RATIONAL AND QUASI-PERMUTATION REPRESENTATIONS OF HOLOMORPHS OF CYCLIC p-GROUPS

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ABSTRACT. For a finite group G, three of the positive integers governing its representation theory over $\mathbb{C}$ and over $\mathbb{Q}$ are $p(G), q(G), c(G)$. Here, $p(G)$ denotes the *minimal degree* of a faithful permutation representation of G. Also, $c(G)$ and $q(G)$ are, respectively, the minimal degrees of a faithful representation of G by quasi-permutation matrices over the fields $\mathbb{C}$ and $\mathbb{Q}$. We have $c(G) \leq q(G) \leq p(G)$ and, in general, either inequality may be strict. In this paper, we study the representation theory of the group $G = \mathrm{Hol}(C_{p^n})$, which is the *holomorph* of a cyclic group of order p^n, p a prime. This group is metacyclic when p is odd and metabelian but not metacyclic when $p = 2$ and $n \geq 3$. We explicitly describe the set of all *isomorphism types* of irreducible representations of G over the field of complex numbers $\mathbb{C}$ as well as the isomorphism types over the field of rational numbers $\mathbb{Q}$. We compute the *Wedderburn decomposition* of the rational group algebra of G. Using the descriptions of the irreducible representations of G over $\mathbb{C}$ and over $\mathbb{Q}$, we show that $c(G) = q(G) = p(G) = p^n$ for any prime p. The proofs are often different for the case of p odd and $p = 2$.

1. Introduction

Throughout this paper, G always denotes a finite group, F denotes a field of characteristic 0 and p denotes a prime number.

The *holomorph* of a group G is the semi-direct product of the group G with its automorphism group, with respect to the obvious natural action, and is denoted by $\mathrm{Hol}(G)$.

Let C_{p^n} denote the cyclic group of order p^n, $n \geq 1$. Let ϕ denote the Euler's phi function. For p odd, the automorphism group of C_{p^n}, is well known to be cyclic group of order $\phi(p^n)$. Therefore, for p odd, $\mathrm{Hol}(C_{p^n})$ is a *split metacyclic* group. It is well known that the automorphism group of C_{2^n} is isomorphic

to the multiplicative group $(\mathbb{Z}/2^n\mathbb{Z})^*$ of units of the ring $\mathbb{Z}/2^n\mathbb{Z}$. For $n = 2$, the multiplicative group $(\mathbb{Z}/4\mathbb{Z})^*$ of units of the ring $\mathbb{Z}/4\mathbb{Z}$ is a cyclic group of order 2, generated by the class -1. For $n \geq 3$, the multiplicative group $(\mathbb{Z}/2^n\mathbb{Z})^*$ of units of the ring $\mathbb{Z}/2^n\mathbb{Z}$ is direct product of $\mathbb{Z}/2\mathbb{Z}$, generated by the class of -1 and a copy of $\mathbb{Z}/2^{n-2}\mathbb{Z}$, generated by the class of 5. Note that for $n = 2$, $\mathrm{Hol}(C_{2^n})$ is the *dihedral group* of order 8, which is a split metacyclic group. For $n \geq 3$, $\mathrm{Hol}(C_{2^n})$ is a *split metabelian* group but not metacyclic.

Frobenius studied representations of finite groups over the field of complex numbers. Schur extended that to study representations of finite groups over subfields of complex numbers, especially the field of rational numbers. He understood that representations of finite groups over the field of rational numbers, in general subfields of complex numbers, in much deeper and involved arithmetic aspects. Schur observed that certain irreducible representations over the field of complex numbers could not be realized over the field of rational numbers, but that a finite multiple of certain irreducible representations could always be realized. The number associated with this multiplicity is called the *Schur index* (see Section 2).

A fundamental result in non-commutative ring theory is the *Artin-Wedderburn theorem* (see [1], [12]). The theorem asserts that the semisimple group ring $F[G]$ is abstractly isomorphic to direct sum of matrix rings over finite dimensional division algebras over F. This is referred to as the *Wedderburn decomposition* of $F[G]$. The *Wedderburn theory*, which came around 1914, shows that questions on representations of a group G over F can be thought of as questions on the algebraic structure of the group algebra $F[G]$.

