes the timing of the lecture was changed to suit me. I found his lectures as well as those of R. Balasubramanian very interesting and inspiring. Along with these two people, Prof. Shorey, Dr. Srinivasan, Mr. Rane etc. made a very lively group of number theorists. That I managed to work for a Phd is due to the encouragement of Prof. K. Ramachandra. Later I also worked as a pool officer under his guidance.

Prof. Ramachandra's only passion seemed to be to work in Mathematics and to encourage people who wished to work in the field. This was amply demonstrated when he felicitated and offered a purse from his own pocket to Mr. Kaprekar. Mr. Kaprekar was a school teacher in Nasik, who worked by himself in number theory and recreational mathematics and tried to inspire school children with the subject.

Later, Prof. Ramachandra was our next door neighbour and both he and his wife Saraswati were very friendly to our youngest daughter Leelavati.

Mangala Narlikar, # 6, Khagol Society,
Panchavati, Pashan, Pune 411 008.

Kishor Bhat

Some Reminiscences about Prof. K. Ramachandra

Talking about Prof. Ramachandra has never done justice to him. I was his final student, working with him from 2005 until his passing. Being with him was an incredible experience, but I suppose many people can say that about their professors. I thought the best thing I could do was to share my experiences with Prof. Ramachandra. When I worked with Prof. Ramachandra, for reasons that do not require too much explanation, I kept a record of my experiences. Below are some edited excerpts:

A Day with Prof. Ramachandra

Today I came into work late. Prof. Ramachandra as usual came on time, and was prompt. Since I am his student, and the only other number theorist at NIAS, he was looking for me.

I met with Prof. Ramachandra when I got in. He wanted to simplify a proof we were working on, and we went off to discuss different ideas about how to simplify our proof. He insisted that everyday, we were to have a discussion on mathematics, and he wanted to make sure I was making progress on my work.

In NIAS, there is a discussion room with three blackboards that the mathematicians like discussing in. We went into the room and suggested certain ways to improve the proof. The progress was not linear. Sometimes we felt we would have already solved it, and sometimes we would not know what to do. At the end of the day, we both improved on our previous result by a small amount, but did not feel ready. This is what working with Prof. Ramachandra was like. He never knew the answers, but he would work with you to get them. He did not like classes, but would encourage his students to do research from day one. "If you want to learn to swim, you must go to the middle of the ocean and swim" was his motto.

Another Day

I met Prof. Ramachandra in the morning. He asked me if there was any news from Acta Aritmetica. For those of you who do not know, Acta Arithmetica was Prof. Ramachandra's favorite periodical. Even though it was a quarterly, he would always anxiously go to the library every Thursday to check for the next copy. In his final year, we were able to access copies on the internet, so I, as his student, would check online for the latest copy. There was no new issue that day.

Prof. Ramachandra and I had a discussion on numbers that are the sum of squares. He gave me some problems to work on the day before, and today he gave me some hints to his exercises, which I confused with some other exercises he gave on square-free numbers. This was common. Ramachandra was one of the most focused mathematicians we knew, but in that focus, there was a huge world. He would tell me not to get confused by people who do high-dimensional analysis. "The first dimension is already very complicated."

He called me during lunch and told me that we were required to write a bi-annual report, which he wrote on behalf of the two of us. He mentioned the paper that we both worked on. Ramachandra would tell people that the work was really mine. This was his way, he really cared about, not just promoting his own students, but also encouraging newcomers to the field. Since he felt I was a novice, he wanted me to do well.

In the evening, Ramachandra called me from his residence and asked if I was thinking about the problem we discussed. I said I was and he was pleased, and wished me a happy Ugadi (Kannada New Year).

Another Day

Today Ramachandra gave a lecture. Here are the notes of that lecture.

Ramachandra, who seemed a little irritated, began his lecture. He commented on the difficulty of notation, but added that the difficulty in working through the theorem will add to the extraordinariness of the theorem. "The more you struggle to get a result, the nicer the result is!" he would say, or "We need to prove a complicated statement to prove a nice theorem." Prof. Ramachandra was always filled with statements like this. For him, the difficulty of work was part of its joy. He would repeat this sentiment many times during the lecture. It was good for me as a student to hear this, as it made my own failures more bearable, and would encourage me to do things, even when in all likelihood the effort would end in frustration.

Ramachandra was one of those exceptional individuals. He may have been one of the few Professors I would habitually refer to as sir (in no small part because he always referred to me as sir, despite our 50 year age difference). Working with Ramachandra was always an experience. I always had my bag filled with Ramachandra stories. Most of them were so fantastic, that people would scarcely believe that there was such a person who could be so focused a mathematician. Those who have had the occasion to meet him, learn that it was no exaggeration. He really was the focused single-minded genius I said he was. It's a pity. After today no one will believe me.

Those of us who have been part of the mathematics group in NIAS miss him. We know that mathematics as a whole suffer his loss. Those of us who work with the numbers that he so dearly loved know that they are not as easy to work with now that he is no longer with us.

Riemann Hypothesis – the Prime Problem!

Dedicated to a great man who was so kind to me

B. Sury

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1. Introduction

In a 1975 inaugural lecture at the university of Bonn, the well-known number theorist Don Zagier said: *"There are two facts about the distribution of prime numbers of which I hope to convince you so overwhelmingly that they will be permanently engraved in your hearts. The first is that, despite their simple definition and role as the building blocks of the natural numbers, the prime numbers grow like weeds among the natural numbers, seeming to obey no other law than* *that of chance, and nobody can predict where the next one will sprout. The second fact is even more astonishing, for it states just the opposite: that the prime numbers exhibit stunning regularity, that there are laws governing their behavior, and that they obey these laws with almost military precision."*

On the other hand, the most important open question in number theory and, possibly, in the whole of mathematics is the Riemann hypothesis. David Hilbert said: *"If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?"* A one million dollar prize has been offered by the Clay Mathematics Institute for a proof of this 'Problem of the Millennium'.

In the theory of prime numbers, there are several hypotheses but the Riemann Hypothesis simply stands out. When we trace our path through classical prime number theory, and try to see how the subject has evolved, we find ourselves led inevitably to the so-called Langlands Program, a sort of *grand unification* theory in mathematics. The Riemann Hypothesis and ideas associated with it seem to light up the path of this discovery.

In 1748, Leonhard Euler wrote down the fundamental theorem of arithmetic as an analytic statement. The so-called Euler product

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$$

valid for all real $r > 1$ is just a rephrasing on the fundamental theorem that every natural number > 1 is a unique product of prime powers. This proves the infinitude of primes in an analytic and quantitative manner since the series on the left diverges at $s = 1$. Needless to say the distribution of prime numbers, being a fundamental problem, fascinated the top mathematicians of each generation. The great Carl Friedrich Gauss – conjectured the so-called 'Prime Number Theorem' in 1794, at the ripe old age of 17(!) Roughly speaking, this is the assertion that the function $\pi(x)$ measuring the number of primes up to a given x behaves like the function

$$Li(x) := \int_1^x \frac{dt}{\log t}.$$

That is, $\pi(x)/\log(x) \to 1$ as $x \to \infty$. So, $\frac{\pi(x)\log x}{x} \to 1$ as $x \to \infty$. Equivalently, the n-th prime p_n satisfies $\frac{p_n}{n\log(n)} \to 1$ as $n \to \infty$. It is amusing to compare Gauss's interest with Fermat's last theorem (FLT) – he is said to have remarked that the problem doesn't interest him as he can come up with many similar questions on Diophantine equations which cannot be answered and that there was nothing particularly special about FLT! Although the infinitude of primes was known from Euclid's time, the infinitude of primes in every arithmetic progression of the form $an+b$ with $(a, b) = 1$ was proved only in 1837 by Lejeune Dirichlet. Unlike Euclid's proof which is a variation of the fundamental theorem of arithmetic and can be (and is being) taught in schools, Dirichlet's proof requires much more sophisticated, *analytic* techniques. One might say that Euler's basic, but deep, observation on Euler product expansion was the key behind Dirichlet's perspective. Dirichlet would need to consider (approximately a) more series similar to Euler's series $\sum_{n=1}^{\infty} \frac{1}{n^s}$. During 1848–50, the Russian mathematician Chebychev proved the beautiful fact that there are certain constants $a, b > 0$ such that

$$a\frac{x}{\log x} \le \pi(x) \le b\frac{x}{\log x}$$

for large x. However, it is Bernhard Riemann's 1859 memoir which turned around prime number theory, introducing novel techniques and giving a 'never-before' impetus to the subject of analytic number theory. Riemann lived less than 40 years (September 17, 1826 – July 20, 1866) and wrote, but one, paper on number theory! Earlier, when Riemann was submitted a Doctoral Dissertation in 1851, Gauss remarked that Riemann had 'Gloriously fertile originality'. Riemann developed what is now known as Riemannian geometry and was the indispensable theory and language used by Einstein for formulating his theory of relativity. Coming back to Riemann's paper in number theory, the key difference between earlier workers and Riemann's paper was the he considered the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ as a function of a complex(!) variable s which varies over the right half-plane $\mathrm{Re}(s) > 1$. This is now called the Riemann zeta function. Riemann proved two basic properties (meromorphic continuation and functional equation to be recalled below). The key point of viewing the zeta function as a function of a complex variable s allowed Riemann to prove an 'explicit formula' connecting the complex zeroes of the zeta function and the set of prime numbers! Riemann also made 5 conjectures, 4 of which were solved in the next 40 years. The one-unproved conjecture is the Riemann Hypothesis. In this article, we start with Riemann's memoir and the development of analytic number theory originating from it. Following that, we point out to various interesting equivalent statements to the Riemann hypothesis. We then turn to evidence towards the truth of the **RH** which leads to the study of zeta functions of curves over finite fields where one had the first success story. This opens the door to more general zeta and L-functions like the Dedekind zeta functions, Dirichlet L-functions, L-functions of elliptic curves and of modular forms. The discussion of Artin L-functions leads us inexorably to representation theory and, finally, the Langlands program which is the unification theme.

2. Riemann's Memoir

The Riemann zeta function defined by the infinite series $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\mathrm{Re}\,(s) > 1$ satisfies:

(I) Meromorphic continuation: $\zeta(s)$ can be defined for all $s \in \mathbf{C}$ as a holomorphic function except for the single point $s = 1$ where it has a simple pole with residue 1; *The key function Riemann uses for this is Jacobi's theta function*

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x}$$

which has the transformation property $\theta(1/x) = \sqrt{x}\theta(x)$ *which is also a harbinger of the connection of* $\zeta(s)$ *with modular forms to be discussed later.*

(II) Functional equation: The continued function (again denoted $\zeta(s)$) satisfies

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s).$$

Here $\Gamma(s)$ is the Gamma function defined by

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$$

for $x > 0$. *The appearance of the Gamma function here was not properly understood until the appearance of John Tate's thesis as late as 1950. From Tate's work, it becomes clear that the Gamma factor is the correct term corresponding to the archimedean place (or 'the infinite prime') of* $\mathbf{Q}$.

The functional equation tells us that the values of $zeta$ at s and at $1-s$ are related. As the Gamma function has poles at all negative integers, the zeta function has zeroes at all $-2n$ for natural numbers n. Also, from the simple pole of $\zeta(s)$ at $s = 1$ and of $\Gamma(s/2)$ at $s = 0$, we obtain $\zeta(0) = -1/2$. Sometimes, this is stated in fancy language (by abusing notation) as

$$1 + 1 + 1 + \cdots = -\frac{1}{2}!$$

Similarly, the value $\zeta(-1) = -\frac{1}{12}$ gives:

$$1 + 2 + 3 + \cdots = -\frac{1}{12}!$$

As a matter of fact, one has

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

where B_r are the Bernoulli numbers! Note that $B_{\mathrm{odd}>1} = 0$ which is related to $\zeta(-\mathrm{even} < 0) = 0$.

Looking at the symmetry in the functional equation, it may be tempting to muse whether all the zeroes of $\zeta(s)$ are on the line of symmetry $\mathrm{Re}\,(s) = 1/2$ but this, in itself may be too simplistic, as there are series with similar symmetry whose zeroes are not on the line of symmetry; so this symmetry by itself is not sufficient reason to conjecture the Riemann hypothesis (to be discussed below). However, these other series do not possess Euler products; so, this still does not rule out the possibility that the symmetry may be the property which prompted Riemann to formulate the Riemann hypothesis.

Riemann's five conjectures (see [6,7]) in his 8-page paper were:

(i) $\zeta(s)$ has infinitely many zeroes in $0 \le \mathrm{Re}\,(s) \le 1$.

(ii) The number of zeroes of $\zeta(s)$ in a rectangle of the form $0 \le \mathrm{Re}\,(s) \le 1$, $0 \le \mathrm{Im}\,(s) \le T$ equals

$$\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

as $T \to \infty$, where the notation $f(T) = O(g(T))$ means $\frac{f(T)}{g(T)}$ is bounded by a constant independent of T.