In this paper, we first determine the set of all isomorphism types of irreducible representations of $\mathrm{Hol}(C_{p^n})$ over the field of complex numbers. Thereafter, using the classical Schur theory on group representations (see [9]), we find the set of all isomorphism types of irreducible representations of $\mathrm{Hol}(C_{p^n})$ over the field of rational numbers. We prove that the Schur index of any irreducible complex representation of $\mathrm{Hol}(C_{p^n})$ with respect to the field of rationals is equal to 1. In addition, we also compute the Wedderburn decomposition of rational group algebra of $\mathrm{Hol}(C_{p^n})$. Note that for $p = 2, n = 1$, $\mathrm{Hol}(C_{p^n})$ is C_2, and for $p = 2, n = 2$, that is D_8, the dihedral group of order 8. For these two groups, all isomorphism types of irreducible representations and Wedderburn decomposition are well known to us. In this paper, when $p = 2$, we consider $n \geq 3$.

Let S_n denote the *permutation group* on n letters. Let $|G|$ denote the order of the group G. The *Cayley's theorem* states that any group G can be embedded in $S_{|G|}$. It is interesting to find the least positive integer n such that G is embedded in S_n. The minimal degree of a *faithful permutation* representation of G is the least positive integer n such that G is embedded in S_n, and is denoted by $p(G)$.

In 1963, analogous to permutation group, Wong (see [18]) defined a quasi-permutation group as follows: If G is a finite linear group of degree n, that is, a finite group of automorphisms of an n-dimensional complex vector space (or, equivalently, a finite group of non-singular matrices of order n with complex coefficients) such that the trace of every element of G is a non-negative integer, then G is called a *quasi-permutation group*. The reason for this terminology is that, if G is a permutation group of degree

n, its elements, considered as acting on the elements of a basis of an n-dimensional complex vector space, induce automorphisms of the vector space forming a group isomorphic to G . The trace of the automorphism corresponding to an element x of G is equal to the number of letters left fixed by x, and so is a non-negative integer. Therefore a permutation group of degree n has a representation as a quasi-permutation group of degree n.

A square matrix over $\mathbb{C}$ is called *quasi-permutation* matrix if it has non-negative integral trace. Thus every permutation matrix over $\mathbb{C}$ is a quasi-permutation matrix. For a group G, let $q(G)$ denote the minimal degree of a faithful representation of G by *quasi-permutation matrices over* $\mathbb{Q}$, and let $c(G)$ be the minimal degree of a faithful representation of G by *quasi-permutation matrices over* $\mathbb{C}$ (see [15]). It is easy to see that for a group G the following inequalities hold:

$$c(G) \leq q(G) \leq p(G).$$

In [15], the case of equality has been investigated for abelian groups. For an abelian group, the invariants $c(G)$ and $q(G)$ coincide, as the Schur index of any irreducible complex representation of an abelian group is equal to 1. In [5], Behravesh and Ghaffarzadeh proved that if G is a finite p-group then $q(G) = p(G)$. They also proved that for a finite p-group G, p odd prime, $q(G) = p(G) = c(G)$. The above quantities have been found for several class of finite groups (see [2, 3, 6, 7, 8, 17, 16]).

In this paper, for $G = \mathrm{Hol}(C_{p^n})$, p a prime, we compute $c(G), q(G), p(G)$, and prove that $c(G) = q(G) = p(G) = p^n$. A point to be noted is that even though the structure of the holomorph is different for the case $p = 2$ from the case of odd primes p, the end result is the same. The method of proof is accomplished by explicitly describing the equivalence classes of irreducible representations over $\mathbb{C}$ and over $\mathbb{Q}$, and by computing the Wedderburn decomposition of the rational group algebra. To compute $c(G)$ and $q(G)$, we use the basic method described by Behravesh in [2].

The main theorems are Theorems 6.1, 7.2, 8.1, 9.1, 9.4, 9.5, *and* 10.10.

2. **Schur Index in the Theory of Group Representations**

Let G be a group, and let F be a field of characteristic 0. Fix an algebraic closure $\overline{F}$ of F. Consider an irreducible representation $\widetilde{\rho}$ of G over $\overline{F}$. If u is the exponent of G, then the character $\widetilde{\chi}$ of $\widetilde{\rho}$ takes values in the field $F(\zeta_u)$, where ζ_u is a primitive u-th root of unity. The *character field* $F(\widetilde{\chi})$ of $\widetilde{\rho}$ over F is the extension field of F obtained by adjoining all the values $\widetilde{\chi}(g)$'s as g varies in G. Let Γ be the Galois group of $F(\widetilde{\chi})$ over F. It is well known and easy to see that for each σ, an element of Γ, the values $\sigma(\widetilde{\chi}(g))$ are also values of a character of a representation of G over $\overline{F}$. In this manner, starting with $\widetilde{\rho}$, we have obtained $|\Gamma|$ distinct representations of G, over $\overline{F}$; these representations are said to be *Galois conjugate*. This gives a class of mutually inequivalent Galois conjugate representations of G which have different characters, but all of them have the same character field. Schur's theory implies that there exists a unique irreducible F-representation ρ of G such that $\widetilde{\rho}$ occurs as an irreducible constituent of $\rho \otimes_F \overline{F}$, with some multiplicity; this multiplicity is called the **Schur index** of $\widetilde{\rho}$ with respect to F.

This Schur index is denoted by $m_F(\widetilde{\rho})$ or $m_F(\widetilde{\chi})$. We summarize the above discussion in the following theorem.

Theorem 2.1. [10, Theorem 2] (**Schur**) *Every irreducible F-representation ρ is completely reducible over $\overline{F}$ into a certain number inequivalent irreducible $\overline{F}$-representations $\widetilde{\rho} = \widetilde{\rho}_1, \widetilde{\rho}_2, \ldots, \widetilde{\rho}_\delta$ with the same multiplicity m given by*

$$m = m_F(\widetilde{\chi}_i), (1 \le i \le \delta),$$

where $\widetilde{\chi}_i$ is the character of $\widetilde{\rho}_i$ with $\widetilde{\chi}_1 = \widetilde{\chi}$. The number δ is given by

$$\delta = [F(\widetilde{\chi}_i) : F], (1 \le i \le \delta).$$

The $\overline{F}$-representations $\widetilde{\rho} = \widetilde{\rho}_1, \widetilde{\rho}_2, \ldots, \widetilde{\rho}_\delta$ are Galois conjugates with respect to F. Conversely, each irreducible $\overline{F}$-representation $\widetilde{\rho}$ occurs in the decomposition of an unique irreducible F-representation ρ of G. Moreover, if ψ is the character corresponding to the irreducible F-representation ρ, and Γ is the Galois group of $F(\widetilde{\chi}_i)$ over F, then

$$\psi = m(\sum_{\sigma \in \Gamma} \widetilde{\chi}^\sigma),$$

where $\widetilde{\chi}^\sigma(g) = \sigma(\widetilde{\chi}(g))$, for all $g \in G$.

Definition 2.2. Let ρ be an irreducible F-representations of G, and χ be its corresponding character. The *kernel* of ρ or χ is defined as $\{g \in G \mid \chi(g) = \chi(1)\}$, and is denoted by $\ker(\rho)$ or $\ker(\chi)$.

Theorem 2.3. *G has a faithful irreducible F-representation if and only if G has a faithful irreducible $\overline{F}$-representation.*

Proof. Let ρ be a faithful irreducible F-representation of G, and ψ be its corresponding character. By Theorem 2.1, we have

$$\psi = m(\sum_{\sigma \in \Gamma} \widetilde{\chi}^\sigma),$$

where $\widetilde{\chi}$ is an irreducible $\overline{F}$-character of G and Γ is the Galois group of $F(\widetilde{\chi}_i)$ over F. Then it is easy to see that $\ker(\psi) = \ker(\widetilde{\chi})$. This completes the proof. $\square$

3. Algorithms for $p(G)$, $c(G)$ and $q(G)$

We begin with some basic results on $p(G)$, $c(G)$ and $q(G)$ due to Behravesh [2] for a finite group G. Recall that $p(G)$ denotes the *minimal degree* of a faithful permutation representation of G, and that the minimal degrees of a faithful representation of G by quasi-permutation matrices over the fields $\mathbb{C}$ and $\mathbb{Q}$ are denoted by $c(G)$ and $q(G)$ respectively.

Lemma 3.1. [2, Lemma 2.1] *Let G be a finite group. Let $H_i \le G$ for $i = 1, 2, \ldots, n$, and $K_i = \cap_{x \in G} H_i^x$, the core of H_i in G. Suppose that $\cap_{i=1}^n K_i = \{1\}$. Then $p(G) \le \sum_{i=1}^n [G : H_i]$.*

Theorem 3.2. [2, Theorem 2.2] *Let G be a finite group. Then*

$$p(G) = min\Big\{ \sum_{i=1}^{n} [G : H_i] | H_i \leq G(1 \leq i \leq n), \cap_{i=1}^{n} \text{core}_G H_i = 1 \Big\}.$$

Definition 3.3. *Let G be a finite group and χ be a irreducible complex character of G. Let $\Gamma(\chi)$ be the Galois group of $\mathbb{Q}(\chi)$ over $\mathbb{Q}$. We define*

(1) $d(\chi) = |\Gamma(\chi)|\chi(1)$.