(iii) The function

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

has an infinite product expansion of the form

$$e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho} e^{s/\rho}\right)$$

for some constants A, B where the product runs over the zeroes of $\zeta(s)$ in the infinite strip $0 \le \mathrm{Re}\,(s) \le 1$.

(iv) If $\Lambda(n)$ is the von Mangoldt arithmetical function defined to be $\log p$ if n is a power of a single prime p and zero otherwise, and if $\psi(x) = \sum_{n \leq x} \Lambda(n)$, then

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{\log(1-x^{-2})}{2}.$$

The value $\frac{\zeta'}{\zeta}(0)$ can be seen to be $\log(2\pi)$ on using the functional equation. Note that the sum over the zeroes is to be interpreted as

$$\lim_{T\to\infty} \sum_{|\rho|\leq T} \frac{x^\rho}{\rho}$$

and is not absolutely convergent.

(v) (**Riemann hypothesis**) All the zeroes of $\zeta(s)$ in the so-called critical strip $0 \leq \mathrm{Re}\,(s) \leq 1$ lie on the vertical line $\mathrm{Re}\,(s) = \frac{1}{2}$.

The conjectures (i), (ii), and (iv) were proved in 1895 by von Mangoldt and (iii) was proved by Hadamard in 1893. Until date, (v) is open. Notice that (iv) gives an explicit relation between prime numbers and *zeroes* of $\zeta(s)$! In fact, in 1893, Hadamard and de la vallé Poussin independently proved that

$$\zeta(s) \neq 0 \ \forall \ \mathrm{Re}\,(s) = 1.$$

This non-vanishing on the vertical line implies immediately that the ratio $\frac{\psi(x)}{x} \to 1$ as $x \to \infty$. This is just another rephrasing of the Prime number theorem -1 Indeed, looking at (iv), we see that $|x^\rho| = x^{\mathrm{Re}\,(\rho)}$ and, therefore, the prime number theorem ($\psi(x) \sim x$) is equivalent to the assertion $\mathrm{Re}\,(\rho) < 1$.

One might say in jest that the prime number theorem is an 'one-line proof' viz., that the Riemann zeta function does not vanish on the one line $\mathrm{Re}\,(s) = 1$!

Incidentally, the key to the proof of the non-vanishing of the Riemann zeta function on the line $\mathrm{Re}\,(s) = 1$ relies on the elementary fact $3 + 4\cos(\theta) + \cos(2\theta) \geq 0$ for all θ.

Another rephrasing is the assertion $\frac{\theta(x)}{x} \to 1$ as $x \to \infty$, where the Chebychev function $\theta(x) = \sum_{p \leq x} \log p$. The above statements are quite easy to see using a very simple elementary idea known as Abel's partial summation formula (see [1]) which states: *For any arithmetic function $a(n)$, consider the partial sums $A(x) = \sum_{n\leq x} a(n)$ (and $A(x) = 0$ if $x < 1$). For any C^1-function f on (y, x) where $0 < y$, one has*

$$\sum_{y<n\leq x} a(n) f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\,dt.$$

For instance, taking $f(x) = \log x$ in the partial summation formula, one can deduce

$$\theta(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}\,dt.$$

One can also use

$$0 \leq \psi(x) = \sum_{m \leq \log_2 x} \theta(x^{1/m})$$

to obtain

$$\frac{\psi(x)}{x} - \frac{\theta(x)}{x} \to 0$$

as $x \to \infty$.

It is not difficult to see that the **RH** itself is equivalent to the assertion:

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x^{1/2}\log x)$$

where $\pi(x)$ counts the number of primes up to x.

An important technique used in analytic number theory to estimate the sum $\sum_{n\leq x} a_n$ for a given $f(s) = \sum_n \frac{a_n}{n^s}$ is the Perron formula (for any $c > 0$):

$$\frac{1}{2\pi i} \int_{\mathrm{Re}\,(s)=c} \frac{x^s}{s}\,ds = 0,\, 1/2 \quad \text{or} \quad 1,$$

according as to whether $0 < x < 1$, $x = 1$ or $x > 1$.

Then, for a suitably chosen $c > 0$, we would have

$$\frac{1}{2\pi i} \int_{\mathrm{Re}\,(s)=c} f(s)\frac{x^s}{s}\,ds = \sum_{n\leq x} a_n.$$

This is used for the logarithmic derivative of $\zeta(s)$ to obtain the prime number theorem in the form $\psi(x) \sim x$. Note $\frac{\zeta'(s)}{\zeta(s)} = -\sum_n \frac{\Lambda(n)}{n^s}$ from the Euler product formula.

Actually, The fact that $\zeta(s) \neq 0$ for $\mathrm{Re}\,(s) > 1$ (which is not obvious from the series expression but becomes clear from the absolute convergence of the Euler product expression) is said to be the key analytical information used in Deligne's first proof of the analogue of the Riemann Hypothesis for varieties over finite fields (mentioned below). We will discuss the place of the **RH** in contemporary mathematics as well as point out results which provide evidence for it to be true. On the way, we will encounter many classes of so-called L-functions of which the Riemann zeta function is a prototype and also mention other hypotheses which imply or are implied by **RH**. Before proceeding towards that, we just mention that a rather simple aspect already makes the **RH** as the main open problem in prime number theory – if **RH** were to fail, it would create havoc in the distribution of prime numbers.

David Hilbert had some interesting views on the **RH**. Comparing the problem of transcendence of e^{π}, Fermat's last theorem and the Riemann Hypothesis, Hilbert felt that **RH** would be proved in a few years, Fermat would take quite a few years but that the transcendence result would not be proved for several hundred years. The opposite situation seems to have prevailed! In fact, Hilbert seems to have expressed conflicting views on **RH**. Once he said that if he were to wake up after a sleep of a thousand years, the first question he would ask is whether the **RH** has been solved! G. H. Hardy grew to be very fond of the **RH**. Once, while beginning a risky journey, he wrote to Harald Bohr that he had solved the **RH** although he had not done so!

It is not difficult to show that the **RH** gives

$$\frac{Li(x)-\pi(x)}{\sqrt{x}\log(x)} \simeq 1+2\sum_{\gamma}\frac{\sin(\gamma)\log(x)}{\gamma}$$

where the sum is over all positive real γ such that $\frac{1}{2}+i\gamma$ is a zero of $\zeta(s)$. Therefore, as the right side is a sum of periodic functions, sometimes people express the **RH** as saying that 'the primes have music in them'!

In fact, there is much more to this. Remarkably, it has been observed by the English physicist Michael Berry and his colleagues (see [2]) that there is a deep connection between the harmonics – the Riemann zeroes – and the allowable energy states of physical systems that are on the border between the quantum world and the everyday world of classical physics.

2.1 Lindelöf Hypothesis and Mertens Conjecture, [7]

A consequence of the **RH** is the Lindelöf hypothesis:

$$\zeta\left(\frac{1}{2}+it\right) = O(t^{\epsilon})$$

as $t \to \infty$. *This is still open.*

Let $\mu(n)$ denote the Mobius function defined for $n > 1$ as $(-1)^r$ if n is a square-free product of r prime numbers, and 0 if n is not square-free. One takes $\mu(1) = 1$. Note that formally, one has

$$\zeta(s)\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = 1.$$

Landau proved that the prime number theorem is equivalent to the assertion

$$\frac{1}{x}\sum_{n\le x}\mu(n) \to 0 \quad \text{as} \quad x \to \infty.$$

It is also easy to show that the **RH** is equivalent to the assertion:

$$\sum_{n\le x}\mu(n) = O(x^{1/2+\epsilon}).$$

Indeed, clearly the **RH** implies this assertion. Conversely, assuming this assertion, partial summation gives us that $\sum_n \frac{\mu(n)}{n^s}$ converges for all s for which $\mathrm{Re}\,(s) > 1/2$. Thus, $\frac{1}{\zeta(s)} = \sum_n \frac{\mu(n)}{n^s}$ has no poles in this half-plane; this is the **RH**.

It is conceivable that it may be easier to work with the function on the left hand side above by some combinatorial method rather than working with $\pi(x)$. Actually, most random sequences of $+1$'s and -1's give a sum upto x which is bounded by $x^{1/2+\epsilon}$ and the Möbius function appears to be fairly random; thus, this is some probabilistic evidence for the **RH** to hold.

Mertens conjectured the stronger:

$$\left|\sum_{n\le x}\mu(n)\right| \le \sqrt{x}$$

for $x > 1$. This was proved to be false by Odlyzko & te Riele in 1985. *It is unknown (although expected to be false) whether the assertion*

$$\sum_{n\le x}\mu(n) = O(x^{1/2})$$

which is stronger than the **RH** *holds.*

2.2 Turan's Theorem

Paul Turan showed the interesting result that if, for every N, the finite sum $\sum_{n=1}^{N}\frac{1}{n^s}$ is non-zero for all $\mathrm{Re}\,(s) > 1$, then the **RH** follows.

However, this approach towards solving the **RH** was doomed to failure as well: Montgomery proved that Turan's hypothesis does not hold; indeed, for each large N, the finite series $\sum_{n=1}^{N}\frac{1}{n^s}$ has a zero in $\mathrm{Re}\,(s) > 1$! A somewhat careful analysis of Turan's proof reveals that positivity of a certain function was used. In the following discussion, such a positivity condition makes it possible to obtain an equivalent rephrasing of the **RH**.

2.3 Weil's Explicit Formula and the **RH**

Let $\rho = \frac{1}{2} + i\gamma$ vary over the zeroes of $\zeta(s)$; here γ is complex, and the **RH** implies that γ is real. Consider any analytic function $h(z)$ on $|\mathrm{Im}\,(z)| \leq \frac{1}{2} + \delta$ satisfying

$$h(-z) = h(z),\ |h(z)| \leq A(1 + |z|)^{-2-\delta}$$

for some $A, \delta > 0$.

Suppose g is the Fourier transform of h; that is,

$$g(u) = \frac{1}{2\pi} \int_{\mathbf{R}} h(z) e^{-izu}\, dz.$$

André Weil proved the so-called *explicit formula*

$$\sum_{\gamma} h(\gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(z) \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{iz}{2} \right) dz + 2h\left(\frac{i}{2} \right)$$
$$- g(0) \log \pi - 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} g(\log n).$$

In other words, *the set of prime numbers and the nontrivial zeroes of $\zeta(s)$ are in duality! As Weil observed, the Riemann Hypothesis is true if and only if $\sum_\gamma h(\gamma) > 0$ for all h of the form $h(z) = h_0(z)\overline{h_0(\bar{z})}$.*

3. Other Equivalent Hypotheses to RH

(i) Hardy & Littlewood proved for the first time in 1918 that infinitely many zeroes of $\zeta(s)$ lie on the critical line $\mathrm{Re}\,(s) = 1/2$. They also showed that the **RH** holds good if and only if

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!\zeta(2n+1)} = O(x^{-1/4}).$$

(ii) In 1977, Redheffer showed that the truth of the **RH** is equivalent to the assertion that for each $\epsilon > 0$, there exists $c(\epsilon) > 0$ so that

$$|\det A_n| < c(\epsilon) n^{1/2+\epsilon},$$

where A_n is the $n \times n$ matrix whose (i, j)-th entry is 1 if either $j = 1$ or $i|j$ and zero otherwise.

(iii) Recently, in 2002, Jeffrey Lagarias proved [8] that **RH** is equivalent to the assertion

$$\sigma(n) \leq H_n + e^{H_n} \log(H_n)$$

where $\sigma(n) = \sum_{d|n} d$, $H_n = \sum_{i=1}^{n} \frac{1}{i}$.

(iv) Functional-analytic approaches seem quite promising in view of Weil's positivity condition. Nyman, and later Baez-Duarte have versions of the **RH**. The latter's results were rephrased by Bhaskar Bagchi [3] to yield the following avataar of the **RH**.

Look at the inner product space $\mathcal{H}$ consisting of all sequences $a := \{a_n\}$ of complex numbers which satisfy $\sum_{n=1}^{\infty} \frac{|a_n|^2}{n(n+1)} < \infty$. Here, we take

$$\langle a, b \rangle = \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n(n+1)}.$$

All bounded sequences are in $\mathcal{H}$. For $k = 1, 2, 3 \ldots$ consider the special elements $a(k) \in \mathcal{H}$ given by $a(k)_n = \left\{\frac{n}{k}\right\}$, the fractional part of $\frac{n}{k}$. Then, the **RH** is equivalent to either of the following statements:

(a) *The constant sequence* $1, 1, 1, \ldots$ *is in the closure of the space spanned by the* $a(k)$*'s;* $k = 1, 2, \ldots$.

(b) *The set of finite linear combinations of the* $a(k)$*'s is dense in* $\mathcal{H}$.