(2)

$$m(\chi) = \begin{cases} 0 & \text{if } \chi = 1_G \\ \Big| \min\Big\{ \sum_{\sigma \in \Gamma(\chi)} \chi^{\sigma}(g) : g \in G \Big\} \Big| & \text{otherwise.} \end{cases}$$

(3) $c(\chi) = \sum_{\sigma \in \Gamma(\chi)} \chi^{\sigma} + m(\chi) 1_G$.

Lemma 3.4. *Let χ be an irreducible complex character of G. Then $m_{\mathbb{Q}}(\chi) \sum_{\sigma \in \Gamma(\chi)} \chi^{\sigma}$, where $m_{\mathbb{Q}}(\chi)$ is the Schur index of χ with respect to $\mathbb{Q}$, is an irreducible rational character of G. Moreover, $ker(\chi) = ker(m_{\mathbb{Q}}(\chi) \sum_{\sigma \in \Gamma(\chi)} \chi^{\sigma})$, and in particular, χ is faithful if and only if $m_{\mathbb{Q}}(\chi) \sum_{\sigma \in \Gamma(\chi)} \chi^{\sigma}$ is also faithful.*

The proof follows from Theorem 2.1 and Theorem 2.3.

With notations as in above, let $\{\mathcal{C}_i \mid 0 \leq i \leq r\}$ be the set of Galois conjugacy classes of irreducible complex characters of G, over $\mathbb{Q}$. For each $i \in \{0, 1, 2, \ldots, r\}$, let ψ_i be a representative of the Galois conjugacy class $\mathcal{C}_i$ with $\psi_0 = 1_G$. Let $m_i = m_{\mathbb{Q}}(\psi_i)$ be the Schur index of ψ_i with respect to $\mathbb{Q}$ and $K_i = ker\psi_i$ be the kernel of ψ_i. Note that both the Schur index and kernel is independent of the choice of the representatives. Let $\Psi_i = \sum_{\chi \in \mathcal{C}_i} \chi$. Clearly $K_i = ker\Psi_i$. For $I \subseteq \{0, 1, \ldots, r\}$, let $K_I = \cap_{i \in I} K_i$.

Theorem 3.5. [2, Theorem 3.6] *Let G be a finite group. Then*

$$c(G) = \min\Big\{ \xi(1) + m(\xi) : \xi = \sum_{i \in I} \Psi_i, K_I = 1, I \subseteq \{1, \ldots, r\}, K_J \neq 1 \text{ if } J \subset I \Big\},$$

$$q(G) = \min\Big\{ \xi(1) + m(\xi) : \xi = \sum_{i \in I} m_i \Psi_i, K_I = 1, I \subseteq \{1, \ldots, r\}, K_J \neq 1 \text{ if } J \subset I \Big\}.$$

Corollary 3.6. [2, Theorem 3.7] *Let $\chi \in \text{Irr}(G)$, the set of all irreducible complex characters of G. Then $\sum_{\alpha \in \Gamma(\chi)} \chi^{\alpha}$ is a rational valued character of G. Moreover $c(\chi)$ is a non-negative rational valued character of G and $c(\chi)(1) = d(\chi) + m(\chi)$.*

Lemma 3.7. [2, Corollary 3.11] *Let G be a finite group with a unique minimal normal subgroup. Then*

(1) $c(G) = min\Big\{ c(\chi)(1) : \chi \text{ is a faithful irreducible complex character of } G \Big\}$;

(2) $q(G) = min\Big\{ m_{\mathbb{Q}}(\chi)c(\chi)(1) : \chi \text{ is a faithful irreducible complex character of } G \Big\}$.

Corollary 3.8. *Let G be a finite group with a unique minimal normal subgroup. If the Schur index of each irreducible complex character with respect to $\mathbb{Q}$ is equal to 1, then $q(G) = c(G)$.*

The following result is known, but for the sake of completeness, we include a proof.

Lemma 3.9. *Let $G = \langle a \rangle \cong C_{p^n}$. For $m \geq 1$, let $\Phi_m(X)$ denote the m-th cyclotomic polynomial and ζ_m a primitive m-th root of unity. Then we have the following description of inequivalent irreducible $\mathbb{Q}$-representations of G.*

(i) *$X^{p^n} - 1 = \displaystyle\prod_{i=0}^{n} \Phi_{p^i}(X)$ is the decomposition into monic irreducible polynomials over $\mathbb{Q}$.*

(ii) *The set of isomorphism types of irreducible $\mathbb{Q}$-representations of G is in a bijective correspondence with the set of monic irreducible factors $\{\Phi_1(X), \Phi_p(X), \ldots, \Phi_{p^n}(X)\}$ of $X^{p^n} - 1$ over $\mathbb{Q}$.*

(iii) *If $\rho_0, \rho_1, \ldots, \rho_n$ are $n+1$ inequivalent irreducible representations of G over $\mathbb{Q}$, then a matrix representation of ρ_i is defined by*

$$\rho_i(a) = C_{\Phi_{p^i}(X)}, i = 0, 1, \ldots, n$$

where $C_{\Phi_{p^i}(X)}$ denotes the companion matrix of $\Phi_{p^i}(X)$.