4. Evidence Towards RH-Zeta Functions of Curves Over Finite Fields [5]

There may be said to be three types of evidence to believe in the possible truth of **RH**. One is, of course, deep analytic methods which show that at least 40 per cent of the zeroes of the nontrivial zeroes lie on the critical line. The other is indirect evidence by virtue of statements which are nontrivial consequences of the **RH** and are either believable for other reasons or have been shown to be true by other means. The Generalized Riemann Hypothesis (which we will mention later) implies things like: (i) Miller's primality test, (ii) Artin's primitive roots conjecture, (iii) knowledge of rate of equidistribution in case of geodesic motion on an arithmetic hyperbolic surface (this is a central topic in quantum chaos). The third, and perhaps the most compelling sort of evidence comes from the proof of **RH** (yes proof!) for analogues of the Riemann zeta function. Let us talk about this third type of evidence. Just as the Riemann zeta function is an Euler product involving all the prime numbers, there is an analogous zeta function for finite fields which involves its irreducible monic polynomials. In fact, a definition of the zeta function of an algebraic curve over a finite field was given by Emil Artin in his 1924 thesis.

He also proved the analogue of the **RH** for some 40 curves. In 1934, Helmut Hasse established that the analogue of **RH** holds for the class of zeta functions associated to elliptic curves (nonsingular cubic curves $y^2 = f(x)$) over finite fields. Andre Weil proved the **RH** for *all* nonsingular curves over finite fields in 1948 by deep methods from algebraic geometry. A simpler proof by Andrei Stepanov in 1969 was further simplified by Enrico Bombieri in 1972 using the Riemann–Roch theorem to a 5-page proof! As a matter of fact, Weil can be thought of as the creator of the subject now known as arithmetic geometry. In 1949, Weil defined a zeta function for any algebraic variety over a finite field and made several conjectures (which came to be known as the Weil conjectures). One of these conjectures is an analogue of the **RH**. Actually, Weil proved all these conjectures (not just the **RH** analogue) in the special case of nonsingular algebraic curves. It is perhaps amazing that the prototype already occurs in the work of Gauss! The Last entry in his famous mathematical diary is a special case of Weil's **RH**: *Let* $p \equiv 1 \bmod 4$ *be a prime. Then, the number of solutions of the congruence* $x^2+y^2+x^2y^2 \equiv 1 \bmod p$ *equals* $p-1-2a$*, where* $p = a^2+b^2$ *and* $a+ib \equiv 1 \bmod 2(1+i)$.

It required tremendous progress in algebraic geometry before Pierre Deligne proved the Weil conjectures in general in 1973. Deligne's journey takes him through the theory of modular forms and a beautiful conjecture due to Ramanujan turns out to be the analogue of the **RH**! Before that, in 1950, Atle Selberg defined another kind of analogue of the zeta function which counts the lengths of closed geodesics in Riemannian manifolds. In a remarkable tour-de-force, Selberg developed a so-called trace formula involving eigenvalues of Laplacian and deduced the analogue of the **RH** for his zeta function! The trace formula resembles Weil's explicit formula above. Selberg had received a Fields medal for his elementary (that is not involving complex analysis) proof of the prime number theorem. His work on the trace formula was perhaps worthy of another Fields medal!

Let C be a nonsingular projective curve over a finite field $\mathbf{F}_q$ where $q = p^e$ and p is a prime. One considers the formal finite sums of the form $D = \sum a_i P_i$ where a_i are integers (of any sign) and the points P_i in C are defined over some finite extensions of $\mathbf{F}_q$ where $\mathrm{Frob}_q(D) = D$. This is called the group $\mathrm{Div}(C)$ of divisors of C. One calls a divisor $D = \sum a_i P_i$ effective (and writes $D > 0$) if $a_i \geq 0$ for all i. The *prime divisors* are those which are not expressible as a sum of effective divisors. Denoting the homomorphism $\sum a_i P_i \to \sum a_i$ by 'deg', Artin-Hasse-Schmidt's definition of the zeta function of C is:

$$\zeta(C,s) := \sum_{D>0} (q^{\deg(D)})^{-s} = \prod_P (1 - q^{\deg(P)})^{-s}.$$

This satisfies the functional equation

$$q^{(g-1)s}\zeta(C,s) = q^{(g-1)(1-s)}\zeta(C,1-s)$$

where g is the genus of C.

The Riemann–Roch theorem implies that $\zeta(C,s)$ is a rational function of q^{-s}; write $\zeta(C,s) = Z(C,t)$ where $t = q^{-s}$ and Z is a rational function of t.

The **RH** is the statement that all zeroes of $\zeta(C,s)$ lie on $\mathrm{Re}\,(s) = \frac{1}{2}$; this is equivalent to the assertion that the numerator polynomial of Z has all zeroes of absolute value $q^{-1/2}$. This is easy to verify for $g = 0$. For $g = 1$, one has the case of elliptic curves and it is Hasse's theorem.

In the Weil conjectures for general algebraic varieties X, the **RH** corresponds to the statement that the zeroes and poles of the corresponding rational function have absolute values $q^{\pm d/2}$ for some integer d. In fact, the roots (even in Hasse's theorem for elliptic curves) are viewed as eigenvalues of the Frobenius automorphism of $\mathbf{F}_q$ acting on the cohomology of the variety X.

5. Dedekind Zeta Functions [10]

For an algebraic number field K (example $K = \{a+ib : a, b \in \mathbf{Q}\}$), with its ring of integers $\mathcal{O}$ (in the above example, it is the ring $\mathbf{Z}[i]$ of Gaussian integers), the 'fundamental theorem of arithmetic' in $\mathbf{Z}$ generalizes to an analogue which asserts that ideals in $\mathcal{O}$ are uniquely products of prime ideals. Moreover, every non-zero ideal I has finite index in $\mathcal{O}$, which is denoted by $N(I)$. Thus, one has the Dedekind zeta function

$$\zeta_K(s) = \sum_{I \neq 0} N(I)^{-s} = \prod_P (1 - N(P)^{-s})^{-1}.$$

The series and the product are absolutely convergent for $\mathrm{Re}\,(s) > 1$. Note that $\zeta_{\mathbf{Q}} = \zeta$ and the Dedekind zeta function of K carries the same information on distribution of prime ideals in $\mathcal{O}$ as does the Riemann zeta function about prime numbers.

The residue of the Riemann zeta function at $s = 1$ is 1 and does not contain any information. However, the corresponding residue for $\zeta_K(s)$ carries subtle information on K like its class number etc. In fact, we have: *Analytic continuation of $\zeta_K(s)$: $\zeta_K(s)$* admits a meromorphic continuation to $\mathrm{Re}\,(s) > 1 - 1/d$ and is holomorphic except for a simple pole at $s = 1$ with residue given by 'the analytic class number formula':

$$\lim_{s \to 1^+} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h(K) \mathrm{Re}\, g(K)}{|\mu(K)|\sqrt{|\mathrm{disc}(K)|}}.$$

There is also a functional equation of the form $\Lambda(s) = \Lambda(1-s)$ which gives in particular the location of the 'trivial zeroes' of $\zeta_K(s)$. For example, it turns out that $\zeta_K(-n) = 0$ for all non-negative integers n if $K \not\subset \mathbf{R}$. Finally, there is the:

Extended Riemann hypothesis. *All the 'nontrivial' zeroes of $\zeta_K(s)$ lie on* $\mathrm{Re}\,(s) = \frac{1}{2}$.

6. Dirichlet L-Functions [10]

The Riemann zeta function can be thought of as one of a class of the Dirichlet L-functions. Dirichlet proved the infinitude of primes in progressions several years before Riemann's work and, so, he looked at all his series in terms of convergence etc. but not in terms of analytic continuation. Suppose we wish to investigate the prime distribution in residue classes modulo q for some natural number q. Dirichlet considered the finite, abelian group $\mathbf{Z}_q^*$ of invertible residue classes mod q and the dual (in the sense of harmonic analysis) group of homomorphisms from this group to $\mathbf{C}^*$. Defining any such homomorphism to be zero on non-invertible residue classes and extending it to the whole of $\mathbf{Z}$ so as to be periodic mod q, one has the notion of Dirichlet characters mod q. For any such Dirichlet character χ mod q, one has a Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

The Euler product expression is valid from the complete multiplicativity property of χ. For example, if $q = 4$, the group has two elements and the nontrivial character is the map which takes the value $\left(\frac{-1}{p}\right)$ – the Legendre symbol – at any odd prime p. Note that $\zeta(s)$ is essentially $L(s,\chi)$ for the trivial character $\chi \equiv 1$. Look at any $a \geq 1$ which is relatively prime to q. Using the Schur's orthogonality property for characters shows:

$$\sum \left\{ \frac{1}{p} : p \leq x,\, p \equiv a \bmod q \right\}$$
$$= \frac{1}{\phi(q)} \sum_{p \leq x} \frac{1}{p} + \frac{1}{\phi(q)} \sum_{\chi \neq 1} \bar{\chi}(a) \sum_{p \leq x} \frac{\chi(p)}{p}.$$

Therefore, the assertion that

$$L(1,\chi) \neq 0 \;\; \forall \;\; \chi \neq 1$$

is equivalent to Dirichlet's theorem that:

$$\sum \left\{ \frac{1}{p} : p \leq x,\, p \equiv a \bmod q \right\} = \frac{1}{\phi(q)} \log\log x + O(1).$$

Here, we have used the relation proved easily by Euler:

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + O(1).$$

Note that in the case of $q = 4$, the nontrivial character χ satisfies

$$L(1,\chi) = 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}.$$

Thus, asympotically, half the number of primes upto x are in each of the two classes 1 mod 4 and 3 mod 4.

Generalized Riemann hypothesis. *All the 'nontrivial' zeroes of $L(s,\chi)$ lie on* $\mathrm{Re}\,(s) = \frac{1}{2}$ *for any Dirichlet character χ.*

In some very interesting works, an explicit connection of the **RH** with the so-called Gauss class number problem was uncovered by Deuring, Hecke and Heilbronn. The number $h(d)$ of equivalence classes of binary quadratic forms of discriminant $d < 0$ is also the class number of the imaginary quadratic field $\mathbf{Q}(\sqrt{d})$. Gauss conjectured that $h(d) \to \infty$ as $-d \to \infty$. If χ_d is the Dirichlet character $p \mapsto \left(\frac{d}{p}\right)$ on primes, then Dirichlet's class number formula gives (for $d < -4$),

$$h(d) = \frac{\sqrt{-d}\, L(1,\chi_d)}{\pi}.$$

Hecke showed that the analogue of the **RH** for $L(s,\chi_d)$ implies $h(d) \to \infty$. Deuring proved that if the usual **RH** were false, then one would have $h(d) > 1$ for large $-d$. This was generalized by Heilbronn who showed that if the **RH** were to be false for any Dirichlet L-function $L(s,\chi)$, then $h(d) \to \infty$. In this manner, Gauss's conjecture was proved!

To indicate a relationship of $\zeta_K(s)$ with Dirichlet L-functions, consider a nontrivial primitive Dirichlet character

χ and look at the quadratic field $K = \mathbf{Q}(\sqrt{\pm q})$ where the sign is the value $\chi(-1)$. Then, one has:

$$\zeta_K(s) = \zeta(s)L(s, \chi).$$

At least, some readers may be surprised to know that this statement contains in it the quadratic reciprocity law of Gauss! A point to be noted is that the right hand side is defined essentially in terms of **Q**. *As a matter of fact, whenever K is a Galois extension of* **Q** *whose Galois group is abelian, the famous theorem of Kronecker-Weber asserting that K is contained in a field of the form* $\mathbf{Q}(e^{2i\pi/m})$ *is equivalent to writing* $\zeta_K(s)$ *as a product of* $L(s,\chi)$ *for certain Dirichlet characters* χ*'s and of* $\zeta(s)$. *Whenever we have a decomposition of some Dedekind zeta function* $\zeta_K(s)$ *as a product of terms like the above involving only information from* **Q**, *this gives a description of 'the primes which split in K'. This is valid when L is an abelian extension field of an algebraic number field K (with the RHS involving data from K) and is known as an 'Artin's reciprocity law' or 'abelian class field theory'. A conjectural form of this idea started with Emil Artin and led to the famous conjectures of Langlands.*

It should be mentioned that there are several concrete applications of this point of view (of viewing the distribution of ideals in $\mathcal{O}$ of norm less than some x in terms of $\zeta_K(s)$). For example, one has:

$$|\{I : N(I) \leq x\}| = (\operatorname{Re} s_{s=1}\zeta_K(s))x + O(x^{1-1/d})$$

as $x \to \infty$, where d is the degree of K over **Q**. As a concrete instance, the analytic properties of $\zeta_K(s)$ for $K = \mathbf{Q}(i)$ implies:

$$\sum_{n \leq x} r_2(n) = \pi x + O(\sqrt{x})$$

as $x \to \infty$, where $r_2(n)$ is the number of ways of writing n as a sum of two squares of integers.

We end this section by merely mentioning two interesting things. The first is that analogous to (and generalizing) Dirichlet characters, there are – associated to a number field K – the so-called Hecke characters defined on the ray class groups of K. The second is that there is a conjecture of Dedekind (which is still open) asserting that when $L \supset K$ are number fields, then the quotient $\frac{\zeta_L(s)}{\zeta_K(s)}$ extends to an entire function of s.

7. *L*-Function of Elliptic Curves [11]

Let E be an elliptic curve defined over **Q**; this means that it is defined by an equation of the form $y^2 = x^3 + ax + b$ with $a, b \in \mathbf{Z}$, such that the roots of the cubic x^3+ax+b are distinct. There is a group law on the points of E which can be described explicitly. For any odd prime number p which does not divide $4a^3 + 27b^2$ (the discriminant of the cubic here), the equation reduces mod p to give an elliptic curve over the finite field $\mathbf{F}_p$. The number $a_p = p + 1 - |E(\mathbf{F}_p)|$ measures the deficiency in the number of points of the curve from the projective line. The famous thesis of Hasse in 1934 where he proves the Weil conjectures for elliptic curves, proves in particular the bound

$$|a_p| \leq 2\sqrt{p}.$$

This was conjectured by Emil Artin and is the **RH** here! To indicate how, let us consider the Frobenius map Frob_p at p. Then, the points of $E(\mathbf{F}_p)$ are those of E fixed by Frob_p. Therefore,

$$|E(\mathbf{F}_p)| = |\mathrm{Ker}(\mathrm{Frob}_p - 1)| = 1 - (\lambda + \lambda') + p$$

where λ, λ' are the roots of the characteristic polynomial.

Even when the prime p is one of the finitely many of bad reduction – those for which the equation defining E does not reduce mod p to give a nonsingular curve (that is, does not have distinct roots) – the nonsingular points form a group and one defines $a_p = p + 1 - |E_{ns}(\mathbf{F}_p)|$. These numbers are encoded in the L-function of E which is defined as

$$L(s, E) = \prod_{p|N_E}\left(1 - \frac{a_p}{p^s}\right)^{-1} \prod_{p\nmid N_E}\left(1 - \frac{a_p}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}$$

where N_E, the conductor of E which we do not define precisely here, is divisible by only the bad primes. One can similarly define the L-function of an elliptic curve defined over a number field K. The analogue of the Hasse inequality is

$$|a_v| \leq 2\sqrt{N(v)}$$

where $N(v)$ is the norm of the prime ideal v. Writing $a_v = 2\sqrt{N(v)}\cos(\theta_v)$, there is a conjecture due to Sato & Tate which predicts how the angles θ_v are distributed as v varies. When E has CM (complex multiplication) – which means that there are group endomorphisms of E other than 'raising to an integral power' – Hecke already showed that uniform distribution theorem of Hermann Weyl holds good for the angles in the interval $[0, \pi]$. On the other hand, when E does not have CM, such a uniform distribution theorem does not hold good for the angles with respect to the usual Lebesgue measure but Sato-Tate conjecture predicts that it does hold good with respect to the measure $\frac{2}{\pi}\sin^2(\theta)d\theta$. A strengthened form of the Sato-Tate conjecture due to Akiyama & Tanigawa predicts: *the number of prime ideals v with norms at the most x and $\theta_v \in (\alpha, \beta)$ is asymptotic (as $x \to \infty$) to $\left(\frac{2}{\pi}\int_\alpha^\beta \sin^2(\theta)d\theta\right)\pi_K(x)$ with an error term $O(x^{1/2+\epsilon})$.*

This conjecture implies the truth of **RH** for all the L-functions.

$$L_n(s) = \prod_{p \nmid N_E}\prod_{j=0}^{n}\left(1 - \frac{e^{i(n-2j)\theta_p}}{p^s}\right)^{-1}.$$

These latter L-functions come from modular forms which we discuss now.

8. *L*-Functions of Modular Forms [5]

Consider a positive integer N and a Dirichlet character χ mod N. We look at the vector space $S_k(\Gamma_0(N), \chi)$ of cusp forms of type (k, χ). Any element f here satisfies the transformation formula

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, and is a holomorphic function on the upper half-plane as well as on all the cusps. In particular, any such f satisfies $f(z+1) = f(z)$ and thus, at $i\infty$, it has a Fourier expansion $f(z) = \sum_{n=1}^\infty a_n q^n$ where $q = e^{2i\pi z}$. One defines the L-function of f as

$$L(s, f) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}.$$

Using the theory of the so-called Hecke operators, Hecke proved that for any $f \in S_k(\Gamma_0(N))$, the L-function $L(s, f)$ extends to an entire function and satisfies a functional equation with a symmetry $s \leftrightarrow k-s$. He also proved that the L-function has an Euler product

$$L(s, f) = \prod_{p|N}\left(1 - \frac{a_p}{p^s}\right)^{-1}\prod_{p\nmid N}\left(1 - \frac{a_p}{p^s} + \frac{\chi(p)}{p^{2s+1-k}}\right)^{-1}$$

which converges for $\mathrm{Re}\,(s) > (k+2)/2$, if and only if, f is a (normalized) common eigenform for all the Hecke operators.

8.1 Ramanujan–Petersson and Selberg Conjectures

In its simplest form, this is the conjecture that the Fourier coefficients $a_n(f)$ of a normalized Hecke eigenform of weight k for $SL_2(\mathbf{Z})$ satisfies

$$|a_p(f)| \leq 2p^{\frac{k-1}{2}} \quad \text{for every prime } p.$$

Hecke's work shows that the Fourier coefficients $a_n(f)$ are just the eigenvalues for the Hecke operators T_n. This conjecture is therefore an analogue of the **RH**, and was proved by Deligne in the work on Weil conjectures alluded to earlier. The analogue of the Ramanujan–Petersson conjecture for Maass forms (that is, forms where the holomorphy assumption is dropped) is the assertion that $a_n(f) = O(n^\epsilon)$ for each $\epsilon > 0$. This is still open. Later, we will mention a much more general version of the conjecture. In a seemingly unrelated work, Selberg made a conjecture. If a Maass form $f(z)$ for $\Gamma_0(N)$ – viewed as a function of two real variables x, y – is an eigenfunction for the non-Euclidean Laplacian $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, Selberg conjectured that the corresponding non-zero eigenvalues λ ($\lambda = 0$ corresponds to a holomorphic form) satisfy $\lambda > \frac{1}{4}$. Selberg proved this for $SL_2(\mathbf{Z})$ and, in the case of general N, he proved the weaker lower bound $\frac{3}{16}$. The general conjecture is still open. The adelic formalism of Satake shows that the Ramanujan–Petersson conjecture and the Selberg conjecture are two sides of the same coin – the latter may be thought of as an archimedean analogue of the former. Both conjectures could be unified as an adelic formulation of the Ramanujan–Petersson conjecture which will be discussed below.

8.2 *Eichler-Shimura Correspondence and Taniyama-Shimura-Weil Conjecture*

If $f \in S_2(\Gamma_0(N))$, it is clear that the differential form $f(z)dz$ is invariant under $\Gamma_0(N)$. Then, for any fixed point z_0 on the upper half-plane, the integral $\int_{z_0}^{z} f(z)dz$ is independent of the path joining z_0 to z. Thus, for any $\gamma \in \Gamma_0(N)$, there is a well-defined function

$$\gamma \mapsto \Phi_f(\gamma) = \int_{z_0}^{\gamma(z_0)} f(z)dz.$$

It is also easy to see that this function does not depend on the choice of z_0.

Theorem. **(Eichler-Shimura).** *When f is a normalized new form with integer coefficients, the set $\{\Phi_f(\gamma)\}$ as γ varies, forms a lattice Λ_f in $\mathbf{C}$. There is an elliptic curve E_f defined over $\mathbf{Q}$ which becomes isomorphic to the complex torus $\mathbf{C}/\Lambda_f$ over $\mathbf{C}$. Moreover*

$$L(s, E_f) = L(s, f).$$

The converse result that every elliptic curve E over $\mathbf{Q}$ comes from a modular form of weight 2 for $\Gamma_0(N_E)$ as above was conjectured by Taniyama-Shimura-Weil and is now a famous theorem of Taylor and Wiles for square-free N and of Breuil, Conrad, Diamond & Taylor for other N.

8.3 Weil's Converse Theorem

This is the basic method used to prove many of the theorems underlying the Langlands philosophy. The latter roughly is the idea that all sorts of L-functions arising 'geometrically' are L-functions of certain modular forms. The results of Weil we are talking about is:

Weil's converse theorem. *Let $\{a_n\}$ be a sequence of complex numbers such that $a_n = O(n^c)$ for some constant $c > 0$. Fix a natural number N, an even natural number k and a sign ϵ. Assume:*

(i) *$\Lambda(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)\sum_n \frac{a_n}{n^s}$ is an entire function which is bounded in vertical strips,*

(ii) *$\Lambda(s) = \epsilon(-1)^{k/2}\Lambda(k-s)$,*

(iii) *for each $(m, N) = 1$, and every primitive character χ,*

$$\Lambda_\chi(s) = (m^2N)^{s/2}(2\pi)^{-s}\Gamma(s)\sum_n \frac{a_n\chi(n)}{n^s}$$

is entire and bounded in vertical strips,

(iv) *$\Lambda_\chi(s) = \epsilon(-1)^{k/2}\chi(-N)\frac{T(\chi)}{T(\bar{\chi})}\Lambda_{\bar{\chi}}(k-s)$, where*

$$T(\chi) = \sum_{l \pmod m} \chi(l)e^{2i\pi l/m}.$$

(v) *$\sum \frac{a_n}{n^s}$ converges absolutely at $s = k - \delta$ for some $\delta > 0$.*

Then, $f(z) = \sum a_n e^{2i\pi nz}$ is a cusp form in $S_k(\Gamma_0(N))$.

9. Artin L-Functions

Now, we introduce one of the most interesting classes of L-functions. Let L/K be a Galois extension of number fields with $\mathrm{Gal}(L/K) = G$, say. For a prime ideal P of $\mathcal{O}_K$, write

$$P\mathcal{O}_L = (P_1P_2\cdots P_g)^e$$

with P_i prime ideals. Consider the decomposition groups

$$D_{P_i} = \{\sigma \in G : \sigma(P_i) = P_i\}.$$

They are all conjugate. Also, there is a surjective natural homomorphism to the Galois group of the residue field extension

$$D_{P_i} \to \mathrm{Gal}\left(\frac{\mathcal{O}_L/P_i}{\mathcal{O}_K/P}\right)$$

whose kernel is the inertia group I_{P_i}. The inertia groups are trivial if P is unramified in L (that is, if $e = 1$) – something which happens for all but finitely many prime ideals P. As the Galois group of an extension of finite fields is cyclic with a distinguished generator, the Frobenius automorphism, there is a conjugacy class σ_P in G corresponding to any unramified prime ideal P. This is also called the Frobenius conjugacy class or the Artin symbol of P.

Whenever one has a finite-dimensional complex representation of G, say, $\rho : G \to GL(V)$, Artin attached an L-function defined by

$$L(s, \rho; L/K) = \prod_P \det(1 - \rho(\sigma_P)N(P)^{-s}|V^{I_P})^{-1}$$

where V^{I_P}, the subspace fixed by I_P is acted on by the conjugacy class σ_P. Artin showed that these L-functions have nice properties like invariance under the induction of representations. He also posed:

Artin's Conjecture. *$L(s, \rho; L/K)$ extends to an entire function when the character of ρ does not contain the trivial character.*

Thus, essentially the pole of a Dedekind zeta function ought to come from that of the Riemann zeta function. Artin's conjecture is still open although it has been proved in a few cases. A consequence of Artin's reciprocity law is the statement (weaker than Artin's conjecture) that these L-functions extend to meromorphic functions for any s.

10. Automorphic L-Functions and Langlands Program [4]

The whole point of view ever since Artin defined his L-functions shifted to viewing everything in the powerful language of representation theory. Classical modular form theory for subgroups of $SL_2(\mathbf{Z})$ can be viewed in terms of representations of $SL_2(\mathbf{R})$. More generally, representations of adele groups (to be defined below) surfaced as the principal objects of study. We have already alluded to the fact that Ramanujan–Petersson conjecture and Selberg conjecture could be unified in the adelic framework. In fact, we also mentioned in passing that Tate's thesis afforded the first understanding of why the Gamma factors appeared in the functional equation for the Riemann zeta function. Let us describe the adelic setting briefly now.