(iv) *G has a unique faithful irreducible representation over $\mathbb{Q}$. If χ is the unique faithful irreducible character of G over $\mathbb{Q}$ then*

$$\chi(a^i) = \begin{cases} -p^{n-1} & \text{if } (i, p^n) = p^{n-1} \\ p^{n-1}(p-1) & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof of (i) follows from the well known result by Gauss that cyclotomic polynomials are irreducible over $\mathbb{Q}$.

Now we prove (ii). Consider the map $\psi : \mathbb{Q}[X] \to \mathbb{Q}[G]$, and is defined by: $f(X) \mapsto f(a)$. Since ψ is a ring epimorphism, $\mathbb{Q}[G] \cong \mathbb{Q}[X]/\langle Ker(\psi) \rangle$. So, we have

$$\mathbb{Q}[G] \cong \frac{\mathbb{Q}[X]}{\langle X^{p^n} - 1 \rangle} = \frac{\mathbb{Q}[X]}{\langle \displaystyle\prod_{i=0}^{n} \Phi_{p^i}(X) \rangle}.$$

From the Chinese remainder theorem, we can write

$$\mathbb{Q}[G] \cong \bigoplus_{i=0}^{n} \frac{\mathbb{Q}[X]}{\langle \Phi_{p^i}(X) \rangle}.$$

Note that under the above isomorphism, a is mapped to $(X + \langle \Phi_1(X) \rangle, X + \langle \Phi_p(X) \rangle, \ldots, X + \langle \Phi_{p^n}(X) \rangle)$. As $\Phi_{p^i}(X)$ is an irreducible polynomial over $\mathbb{Q}$, then $\mathbb{Q}[X]/\langle \Phi_{p^i}(X) \rangle$ is a field. Hence $\mathbb{Q}[G]$ has $n+1$ simple components $\mathbb{Q}, \mathbb{Q}(\zeta_p), \ldots, \mathbb{Q}(\zeta_{p^n})$, as a consequence of that G has $n+1$ irreducible representations over $\mathbb{Q}$, and also they have a bijective correspondence with the set of monic irreducible factors $\{\Phi_1(X), \Phi_p(X), \ldots, \Phi_{p^n}(X)\}$ of $X^{p^n} - 1$ over $\mathbb{Q}$. This completes the proof.

Let ρ_i be the irreducible representation corresponding to the simple component $\dfrac{\mathbb{Q}[X]}{\langle \Phi_{p^i}(X) \rangle}$. We consider $\mathbb{Q}[X]/\langle \Phi_{p^i}(X) \rangle$ as the representation space of ρ_i. Then ρ_i is defined explicitly as follows.

$$\rho_i(a) : \frac{\mathbb{Q}[X]}{\langle \Phi_{p^i}(X) \rangle} \to \frac{\mathbb{Q}[X]}{\langle \Phi_{p^i}(X) \rangle} \text{ acts by multiplication of } [X] = X + \langle \Phi_{p^i}(X) \rangle.$$

Note that the degree of the representation ρ_i is equal to the degree of $\Phi_{p^i}(X)$, that is, $\phi(p^i)$. We can take $[1], [X], \ldots, [X^{\phi(p^i)-1}]$ to be an ordered basis of $\frac{\mathbb{Q}[X]}{\langle \Phi_{p^i}(X) \rangle}$, and with respect to that basis the matrix for $\rho_i(a)$ is $C_{\Phi_{p^i}(X)}$, where $C_{\Phi_{p^i}(X)}$ denotes the companion matrix of $\Phi_{p^i}(X)$. This completes the proof of (iii).

It is easy to see that G has $\phi(p^n)$ many faithful irreducible complex representations, and they are all Galois conjugates with respect to $\mathbb{Q}$. Their direct sum gives the unique faithful irreducible representation of G over $\mathbb{Q}$. The values of the character χ corresponding to the unique faithful irreducible $\mathbb{Q}$-representation can be obtained from (iii), by taking traces of the various powers of the companion matrix $C_{\Phi_{p^n}(X)}$. This completes the proof of the theorem. $\qquad\square$

4. Wigner-Mackey Method of Little Groups

The groups we study are holomorphs that are semi-direct products of a normal abelian subgroup by a subgroup. To find the set of irreducible complex representations of such a group, we describe the Wigner-Mackey method of little groups.