10.1 Basics on Adeles

A basic tenet is that to do number theory (to 'know' any algebraic number field K) is to look at all possible notions of distance on K. For example, the usual notion of distance on $\mathbf{Q}$ as a subset of $\mathbf{R}$ keeps the number theory of $\mathbf{Q}$ hidden. If p is a prime number, there is a 'p-adic distance' defined as

$$|x-y|_p = p^{-\mathrm{ord}_p(x-y)},$$

where $\mathrm{ord}_p(p^t a/b) = t$ for any non-zero rational number $p^t a/b$ where a, b are indivisible by p. One takes $\mathrm{Ord}_p(0) = \infty$ and $|0|_p = 0$. So, a number which is divisible by a high power of p is close to zero in this distance! Our usual intuition based on the geometry of Euclidean spaces takes a beating here – for instance, every triangle is isosceles, every point inside a disc is its center etc.! This is because the p-adic distance has the property that it is nonarchimedean, that is,

$$|x-y|_p \leq \max(|x-z|_p, |z-y|_p \quad \text{for any } x, y, z;$$

in particular, $|nx| = |x|$ for all natural numbers n. So, given x, y with $|x|_p < |y|_p$, one cannot choose n such that $|nx|_p$ is bigger than $|y|_p$. Ostrowski showed that the possible distinct notions of distance on $\mathbf{Q}$ are the usual archimedean one coming from $\mathbf{R}$ and the p-adic ones for primes p. Also, just as $\mathbf{R}$ is constructed from $\mathbf{Q}$ by a process of completion with respect to the usual distance, there are p-adic completions of $\mathbf{Q}$ to which the notion of p-adic distance extends. These are fields called the p-adic numbers $\mathbf{Q}_p$. They are locally compact like $\mathbf{R}$ is. But, they are different (much nicer!) from $\mathbf{R}$ in many ways. The nonarchimedean-ness shows, for example, that a series in $\mathbf{Q}_p$ converges if and only if the terms converge to 0. Unlike $\mathbf{R}$, there is a subring of $\mathbf{Q}_p$ is called the p-adic integers which form a compact subset. Akin to viewing real numbers as decimals, one may think of $\mathbf{Q}_p$ as consisting formally of series of the form $\sum_{n=-r}^{\infty} a_n p^n$ where r is an integer and the 'digits' a_n's are between 0 and $p-1$. While adding and multiplying such numbers, one adds them and multiplies as if they were formal series but one has to rewrite the expressions so that the resulting expression has digits between 0 and $p-1$. The p-adic integers $\mathbf{Z}_p$ can be thought of as those series which have no terms of negative degree. Note that $\mathbf{Q}_p = \bigcup_{n\geq 1} p^n \mathbf{Z}_p$. The p-adic integers is also the closure of $\mathbf{Z}$ in $\mathbf{Q}_p$ for the p-adic completion.

More generally, for any number field K (with ring of integers $\mathcal{O}$), one has the P-adic completion for each prime ideal of $\mathcal{O}$. The point is that every ideal is uniquely a product of prime ideals even if a similar unique decomposition does not hold good for elements of $\mathcal{O}$. These P-adic topologies are nonarchimedean and are all mutually inequivalent. There are also $[K : \mathbf{Q}]$ archimedean distance functions on K extending the usual one on $\mathbf{Q}$ although some of them may be equivalent. The inequivalent ones are called places of K. It is in this regard that Tate's thesis tells us that the Gamma factors in the functional equation for $\zeta_K(s)$ are 'Euler factors' corresponding to the archimedean places of K. If v is a place of K, the completion K_v is a locally compact field – it is $\mathbf{R}$ or $\mathbf{C}$ when v is archimedean and a finite extension field of $\mathbf{Q}_p$ when v corresponds to a prime ideal P and $P \cap \mathbf{Z} = p\mathbf{Z}$. The closure of $\mathcal{O}$ in K_v is a compact subring $\mathcal{O}_{\sqsubseteq}$ when v is nonarchimedean. The best way to study K is to introduce the adele ring $\mathbf{A}_K$ of K which is a certain locally compact ring. The adele ring of K is defined as the set of all tuples $(x_v)_v$ with $x_v \in K_v$ where all but finitely many of the x_v are in $\mathcal{O}_v$. Note that for any $x \in K$, the 'diagonal embedding' $(x, x, \dots)$ is in $\mathbf{A}_K$. To define the topology on adeles, consider any finite set S of places of K containing all the archimedean ones. The product ring $\prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v$ is locally compact as S is finite. As S varies, these products form a basis of neighbourhoods of zero for a unique topology on $\mathbf{A}_K$ for which it is locally compact. The addition and multiplication on $\mathbf{A}_K$ are continuous functions for the adelic topology. The diagonal embedding $x \mapsto (x, x, \dots,)$ maps K as a discrete subgroup of $\mathbf{A}_K$.

A natural way to arrive at the adeles is via harmonic analysis. For example, if the abelian group $\mathbf{Q}$ *is regarded with* *discrete topology, then the compact abelian group which is dual to it (its group of continuous characters) can be computed from first principles. It turns out to be the quotient group* $\mathbf{A_Q}/\mathbf{Q}$. *Here* $\mathbf{Q}$ *is viewed via its diagonal embedding. The situation is a generalization of the duality between* $\mathbf{Z}$ *and* $\mathbf{R}/\mathbf{Z}$.

More generally, when we have a matrix group like $GL_n(K)$ (or more generally an algebraic subgroup $G \subset GL_n$ defined over K), one can naturally consider the groups $G(K_v)$ and $G(\mathcal{O}_v)$ for all places v of K. The groups $GL_n(K_v)$ are locally compact and although $GL_n(\mathcal{O}_v)$ is not compact, it is compact modulo the scalar matrices. In particular, one has the 'adelic group' $G(\mathbf{A}_K)$ which has a basis of neighbourhoods of the identity given by $\prod_{v\in S} G(K_v) \times \prod_{v\notin S} G(\mathcal{O}_v)$ as S varies over finite sets of places containing all the archimedean places of K. The diagonal embedding of $G(K)$ in $G(\mathbf{A}_K)$ embeds it as a discrete subgroup. Unlike $\mathbf{A}_K/K$ which is compact, the quotient space $GL_n(\mathbf{A}_K)/GL_n(K)$ (not a group) is not compact; it does not even have finite 'measure' for a Haar measure of the adele group. However, the finiteness of measure holds modulo the group $Z = \{\mathrm{diag}(t, t, \ldots, t) \in GL_n(\mathbf{A}_K)\}$ of scalar matrices in $GL_n(\mathbf{A}_K)$.

Therefore, for a Grossencharacter ω (a character of the group $GL_1(\mathbf{A}_K)/GL_1(K)$), it makes sense to consider the following Hilbert space consisting of measurable functions on the quotient space $GL_n(\mathbf{A}_K)/GL_n(K)$ with certain properties which remind us of transformation properties of modular forms. This is the Hilbert space $L^2(GL_n(\mathbf{A}_K)/GL_n(K), \omega)$ of those measurable functions ϕ which satisfy:

(i) $\phi(zg) = \omega(z)\phi(g)$, $z \in Z$,
(ii) $\int_{GL_n(\mathbf{A}_K)/Z\cdot GL_n(K)} |\phi(g)|^2 dg < \infty$.

The subspace $L_0^2(GL_n(\mathbf{A}_K)/GL_n(K), \omega)$ of cusp forms is defined by the additional conditions corresponding to any parabolic subgroup. The latter are conjugates in GL_n of 'ladder' groups of the form

$$P_{n_1,\ldots,n_r} = \left\{ \begin{pmatrix} g_1 & \cdots & \cdots & \\ 0 & g_2 & \cdots & \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & g_r \end{pmatrix} \right\}$$

where g_i is an $n_i \times n_i$ invertible matrix. The standard parabolic $P_{n_1,\ldots,n_r}$ is a semidirect product of its unipotent radical

$$U = \left\{ \begin{pmatrix} I_{n_1} & \cdots & \cdots & \\ 0 & I_{n_2} & \cdots & \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{n_r} \end{pmatrix} \right\}$$

and $GL_{n_1} \times \cdots \times GL_{n_r}$. Any parabolic subgroup P has a similar semidirect product decomposition $P = M \propto U$. The parabolic subgroups are characterized by the condition that they are closed subgroups such that $GL_n(\mathbf{C})/P(\mathbf{C})$ is compact. The additional 'cuspidality' condition for a parabolic subgroup P is

$$\int_{U_P(\mathbf{A}_K)/U_P(K)} \phi(ug)du = 0 \ \forall \ g \in GL_n(\mathbf{A}_K).$$

The adele group acts as unitary operators by right multiplication on the Hilbert space $L^2(GL_n(\mathbf{A}_K)/GL_n(K), \omega)$. It leaves the space of cusp forms invariant. By definition, a subquotient of this representation is called *an automorphic representation* of $GL_n(\mathbf{A}_K)$. Moreover, a subrepresentation of the representation on cusp forms is said to be a *cuspidal automorphic representation*. One further notion is that of an *admissible* representation of the adele group – this is one which can contain any irreducible representation of a maximal compact subgroup of the adele group only finitely many times. It is a theorem of D. Flath which tells us that any irreducible, admissible representation of the adele group is a 'restricted' tensor product of unique irreducible representations of $GL_n(K_v)$. Further, for an admissible automorphic representation $\pi = \otimes_v \pi_v$, the representation π_v belongs to a special class known as the unramified principal series for all but finitely many v. An unramified principal series representation π_v is one whose restriction to $GL_n(\mathcal{O}_v)$ contains the trivial representation; a certain isomorphism theorem due to Satake shows that corresponding to π_v, there is a conjugacy class in $GL_n(\mathbf{C})$ of a diagonal matrix of the form

$$A_v = \mathrm{diag}(N(v)^{-z_1}, \ldots, N(v)^{-z_n})$$

for some n-tuple $(z_1, \ldots, z_n) \in \mathbf{C}^n$.

Corresponding to an admissible, automorphic representation $\pi = \otimes_v \pi_v$, Langlands defined an L-function. If S is the finite set of places outside of which π_v is unramified principal series, define for $v \notin S$,

$$L(s, \pi_v) = \det(1 - A_v N(v)^{-s})^{-1}.$$

If $L_S(s,\pi) := \prod_{v\notin S} L(s,\pi_v)$, then Langlands proved that this product has a meromorphic extension to the whole complex plane. Defining $L(s,\pi_v)$ for $v \in S$ in a suitable manner, it also follows that $L(s,\pi) = \prod_v L(s,\pi_v)$ has meromorphic continuation, and a functional equation. If π is cuspidal also, then Godement & Jacquet showed that $L(s,\pi)$ is an entire function unless $n = 1$ and $\pi = |.|^t$ for some $t \in \mathbf{C}$.

Ramanujan–Petersson conjecture. *If π is cuspidal automorphic, then the eigenvalues of A_v have absolute value 1 for all v. Equivalently, for such a π, the matrix coefficients of π_v, for each prime p, belongs to $L^{2+\epsilon}(GL_n(\mathbf{Q}_p)/Z(\mathbf{Q}_p))$ for any $\epsilon > 0$.*

Note that Selberg conjecture can be interpreted as asserting that π_∞ is a tempered representation of $GL_n(\mathbf{R})$.

Langlands Reciprocity conjecture. *Let L/K be a Galois extension of number fields and let G be the Galois group. Let (ρ, V) be an n-dimensional complex representation of G. Then, there is a cuspidal automorphic representation π of $GL_n(\mathbf{A}_K)$ such that $L(s,\rho;L/K) = L(s,\pi)$.*

This is just Artin's reciprocity law (a theorem!) when ρ is 1-dimensional. There are also other conjectures of Langlands which imply that the automorphic L-functions multiplicatively generate all the L-functions like the Dedekind zeta functions, Hasse-Weil zeta functions etc. One has:

Grand Riemann Hypothesis. *All the zeroes of $L(s,\pi)$ for a cuspidal automorphic representation π, lie on $\mathrm{Re}\,(s) = 1/2$.*

The Grand Riemann Hypothesis has several concrete number-theoretic consequences. For instance, it implies the Artin primitive root conjecture which asserts that any non-square $a \neq -1$ is a primitive root for infinitely many primes.

11. Selberg's Program [9]

For what general L-functions can the **RH** be formulated meaningfully? The final section discusses this and it is a program started by Selberg. One defines the *Selberg class* $\mathcal{S}$ consisting of those complex functions $F(s)$ which satisfy the following hypotheses:

(i) $F(s) = 1 + \sum_{n\geq 2} \frac{a_n}{n^s}$ for $\mathrm{Re}\,(s) > 1$.

(ii) $F(s)$ has a meromorphic continuation to the whole complex plane and there is some m so that $(s-1)^m F(s)$ is holomorphic of finite order.

(iii) There are positive real Q, α and complex r_i with $\mathrm{Re}\,(r_i) > 0$ and a complex number w of absolute value 1 such that the function

$$\Phi(s) := Q^s F(s) \prod_{i=1}^{d} \Gamma(\alpha_i s + r_i)$$

satisfies the functional equation

$$\Phi(s) = w\overline{\Phi(1-\bar{s})}.$$

(iv) $F(s) = \prod_p exp(\sum_{k=1}^{\infty} b_{p^k} p^{-ks})$ with $b_{p^k} = O(p^{k\theta})$ for some $\theta < 1/2$.

(v) (**Ramanujan/Riemann Hypothesis**) For any $\epsilon > 0$, one has $a_n = O(n^\epsilon)$.

It is an expectation that the class of functions satisfying the first 4 axioms automatically satisfy the fifth. If this turns out to be true, then we would have a characterization of all Dirichlet series for which the Riemann Hypothesis holds good.