Let G be a semidirect product of a normal abelian subgroup N by a subgroup H. Since N is abelian, the set of all inequivalent irreducible complex representations of N are 1-dimensional. Let $\widehat{N}$ denote the set of all complex irreducible representations of N.

Now G acts on N by conjugation, and this action induces an action on $\widehat{N}$ as follows:

$$\chi \in \widehat{N}, g \in G, \text{ for all } a \in N, \text{we have } \chi^g(a) = \chi(g^{-1}ag).$$

By the definition of conjugate representations, χ^g is the same as conjugate representation of χ by $g \in G$. Let I_χ be the stabiliser subgroup of χ in G. Note that since N is abelian, we have that $N \leq I_\chi$. Let $H_\chi := I_\chi \cap H$. Then I_χ is a semidirect product of N by H_χ. It can be shown that any $\chi \in \widehat{N}$ can be extended to a homomorphism from I_χ to $\mathbb{C}^*$, in such a way that this extended homomorphism is the trivial map when restricted to H_χ. So we can regard χ as an element of $\widehat{I_\chi}$, the set of all one dimensional complex representations of I_χ.

Further if ρ is a complex representation of H_χ and the canonical projection of I_χ on H_χ is composed with ρ, then we get a representation of I_χ. Thus we can form the tensor product representation $\chi \otimes \rho$ of I_χ and $\deg(\chi \otimes \rho) = \deg \rho$.

Theorem 4.1. [13, Proposition 2.5] *Let G be a finite group which is a semi-direct product of an abelian group N by a subgroup H. Let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_t$ be the distinct orbits under the action of G on $\widehat{N}$ and let χ_j be a representative of the orbit O_j. Let I_j denote the stabiliser of χ_j and let $H_j := I_j \cap H$. For*

any irreducible representation of H_j, let $\theta_{j,\rho}$ denote the representation of G induced from the irreducible representation $\chi_j \otimes \rho$ of I_j. Then

(1) $\theta_{j,\rho}$ is irreducible.

(2) If $\theta_{j,\rho}$ and $\theta_{j',\rho'}$ are isomorphic, then $j = j'$ and ρ is isomorphic to ρ'.

(3) Every irreducible representation of G is isomorphic to one of the $\theta_{j,\rho}$.

5. A Presentation of $\mathbf{Hol}(C_{p^n})$, p a Prime

In this section, we describe presentations of $\mathrm{Hol}(C_{p^n})$ which will be useful to us; we consider the cases of p odd and $p = 2$ separately. We first note:

Remark 5.1. (1) For p odd, $\mathrm{Hol}(C_{p^n}) \cong C_{p^n} \rtimes C_{p^{n-1}(p-1)}$ is a split metacyclic group of order $p^{2n-1}(p-1)$.

(2) For $n \geq 3$, $\mathrm{Hol}(C_{2^n}) \cong C_{2^n} \rtimes (C_{2^{n-2}} \times C_2)$ is a metabelian 2-group (which is not metacyclic) of order 2^{2n-1}.

Let p be an odd prime and $C_{p^n} = \langle x \rangle$. Then $\mathrm{Aut}(C_{p^n}) = \langle f_r | f_r : x \mapsto x^r \rangle$, where r is coprime to p^n. It is well known that $\mathrm{Aut}(C_{p^n}) \cong (\mathbb{Z}/p^n\mathbb{Z})^*$, the group of units of the ring $\mathbb{Z}/p^n\mathbb{Z}$ and $\mathrm{Aut}(C_{p^n}) \cong A \times B$, where $A = \langle \tau : x \mapsto x^s \rangle \cong C_{p-1}$, s coprime to $p-1$ and $B = \langle \sigma : x \mapsto x^{1+p} \rangle \cong C_{p^{n-1}}$.

Then, for odd primes p, $\mathrm{Hol}(C_{p^n})$ has a presentation:

$$\mathrm{Hol}(C_{p^n}) = \langle a, b \,|\, a^{p^n} = b^{p^{n-1}(p-1)} = 1, b^{-1}ab = a^r \rangle,$$

where $p^{n-1}(p-1)$ is the order of r modulo p^n.

Let $p = 2$ and $C_{2^n} = \langle x \rangle$. Then $\mathrm{Aut}(C_{2^n}) = \langle f_r \,|\, f_r : x \mapsto x^r \rangle$, where r is coprime to 2^n.

For $n \geq 3$, $\mathrm{Aut}(C_{2^n}) \cong (\mathbb{Z}/2^n\mathbb{Z})^*$, the group of units of the ring $\mathbb{Z}/2^n\mathbb{Z}$ and $\mathrm{Aut}(C_{2^n}) \cong A \times B$, where $A = \langle \sigma : x \mapsto x^5 \rangle \cong C_{2^{n-2}}$ and $B = \langle \tau : x \mapsto x^{-1} \rangle \cong C_2$.

Then, for $n \geq 3$, $\mathrm{Hol}(C_{2^n})$ has a presentation:

$$\mathrm{Hol}(C_{2^n}) = \langle a, b, c \,|\, a^{2^n} = b^{2^{n-2}} = c^2 = 1, b^{-1}ab = a^5, c^{-1}ac = a^{-1}, c^{-1}bc = b \rangle.$$

6. Complex Representations of $\mathbf{Hol}(C_{p^n})$, p Odd

In this section, we find the set of all inequivalent irreducible representations of $\mathrm{Hol}(C_{p^n})$, p odd, over $\mathbb{C}$.

Let $G = \mathrm{Hol}(C_{p^n})$, p odd. As we have seen before G has a presentation:

$$G = \langle a, b \,|\, a^{p^n} = 1, \;\; b^{p^{n-1}(p-1)} = 1, \;\; b^{-1}ab = a^r \rangle,$$

where r has multiplicative order $p^{n-1}(p-1)$ modulo p^n.

Let $N = \langle a \rangle$. Let $\widehat{N}$ denote the set of all irreducible complex representations of N, and which are given by

$$\widetilde{\chi}_i(a) = \zeta^i, i = 0, 1, 2, \ldots, p^n - 1$$

where ζ is a primitive p^n-th root of unity.

Now, G acts on $\widehat{N}$ by conjugation, with the subgroup N acting trivially. So, this induces an action of G/N on $\widehat{N}$. There are $n+1$ orbits, say, $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_n$ which are described as follows.

$$\mathcal{O}_0 := \text{the orbit of } \widetilde{\chi}_0.$$

$$\text{For } 1 \leq k \leq n, \mathcal{O}_k := \text{the orbit of } \widetilde{\chi}_{p^{n-k}}.$$

Explicitly,

$$\mathcal{O}_0 = \{\widetilde{\chi}_0\}.$$

and

$$\mathcal{O}_k = \{\widetilde{\chi}_{p^{n-k}}, \widetilde{\chi}_{r.p^{n-k}}, \ldots, \widetilde{\chi}_{r^{\phi(p^k)-1}.p^{n-k}}\}, k = 1, 2, \ldots, n.$$

Further, the stabilizer subgroup of $\widetilde{\chi}_0$ is G itself and the stabilizer subgroup of $\widetilde{\chi}_{p^{n-k}}$ (for $1 \leq k \leq n$) is $\langle a, b^{p^{(n-1)-(n-k)}(p-1)} \rangle$.

Corresponding to the the orbit $\mathcal{O}_0$, the irreducible complex representations of G are given by

$$\widetilde{\rho}_{0,\omega}(a) = 1, \widetilde{\rho}_{0,\omega}(b) = \omega,$$

where ω is any $p^{n-1}(p-1)$-th root of unity. In this manner, we obtain $\phi(p^n)$ degree 1 representations of G; these exhaust all the degree 1 complex representations of G.

Further, corresponding to each orbit $\mathcal{O}_k$ (for $1 \leq k \leq n$), there are $o(b)/|\mathcal{O}_k|$ irreducible complex representations of G, and they are of degree $\phi(p^k)$. These irreducible complex representations are given explicitly as:

$$\widetilde{\rho}_{k,\omega}(a) = \begin{bmatrix} \zeta^{p^{n-k}} & 0 & 0 & \cdots & 0 \\ 0 & \zeta^{rp^{n-k}} & 0 & \cdots & 0 \\ 0 & 0 & \zeta^{r^2p^{n-k}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & \zeta^{r^{\phi(p^k)-1}p^{n-k}} \end{bmatrix}, \quad \widetilde{\rho}_{k,\omega}(b) = \begin{bmatrix} 0 & 0 & 0 & \cdots & \omega \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

where ζ is a primitive p^n-th root of unity and ω is any $o(b)/|\mathcal{O}_k| = p^{n-k}$-th root of unity.

Noting that $\phi(p^n).1 + \sum_{k=1}^{n} p^{n-k}|\phi(p^k)|^2 = |G|$, it follows that $\widetilde{\rho}_{k,w}$'s (for $0 \leq k \leq n$) exhaust all the inequivalent irreducible complex representations of G.

We summarise the above discussion in the following theorem.