All the familiar L-functions studied so far are in the Selberg class or are conjectured to be so. Any function in the Selberg class can be factorized into 'primitive' functions – this is a theorem due to Selberg, Conrey and Ghosh. Selberg predicted a certain type of orthonormal system in $\mathcal{S}$; this has consequences like the uniqueness of the factorization into primitives!

Selberg's Conjecture. *For any primitive function $F \in \mathcal{S}$, one has*

$$\sum_{p\leq x} \frac{|a_p(F)|^2}{p} = \log\log x + O(1).$$

For primitive functions $F \neq G \in \mathcal{S}$, one has

$$\sum_{p\leq x} \frac{a_p(F)\overline{a_p(G)}}{p} = O(1).$$

Artin's conjecture on the entirety of the Artin L-functions is a consequence of the Selberg conjectures. There are quite a few results in operator theory and noncommutative geometry related to the theme of Riemann Hypothesis that we have not touched upon but that is inevitable as must be with a fundamental theme as this. In conclusion, one might say that the Riemann Hypothesis really is not an isolated problem whose solution is an end in itself but a beacon which shines its light on all of mathematics, generating new and beautiful byproducts.

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The Algebra Generated by Three Commuting Matrices

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Abstract. We present a survey of an open problem concerning the dimension of the algebra generated by three commuting matrices.

This article concerns a problem in algebra that is completely elementary to state, yet, has proven tantalizingly difficult and is as yet unsolved. Consider $\mathbb{C}[A, B, C]$, the $\mathbb{C}$-subalgebra of the $n \times n$ matrices $M_n(\mathbb{C})$ generated by three commuting matrices A, B, and C. Thus, $\mathbb{C}[A, B, C]$ consists of all $\mathbb{C}$-linear combinations of "monomials" $A^i B^j C^k$, where i, j, and k range from 0 to infinity. Note that $\mathbb{C}[A, B, C]$ and $M_n(\mathbb{C})$ are naturally vector-spaces over $\mathbb{C}$; moreover, $\mathbb{C}[A, B, C]$ is a subspace of $M_n(\mathbb{C})$. The problem, quite simply, is this: Is the dimension of $\mathbb{C}[A, B, C]$ as a $\mathbb{C}$ vector space bounded above by n?

Note that the dimension of $\mathbb{C}[A, B, C]$ is at most n^2, because the dimension of $M_n(\mathbb{C})$ is n^2. Asking for the dimension of $\mathbb{C}[A, B, C]$ to be bounded above by n when A, B, and C commute is to put considerable restrictions on $\mathbb{C}[A, B, C]$: this is to require that $\mathbb{C}[A, B, C]$ occupy only a small portion of the ambient $M_n(\mathbb{C})$ space in which it sits.

Actually, the dimension of $\mathbb{C}[A, B, C]$ is already bounded above by something slightly smaller than n^2, thanks to a classical theorem of Schur ([16]), who showed that the maximum possible dimension of a commutative $\mathbb{C}$-subalgebra of $M_n(\mathbb{C})$ is $1 + \lfloor n^2/4 \rfloor$. But n is small relative even to this number.

To understand the interest in n being an upper bound for the dimension of $\mathbb{C}[A, B, C]$, let us look more generally at the dimension of the $\mathbb{C}$-subalgebra of $M_n(\mathbb{C})$ generated by k-commuting matrices. Let us start with the $k = 1$ case: note that "one commuting matrix" is just an arbitrary matrix A. Recall that the Cayley-Hamilton theorem tells us that A^n is a linear combination of $I, A, \ldots, A^{n-1}$, where I stands for the identity matrix. From this, it follows by repeated reduction that A^{n+1}, A^{n+2}, etc. are all linear combinations of $I, A, \ldots, A^{n-1}$ Thus, $\mathbb{C}[A]$, the $\mathbb{C}$-subalgebra of $M_n(\mathbb{C})$ generated by A, is of dimension at most n, and this is just a simple consequence of Cayley-Hamilton theorem.

The case $k = 2$ is therefore the first significant case. It was treated by Gerstenhaber ([4]) as well as Motzkin and Taussky-Todd ([13], who proved independently that the variety of commuting pairs of matrices is irreducible. It follows from this that if A and B are two commuting matrices, then too, $\mathbb{C}[A, B]$ has dimension bounded above by n. (We will study their sequence of ideas in some depth later in this article.)

Thus, for both $k = 1$ and $k = 2$, our algebra dimension is bounded above by n. Hopes of the dimension of the algebra generated by k commuting matrices being bounded by n for much wider ranges of k were dashed by Gerstenhaber himself: He cited an example of a subalgebra of $M_n(\mathbb{C})$, for $n \geq 4$, generated by $k \geq n$ commuting matrices whose dimension is greater than n. His example easily extends, for each $n \geq 4$ and $k \geq 4$, to a subalgebra of $M_n(\mathbb{C})$ generated by k commuting matrices whose dimension is greater than n, and we give this example here: Write $E_{i,j}$ for the matrix that has zeroes everywhere except for a 1 in the (i, j) slot. (These matrices form a $\mathbb{C}$-basis of $M_n(\mathbb{C})$.) Assume $n \geq 4$, and take $A = E_{1,3}$, $B = E_{1,4}$, $C = E_{2,3}$, and $D = E_{2,4}$. Then A, B, C, and D are linearly independent, and the product of any two of them is zero. In particular, they commute pairwise, and the linear subspace spanned by A, B, C, and D is closed under multiplication. Adding the identity matrix to the mix to get a "1" in our algebra, we find that $\mathbb{C}[A, B, C, D]$ is the $\mathbb{C}$ subspace of $M_n(\mathbb{C})$ with basis I, A, B, C, and D—a five-dimensional algebra. Thus, when $n = 4$, we already have our counterexample for the $k = 4$ case. For larger values of n, this example can be modified by taking A to have 1 in the slot $(1, 3)$ along with nonzero elements $a_5, \ldots, a_n$ in the diagonal slots $(5, 5), \ldots, (n, n)$, chosen so that $a_5^2, \ldots, a_n^2$ are pairwise distinct. The matrices A, B, C, and D will still commute, and it is a short calculation (a Vandermonde matrix will appear!) that $\mathbb{C}[A, B, C, D]$ will have basis I, A, B, C, D along with $A^2, \ldots, A^{n-3}$—an $(n + 1)$-dimensional algebra.

Further, taking $E = F = \cdots = D$ in the example above, we find trivially that for any $k \geq 4$, there exists k commuting matrices which generate a $\mathbb{C}$-algebra of dimension greater than n.

This, then, is the source of our open problem: Yes, the algebra dimension is bounded by n for $k = 1$, and $k = 2$. No, the algebra dimension is not bounded by n for $k \geq 4$. So, what happens for $k = 3$?

Note that without the requirement that A, B, and C commute pairwise, this question would have an immediate answer: already, with $k = 2$, there are easy examples of matrices A and B (that do not commute) for which the algebra they generate is the whole algebra $M_n(\mathbb{C})$, so in particular, of dimension n^2. For instance, take A to be a diagonal matrix with entries that are pairwise distinct, and let B be the permutation matrix corresponding to the cyclic permutation $(1\ 2 \cdots n)$, i.e., the matrix with 1 in the slots $(i, i - 1)$ for $i = 2, \ldots, n - 1$ and in the slot $(1, n)$, and zeros everywhere else. One checks (Vandermonde again!) that the matrices $A^i B^j$, for $i, j = 0, \ldots, n - 1$ are linearly independent, thus giving an algebra of full dimension n^2.

It is worth noting that matrices of the form $E_{1,3}$, $E_{1,4}$, $E_{2,3}$, $E_{2,4}$ of $M_4(\mathbb{C})$ that arise in the example above quoted by Gerstenhaber play a significant role in the context of commutative subalgebras of $M_n(\mathbb{C})$. More generally, we may partition our $n \times n$ matrix into four blocks of equal (or nearly equal) sizes and consider the "north-east" block: If $n = 2m$, our north-east block will consist of slots form the first m rows and last m columns. If $n = 2m + 1$, our north-east block will consist of slots from the first m rows and last $m + 1$ columns, or else, from the first $m + 1$ rows and last m columns (we may pick either one). If we consider the $\lfloor n^2/4 \rfloor$ matrices $E_{i,j}$ corresponding to the various slots (i, j) in this block, then it is clear that they are linearly independent and the product of any two of these matrices is zero. These matrices hence commute, and the linear subspace of $M_n(\mathbb{C})$ spanned by them is closed under multiplication. Adding constant multiples of the identity to this space so as to have a "1," we therefore get a commutative subalgebra of $M_n(\mathbb{C})$ of the maximum dimension $1 + \lfloor n^2/4 \rfloor$ possible by Schur's theorem. Schur had also shown that any commutative subalgebra of dimension $1 + \lfloor n^2/4 \rfloor$ must be similar to the algebra generated as above by the matrices $E_{i,j}$ coming from the north-east block.

(In basis-free terms, this corresponds to taking a decomposition of $V \cong \mathbb{C}^n = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of dimensions as equal as possible, and considering all $\{f \in \mathrm{End}_{\mathbb{C}}(V) | f(V_2) \subseteq V_1, f(V_1) = 0\}$, along with the endomorphisms representing multiplication by constants.)

Jacobson ([10]) later gave an alternative proof of Schur's theorem on the maximum dimension of a commutative subalgebra that is valid for any field F, and showed that if

F is not imperfect of characteristic two, then too, any commutative subalgebra F-subalgebra of $M_n(F)$ of the maximum dimension $1 + \lfloor n^2/4 \rfloor$ is conjugate to the algebra generated as above by the matrices $E_{i,j}$ coming from the north-east block.

It is worth remarking in this context that Schur's result was further generalized to the case of artinian rings by Cowsik ([2]): he showed that if A is an artinian ring with a faithful module of length n, then A has length at most $1 + \lfloor n^2/4 \rfloor$. Cowsik was answering a question raised by Gustafson, who had given ([7]) a representation-theoretic proof of Schur's theorem; Gustafson had also proven a related interesting fact: the dimension of a *maximal* commutative subalgebra of $M_n(\mathbb{C})$ is at least $n^{2/3}$.

Other proofs of Schur's theorem have also been given. See [1,11,14], or [19], for instance.

An open problem can be interesting (and significant) because it represents a critical gap in a larger conceptual framework that must be filled before the framework can stand: the missing link in a big theory. Alternatively, an open problem could be interesting because its solution has the potential to involve techniques from other areas and to shed light on and raise new questions in other areas. The problem on the bound of the dimension of $\mathbb{C}[A, B, C]$ falls into the second category. Quite specifically, the most significant attacks on this problem have involved the analysis of the algebraic variety of commuting triples of matrices, and interestingly, have spun off investigations into jet schemes of determinantal varieties and of commuting pairs of matrices.

To get a feel for the connection between our open question and matrix varieties (i.e., the solution set in some large dimensional affine space to polynomial equations defined by matrices–we will see examples below), let us consider the proofs of Taussky-Todd and Motzkin, and of Gerstenhaber that the algebra $\mathbb{C}[A, B]$ generated by two commuting $n \times n$ matrices A and B is of dimension at most n. View pairs of matrices (A, B) as points of affine $2n^2$-dimensional space $\mathbb{C}^{2n^2}$ by viewing the set of entries of A and of B strung together in some fixed order as coordinates of the corresponding point. The set of *commuting* pairs (A, B) correspond to solutions of the n^2 equations arising from the entries of $XY - YX = 0$, where X and Y are generic matrices with entries $x_{i,j}$ and $y_{i,j}$. These equations are polynomial equations in the $x_{i,j}$ and $y_{i,j}$ (in fact, they are bilinear in the $x_{i,j}$ and $y_{i,j}$). Thus, the set of commuting pairs of $n \times n$ matrices (A, B) naturally has the structure of an algebraic variety, which we will denote $C(2, n)$.

Both Taussky-Todd and Motzkin, and Gerstenhaber actually proved that $C(2, n)$ is irreducible. Let us see how their analysis of $C(2, n)$ leads to our desired bound on the dimension of $\mathbb{C}[A, B]$. The proof based on $C(2, n)$ that $\mathbb{C}[A, B]$ has dimension at most n proceeds along the steps below. Both sets of authors use essentially the same set of ideas, with the slight difference that Taussky-Todd and Motzkin use matrices with distinct eigenvalues instead of "1-regular" matrices in steps (1) and (2):

(1) Show first that $\mathbb{C}[A, B]$ has dimension exactly n if each eigenvalue of A appears in exactly one Jordan block. (Recall from elementary matrix theory that A has this property precisely when the minimal polynomial of A coincides with the characteristic polynomial of A, i.e., if the algebra $\mathbb{C}[A]$ has dimension exactly n. Such a matrix A is said to be 1-regular.)
(2) Show that the set U of points (A, B) where A is 1-regular is a dense subset (in a suitable topology) of $C(2, n)$. (This is the step that shows the irreducibility of $C(2, n)$; we will consider irreducibility later.)
(3) Show that if $\mathbb{C}[A, B]$ has dimension exactly n, and therefore at most n, for all points (A, B) in a dense subset of $C(2, n)$, then $\mathbb{C}[A, B]$ must have dimension at most n on all of $C(2, n)$.

The topology used is the well known Zariski topology on $\mathbb{C}^{2n^2}$, where a set is closed iff it is the solution set of a system of polynomial equations (in $2n^2$ variables). An open set in this topology is thus the union of sets $D(f)$, where f is a polynomial, and $D(f)$ consists of all points where f is nonzero. To say that the set U in (2) above is dense in $C(2, n)$ in the Zariski topology is therefore to say that if a polynomial vanishes identically on U, then it must vanish identically on $C(2, n)$.

We will describe steps (1), (2), and (3) below and indicate the difficulties in extending these steps to the corresponding variety of commuting triples of matrices.

Step 1. The form of a typical matrix in the centralizer of a given matrix A (when A is described in Jordan form) is well-known and very concrete (we will not reproduce it here, but see [3] for instance), and it follows from this description that if A is 1-regular, then any matrix B that commutes with A must be a polynomial in A. Described differently, B is already in the algebra $\mathbb{C}[A]$, that is $\mathbb{C}[A, B] = \mathbb{C}[A]$. But $\mathbb{C}[A]$ is of dimension n as A is 1-regular, so $\mathbb{C}[A, B]$ is of dimension n.

Step 2. This is the key step. Once again, one refers to the known form of matrices centralizing a given matrix to observe that given any matrix B, one can find a 1-*regular* matrix A' that commutes with B. (Determining such an A' is actually very easy, although we will not give a recipe for doing this here). So, given an arbitrary point (A, B) in $C(2, n)$, i.e., a commuting pair of matrices (A, B), consider the line L in $\mathbb{C}^{2n^2}$ described by $((1 - \lambda)A' + \lambda A, B)$, where λ varies through $\mathbb{C}$ and A' is some 1-regular matrix that commutes with B. Since B and A' commute, the matrices B and $(1 - \lambda)A' + \lambda A$ also commute for any λ, i.e, the entire line L lies in $C(2, n)$.

Now consider what it means for a matrix A to be 1-regular. It means that $\mathbb{C}[A]$ must be of dimension n, that is, the matrices $1, A, \dots, A^{n-1}$ must be linearly independent. In particular, writing each of the matrices $1, A, A^2, \dots$ as an $n^2 \times 1$ (column) vector and assembling all n vectors together, we get an $n^2 \times n$ matrix $M(A)$, and to say that $1, A, \dots, A^{n-1}$ should be linearly independent is to say that $M(A)$ must have rank n. Thus, $M(A)$ should have the property that at least one of its $n \times n$ minors should be nonzero. Since these minors are polynomials in the entries of $M(A)$, which in turn are polynomials in the entries of A, this translates into an open set condition in the Zariski topology: A, viewed as a point in $\mathbb{C}^{n^2}$, must live in the union of the various open sets in which some $n \times n$ minor of $M(A)$ is nonzero.

Let us apply these ideas to the first coordinates $(1 - \lambda)A' + \lambda A$ of the line L above. Let us consider those values of λ for which $(1 - \lambda)A' + \lambda A$ is 1-regular. This certainly happens when $\lambda = 0$, by our choice of A'. But more is true: taking $(1-\lambda)A' + \lambda A$ for A in the paragraph above, the various $n \times n$ minors of $M((1 - \lambda)A' + \lambda A)$ are now polynomials in λ. At least one of these polynomials is nonzero, since $\lambda = 0$ is not a solution to at least one of them. But a nonzero polynomial in one variable has only finitely many roots, and hence, all but finitely many λ are nonroots of this polynomial. Put differently, for all but finitely many λ, our matrix $(1 - \lambda)A' + \lambda A$ must be 1-regular. Thus, almost all points of L are in U.

Finally, we will show that any point (A, B) in $C(2, n)$ is in the closure of U. Let A' and L be as in the arguments above. Then almost all points of L are in U as we have seen. Let f be any polynomial (in $2n^2$ variables) that is zero on U. Substituting the general point of L into f, we get a new polynomial g in the single variable λ. Since all but finitely many points of L are in U, we find that g is zero for almost all values of λ. Invoking the fact that a nonzero polynomial in a single variable has only finitely many zeroes, we find $g(\lambda)$ is identically zero. Put differently, f must be zero on the entire line L, and in particular, on the point (A, B) corresponding to $\lambda = 1$. Since (A, B) was arbitrary in $C(2, n)$, we find that any polynomial (in $2n^2$ variables) that vanishes on U must vanish on $C(2, n)$, that is, U is indeed dense in the Zariski topology in $C(2, n)$. (In particular, the closure of U in $C(2, n)$ is all of $C(2, n)$.)

Step 3. We only need to show that the condition that $\mathbb{C}[A, B]$ have dimension at most n is equivalent to a set of polynomial conditions on the corresponding point (A, B) of the variety $C(2, n)$. Then, if these conditions are satisfied on any dense subset S of $C(2, n)$, they must be satisfied on the (Zariski) closure of S, i.e., on all of $C(2, n)$. (In particular, since these polynomial conditions are satisfied on our open set U (by (1)), and since the closure of U in $C(2, n)$ is all of $C(2, n)$ (by (2)), they will be satisfied on all of $C(2, n)$. Thus, the dimension of $\mathbb{C}[A, B]$ will indeed by bounded by n for all commuting pairs (A, B).) To see how the upper bound on the dimension translates to a set of polynomial conditions, we repeat the ideas in step (2) above. Observe that $\mathbb{C}[A, B]$ is spanned by the n^2 products $A^i B^j$ for $0 \le i, j \le n - 1$ (note that A and B commute, and by Cayley-Hamilton, powers A^j and B^j for $i \ge n$ can be written as linear combinations of the powers A^i and B^i respectively, for $0 \le i \le n - 1$–a fact we have already considered above). As in the proof of (2) above, collect each $A^i B^j$ as an $n^2 \times 1$ (column) vector, and assemble all n^2 such vectors into an $n^2 \times n^2$ matrix $M(A, B)$. Then, the condition that $\mathbb{C}[A, B]$ has dimension at most n translates to $M(A, B)$ having rank at most n, which is now equivalent to all $(n+1) \times (n+1)$ minors of $M(A, B)$ *vanishing*. The vanishing of each of these minors is of course a polynomial condition on the entries of A and B. This concludes step (3).

Since the dimension problem for three commuting matrices is still open, these arguments must somehow fail, or at least not extend in any obvious manner, when we consider the corresponding algebraic variety $C(3, n)$ of commuting triples of matrices. What fails? Steps (1) and (3) go through easily: if A is 1-regular and if B and C commute with A, then both B and C are in $\mathbb{C}[A]$, and $\mathbb{C}[A, B, C]$ is hence of dimension at most n; similarly, $\mathbb{C}[A, B, C]$ is spanned by the matrices $A^i B^j C^k$ for $0 \leq i, j, k \leq n-1$, and collecting each of $A^i B^j C^k$ into an $n^2 \times 1$ vector and assembling all n^3 such into an $n^2 \times n^3$ matrix $M(A, B, C)$, it is clear that the condition that $\mathbb{C}[A, B, C]$ be of dimension at most n translates into the condition that the $(n+1) \times (n+1)$ minors of $M(A, B, C)$ vanish, which is a set of polynomial conditions on the entries of A, B, and C. It turns out, however, that step (2) actually *fails* when we consider three commuting matrices! The corresponding set U consisting of triples (A, B, C) where A is 1-regular is no longer dense in $C(3, n)$, at least, for most values of n. This makes the problem hard and interesting!

Here, precisely, is what is known. Let us bring in irreducibility: recall that an algebraic variety X is said to be irreducible if it cannot be written as $X_1 \cup X_2$ where X_1 and X_2 are themselves algebraic varieties, i.e., solution sets of systems of polynomial equations. If X is not irreducible, we say X is reducible. (Every algebraic variety is a finite union of irreducible varieties, so we may think of irreducible varieties as analagous to prime numbers in the sense of their being building blocks.) Writing $C(k, n)$ for the variety of commuting k-tuples of $n \times n$ matrices for general k, and writing $U(k)$ for the corresponding subset of k-tuples where the first matrix is 1-regular, it turns out that $U(k)$ being dense in $C(k, n)$ is equivalent to $C(k, n)$ being irreducible. (We will not show this equivalence here; as we have already noted, step (2) above effectively proves that $C(2, n)$ is irreducible.) Guralnick ([5]) showed using a very pretty argument that $C(3, n)$ is *reducible* for $n \geq 32$. (Holbrook and Omladič ([9]) later observed that Guralnick's proof really shows that $C(3, n)$ is reducible for $n \geq 29$.) On the other hand, due to the work of several authors ([5,6,9,8], and most recently, Šivic in [18]), it is known that $C(3, n)$ is *irreducible* for all n upto 10. (Thus, the algebra generated by three commuting $n \times n$ matrix for $n \leq 10$ is indeed bounded by n.)

The irreducibility of $C(3, n)$ is thus itself an open problem for $11 \leq n \leq 28$. It would be very useful if the components (the irreducible constituents) of $C(3, n)$ for $n \geq 29$ can be concretely described, for then, one could potentially analyze the dimension of $\mathbb{C}[A, B, C]$ on each component. But such a description seems hopelessly difficult at this point, because the variety $C(3, n)$ has not yielded much structure that might facilitate a concrete listing of its components.

Working in a different direction, Neubauer and this author ([14]) showed that the variety of commuting pairs in the centralizer of a 2-regular matrix is irreducible. (A matrix is r-regular if each eigenvalue appears in at most r blocks.) This variety shows up naturally as a subvariety of $C(3, n)$: it is the variety of all commuting triples (A, B, C) where one of the matrices, say C, has been *fixed* to be a specific 2-regular matrix. The irreducibility of this subvariety then shows (using essentially the same arguments described above behind the proof that the algebra generated by two commuting matrices is at most n-dimensional) that the dimension $\mathbb{C}[A, B, C]$ is indeed bounded by n if one of A, B, or C is 2-regular (and more generally, if any two of A, B, C commute with a 2-regular matrix). It turned out that this particular variety is related to the variety of jets over certain determinantal varieties (determinantal varieties are varieties defined by the vanishing of minors of a certain size of a generic $n \times n$ matrix, and jets over such varieties are like algebraic tangent bundles over such varieties). This was very pleasing, and led this author to a broader study of such jet varieties ([12]). Meanwhile, Šivic ([18]) showed that the variety of commuting pairs in the centralizer of a 3-regular matrix is also irreducible (which implies a result for the dimension of $\mathbb{C}[A, B, C]$ analagous to the result in the 2-regular case above), but the variety of commuting pairs in the centralizer of an r-regular matrix is *reducible* for $r \geq 5$. The $r = 4$ case is open, although, there are some partial results in [18].

Working in yet a different direction, Šivic and this author ([17]) considered jet schemes over the commuting pairs variety $C(2, n)$. These varieties also appear naturally as subvarieties of $C(3, n)$, as the set of triples where one of the matrices is a fixed nilpotent matrix whose Jordan blocks all have the same size. They showed that for large enough n, these subvarieties are all reducible, but are indeed irreducible if $n \leq 3$.

To the best of this author's knowledge, this is the current state of the art in the subject. The variety $C(3, n)$ has indeed proved to be a very hard object to tackle, even as it has thrown off interesting subproblems, and in special cases, has exhibited connections to other interesting varieties like jet schemes over determinantal varieties and over the commuting pairs variety. The analysis of $C(3, n)$, as well as the original problem, namely whether $\mathbb{C}[A, B, C]$ has dimension at most n when A, B, and C commute, is in need of fresh ideas and approaches.

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Baer Sum Constructions and Extension of Modules

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Abstract. In an earlier paper in this Newsletter, we defined a *matrix of extension* corresponding to extension of finitely generated torsion modules over a Euclidean domain. Using the results from the earlier paper, we show explicitly that $\mathrm{Ext}^1_{\mathbb{Z}}(L, N)$ for any such modules L and N is isomorphic to a finite abelian group depending only on the type of L and N.

In [1], Corollary 3, we had established a bijection between *matrics of extension* and $\mathrm{Ext}^1_{\mathbb{Z}}(L, N)$ for any pair of finitely generated p-torsion $\mathbb{Z}$-modules. We will now show that the bijection defined in Corollary 3 is actually a group isomorphism. Consider two different classes of extensions of finitely generated torsion $\mathbb{Z}$-modules L and N.

$$0 \to L \to M_1 \xrightarrow{\pi_1} N \to 0$$
$$0 \to L \to M_2 \xrightarrow{\pi_2} N \to 0.$$

Let the corresponding *matrices of extension* be denoted by A_1 and A_2 respectively. The *sum* of these two extensions in $\mathrm{Ext}^1_{\mathbb{Z}}(L, N)$ is given by the Baer sum as follows. Form the quotient of Γ the *fiber product* $M_1 \times_N M_2$, i.e.

$$\Gamma = \{(m_1, m_2) \in M_1 \oplus M_2 | \pi_1(m_1) = \pi_2(m_2)\}$$

by the equivalence relation $M_3 = \Gamma/\{(l, 0) - (0, l)\}$. Now the extension

$$0 \to L \to M_3 \to N \to 0$$

given by $l \mapsto [(l, 0)] \sim [(0, l)]$ and $[(m_1, m_2)] \mapsto \pi_1(m_1) = \pi_2(m_2)$ is called the Baer sum of extensions of M_1 and M_2.

Theorem 1. *The matrix of extension corresponding to M_3 is given by $A_1 + A_2$.*

Proof. As before, let $y_1, \ldots, y_l$ be a generating set for L and $x_1, \ldots, x_m$ a generating set for N. Also, let $n^{(j)}_{x_i} \in M_j$ be a section over x_i for $j = 1, 2$. Then, the set $\{[(y_1, 0)] \sim [(0, y_1)], \ldots, [(y_l, 0)] \sim [(0, y_l)]; [(n^{(1)}_{x_1}, n^{(2)}_{x_1})], \ldots, [(n^{(1)}_{x_m}, n^{(2)}_{x_m})]\}$ is in fact a generating set for M_3. The second set of generators still have p^{μ_j} torsion, i.e.

$$p^{\mu_j}[(n^{(1)}_{x_j}, n^{(2)}_{x_j})] = [p^{\mu_j} n^{(1)}_{x_j}, p^{\mu_j} n^{(2)}_{x_j}] = 0$$

We further know the relations that $p^{\mu} n^{(i)}_x = A_i$ for $i = 1, 2$, i.e. for all $j \in 1, \ldots, m$, $[p^{\mu_j} n^{(1)}_{x_j}, p^{\mu_j} n^{(2)}_{x_j}] = [(a^{(1)}_{j1} y_1 + a^{(1)}_{j2} y_2 + \cdots + a^{(1)}_{jl} y_l), (a^{(2)}_{j1} y_1 + a^{(2)}_{j2} y_2 + \cdots + a^{(2)}_{jl} y_l)]$ for $a^{(1)}_{ji} \in A_1$ and $a^{(2)}_{ji} \in A_2$. But by the equivalence relation defined in the Baer sum, this class in M_3 is actually given by $[a^{(1)}_{j1} y_1 + \cdots + a^{(1)}_{jl} y_l + a^{(2)}_{j1} y_1 + \cdots + a^{(2)}_{jl} y_l, 0] \sim [0, a^{(2)}_{j1} y_1 + \cdots + a^{(2)}_{jl} y_l + a^{(1)}_{j1} y_1 + \cdots + a^{(1)}_{jl} y_l]$.

This is precisely the j-th row of the sum of $A_1 + A_2$ and hence proves our theorem. □

Further, the 0 matrix of extension corresponds to the split extension which is the identity element in the $\mathrm{Ext}^1_{\mathbb{Z}}$ group. The *matrics of extension* have a natural group operation given by component-wise addition with the 0 matrix as identity. Since each $a_{ij} \in \mathbb{Z}/p^{\min(\mu_i, \lambda_j)}\mathbb{Z}$ the group of matrices of extension is isomorphic to $\oplus_{1 \le i \le m, 1 \le j \le l} \mathbb{Z}/p^{\min(\mu_i, \lambda_j)}$.

Corollary 1. *The group of equivalent extensions of any two finitely generated p-torsion $\mathbb{Z}$-modules L and N is isomorphic to $\oplus_{1 \le i \le m, 1 \le j \le l} \mathbb{Z}/p^{\min(\mu_i, \lambda_j)}$ where L is of type $(\lambda_1, \ldots, \lambda_l)$ and N is of type $(\mu_1, \ldots, \mu_m)$.*

Proof. We know that this is precisely $\mathrm{Ext}^1_{\mathbb{Z}}(L, N)$. The corollary then follows easily follows from Corollary 3 of [1] and the theorem above. □

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30 January – 4 May, 2012

Venue: ICERM, Providence, Rhode Island.

Description: The goal of this program is to bring together researchers in complex dynamics, arithmetic dynamics, and related fields, with the purpose of stimulating interactions, promoting collaborations, making progress on fundamental problems, and developing theoretical and computational foundations on which future work will build.

For Further Information Please Visit:

http://icerm.brown.edu/sp-s12/

AIM Workshop: Stochastic Dynamics of Small Networks of Neurons

20–24 February, 2012

Venue: American Institute of Mathematics, Palo Alto, California.

Description: This workshop, sponsored by AIM and the NSF, will be devoted to the area between carefully crafted stochastic models of single neurons and large networks of simpler ones.

For Further Information Please Visit:

http://www.aimath.org/ARCC/workshops/neuronnetwork.html

IAENG: International Conference on Operations Research 2012

14–16 March, 2012

Venue: Royal Garden Hotel, Kowloon, Hong Kong.

For Further Information Please Visit:

http://www.iaeng.org/IMECS2012/ICOR2012.html

AIM Workshop: Cohomological Methods in Abelian Varieties

26–30 March, 2012

Venue: American Institute of Mathematics, Palo Alto, California.

Description: This workshop, sponsored by AIM and the NSF, will be devoted to the integral motive, Chow groups and etale cohomology of abelian varieties, and applications to arithmetic geometry.

For Further Information Please Visit:

http://www.aimath.org/ARCC/workshops/cohomabelian.html

Refresher Course in Differential Equations

Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Sponsored by UGC-Academic Staff College

Ramanujan Institute for Advanced Study in Mathematics, University of Madras has been conducting refresher course every year in different topics under Academic Staff College Program of UGC. This year it is planned to conduct refresher course (Batch XXVII) on Differential Equations from 11-11-2011 to 01-12-2011. Experts in Differential Equations from leading institutions will deliver lectures. For application and other information, please contact:

Director
UGC-Academic Staff College, University of Madras
Chennai 600 005
Phone: 2536 8778 Ext: 269, 343

If You have Further Questions, You May also Contact:

Dr. R. Sahadevan, Coordinator
RIASM, University of Madras, Chennai 600 005
Phone: 2536 0357
E-mail: sahadevan@unom.ac.in

International Workshop on Fuzzy Sets, Uncertainty Analysis and Applications

21–25 November, 2011

Visit: http://www.fruaa11.in; www.nitdgp.ac.in; www.isical.ac.in

Contact Address:

Organizing Secretary – FRUAA 2011
Dr. Samajit Kar, Department of Mathematics
National Institute of Technology Durgapur
Durgapur, Burdwan, W.B 713 209, India
Telephone: +91-343-2754582
Cell: +91-9434788032
E-mail: skar@fruaa11.in; contact@fruaa11.in

Nonlinear Functional Analysis and Applications

18–20 January, 2012

To celebrate Prof. S. Kesavan's 60th birthday IMSc is organizing a conference to highlight recent developments in this topic and to inspire young research scholars to pursue research in this area.

Venue: The Institute of Mathematical Sciences, Chennai 600 113.

Contact: krishna@imsc.res.in

Visit: www.imsc.res.in/nfa/

14th International Conference of International Academy of Physical Sciences (CONIAPS-XIV)

22–24 December, 2011

Title of the Conference: Physical Sciences Interface with Humanity (A Golden Jubilee Initiative).

Venue: S. V. National Institute of Technology, Ichchhanath, Surat 395 007, Gujarat.

Contact:

Dr. Dhananjay Gopal or Dr. Lalit Kumari Saini at
`coniapsxiv@gmail.com`
or Dr. P. N. Pandey at
`pnpiaps@rediffmail.com`

Visit: `http://www.svnit.ac.in`

School and Workshop on Cocompact Imbeddings, Profile Decompositions, and their Applications to PDE

3–12 January, 2012

Venue: Tata Institute of Fundamental Research – Centre For Applicable Mathematics, Bangalore.

Contact: `ccpd2012@math.uu.se`

Program Link:
`http://www.icts.res.in/program/ccpd2012`

2nd International Conference on Rough Sets, Fuzzy Sets and Soft Computing (ICRFSC12)

14–16 November, 2012

Topics to Cover Include: Fuzzy Set, Rough Set and Soft Computing.

Venue: Department of Mathematics, Tripura University, Suryamaninagar, Agartala, Tripura 799 130.

Contact for Further Details:

`anjan2002_m@yahoo.co.in;`
`halder_731@rediffmail.com;`
`subrata_bhowmik_math@rediffmail.com`

or write to one of the following at the above address:

A. Mukherjee (Phone: +91-9436123802)

Mrs. S. Bhattacharya (Phone: +91-9436927698)

S. Bhowmik (Phone: +91-9862088913)

Instructional Workshop on the Functional Analysis of Quantum Information Theory

Instructional Workshop on
26 December, 2011 – 6 January, 2012

Venue: The Institute of Mathematical Sciences, Chennai 600 113.

Description: This workshop will be devoted to the mathematical framework of quantized functional analysis (QFA), and illustrate its applications to problems in quantum communication. The lecturers at this workshop will be Ed Effros of UCLA, Gilles Pisier of Paris and Texas A & M, and Vern Paulsen of Houston.

Topics: Topics hoped to be discussed include:

(1) Operator spaces (quantized Banach spaces) and completely bounded maps.
(2) Operator systems (quantized function systems) and completely positive maps.

Participants: Participants need only be familiar with the elements of classical functional analysis including the spectral theorem for bounded self-adjoint operators, and a superficial acquaintance with the matrix models for observables and states. People wishing to participate in the workshop should send an E-mail to `sunder@imsc.res.in` stating their interest in participating in the workshop and also include a line or two describing their level of preparedness for such participation.

International Conference on Advanced Computing, Networking and Security (ADCONS-2011)

16–18 December, 2011

Venue: NITK-Surathkal, Mangalore.

Contact Address:

Dr. P. Santhi Thilagam
Dept. of Computer Science & Engg., NITK Surathkal
Srinivasnagar 575 025, Mangalore, India
E-mail: `adcons2011@gmail.com`
Web: `http://www.adcons2011.com`

National Seminar on Recent Advances in the Application of Mathematical Analysis and Computational Techniques in Applied Sciences

2–4 December, 2011

Topitcs to be Covered: This seminar intends to cover broadly a number of important areas of mathematics.

Contact Address:

Ajit De, Convenor, Department of Mathematics
Siliguri College, Siliguri 734 001
Dist. Darjeeling, West Bengal
Mobile: 09733051852
E-mail: math_seminar@rediffmail.com

Human Resource Development in Mathematics

A Project Sponsored by the

Department of Science and Technology, Government of India

The Institute of Mathematical Sciences, Chennai 600 113

The above mentioned project has been sanctioned by the Department of Science and Technology (DST) of the Government of India to foster the development of mathematics in India. The project consists of various schemes meant to support initiatives from universities and institutions in India to further the development of mathematics in the country. Under this project, about 29 satellite conferences were arranged by mathematicians all over the country during July–September 2010, centered around the International Congress of Mathematicians (ICM2010) held at Hyderabad during August 19–27, 2010. Around 500 mathematicians (faculty and students) drawn from various universities and research institutions in India were given financial support to participate in the the ICM2010.

Another scheme under this project is that to promote the mobility of human resources in mathematics within India. Three kinds of activities are envisaged under this scheme.

1. **Adjunct/Visiting Professors from abroad:**

   Mathematics departments of various Indian universities/teaching institutions are encouraged to identify mathematicians of repute from abroad and to invite them to visit their departments as adjunct/visiting faculty for a fixed period of two months to deliver lecture courses and to interact with the local faculty. The financial support for this consists of an honorarium of ₹35,000 per month for the adjunct/visiting faculty. Travel expenses and local hospitality will be covered, in addition to the payment of honorarium.

2. **Visiting Professors from India:**

   This is along the lines of the preceding activity, except that visiting professors are Indian mathematicians of repute and the length of each visit is between one and three months and the honorarium is ₹20,000 per month. Other conditions remain unchanged.

3. **Travel grants for Indian Researchers:**

   The scheme will provide for coverage of travel expenses (as per eligibility) of Indian researchers from universites and teaching institutions who wish to either visit a mathematical centre of excellence or attend a conference/workshop in India. It is expected that the local expenses are covered by either the home or the host institution.

Proposals for the financial year 2011–2012 are invited from departments intending to host faculty in case of Schemes 1 and 2 above and from interested researchers under Scheme 3. Proposals may be sent by E-mail to the following address:

humresdst@imsc.res.in

and by post to:

Prof. Krishna Maddaly
The Institute of Mathematical Sciences,
CIT Campus, Taramani, Chernnai 600 113

at least three months prior to the proposed visit. For the format of proposals, please contact the above address.

The readers may download the Mathematics Newsletter from the RMS website at

www.ramanujanmathsociety.org
www.rmsconfmathau.org