Theorem 6.1. *Let p be an odd prime and consider the group $G = Hol(C_{p^n})$ for $n \geq 1$. Consider the presentation of G:*

$$G = \langle a, b \mid a^{p^n} = 1, \ b^{p^{n-1}(p-1)} = 1, \ b^{-1}ab = a^r \rangle,$$

where r has multiplicative order $p^{n-1}(p-1)$ modulo p^n. Then, with notations $\widetilde{\rho}_{k,\omega}$, and $\mathcal{O}_k$, $0 \leq k \leq n$ as above, we have the following.

1. *G has $\phi(p^n)$ many degree 1 complex representations, and they are: $\widetilde{\rho}_{0,\omega}$, where ω varies over the set of $p^{n-1}(p-1)$-th roots of unity.*

2. *Corresponding to orbit $\mathcal{O}_k$, $k = 1, 2, \ldots, n$, G has p^{n-k} number of irreducible complex representations of degree $\phi(p^k)$ and they are: $\widetilde{\rho}_{k,\omega}$, where ω varies over the set of p^{n-k}-th roots of unity.*

3. *G has $(1 + p + \cdots + p^n) - p^{n-1}$ many inequivalent irreducible complex representations.*

4. *Corresponding to the orbit $\mathcal{O}_n$, there is only one irreducible complex representation, which is of degree is $\phi(p^n)$. In fact, this is the unique faithful irreducible complex representation of G.*

We briefly illustrate this theorem through an example.

Example 6.2. *Let $G = Hol(C_9)$. G has a presentation:*

$$G = \langle x, y \mid a^9 = b^6 = 1, b^{-1}ab = a^2 \rangle.$$

The set of all inequivalent irreducible complex representations of $N = \langle a \rangle$ are given by

$$\widetilde{\chi}_i(a) = \zeta^i, i = 0, 1, 2, \ldots, 8$$

where ζ is a primitive 3^2-th root of unity. G is acting on the set of all inequivalent irreducible complex representations of N by conjugation. There are three orbits under this action, and they are given by $\mathcal{O}_0 = \{\widetilde{\chi}_0\}$; $\mathcal{O}_1 = \{\widetilde{\chi}_3, \widetilde{\chi}_6\}$; $\mathcal{O}_2 = \{\widetilde{\chi}_1, \widetilde{\chi}_2, \widetilde{\chi}_4, \widetilde{\chi}_8, \widetilde{\chi}_7, \widetilde{\chi}_5\}$.
Note that 2 has order $\phi(9)$ modulo 9.
Corresponding to $\mathcal{O}_0$, there are six irreducible representations of degree 1, and they are defined by

$$\widetilde{\rho}_{0,\zeta_6^i} : a \mapsto 1, b \mapsto \zeta_6^i \ (0 \leq i \leq 5),$$

where ζ_6 is a primitive 6-th root of unity.
Corresponding to $\mathcal{O}_1$, there are three irreducible representations of degree 2, and they are defined by

$$[\widetilde{\rho}_{1,\omega^i}(a)] = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}, [\widetilde{\rho}_{1,\omega^i}(b)] = \begin{bmatrix} 0 & \omega^i \\ 1 & 0 \end{bmatrix} \ (0 \leq i \leq 2),$$

where ω is a primitive cube root of unity.
Corresponding to $\mathcal{O}_2$, there is one irreducible representation of degree 6, and which is defined by

$$[\widetilde{\rho}_{2,1}(a)] = \begin{bmatrix} \zeta & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta^8 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta^7 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta^5 \end{bmatrix}, [\widetilde{\rho}_{2,1}(b)] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

where ζ is a primitive 9-th root of unity. Thus G has $1+3+3^2-3 = 10$ inequivalent irreducible complex representations.

7. **Rational Representations of Hol(C_{p^n}), p Odd**

In this section, we consider as in the previous section, the groups $\mathrm{Hol}(C_{p^n})$, p odd but now we describe the set of all inequivalent irreducible representations over $\mathbb{Q}$. We will use information on Schur index. We use the same notations as in the previous section.

Lemma 7.1. *The Schur index of any irreducible complex representation of $Hol(C_{p^n})$, p odd, with respect to $\mathbb{Q}$ is equal to 1.*

Proof. Let $G = \mathrm{Hol}(C_{p^n})$, p odd. It is clear that the Schur index of any degree 1 complex representation of G with respect to $\mathbb{Q}$ is equal to 1. Now we show that the Schur index of any non-linear irreducible complex character of G with respect to $\mathbb{Q}$ is equal to 1.

Let $H = \langle a, b^{p-1} \rangle$. Note that H is a normal Hall subgroup of G and $G/H \cong C_{p-1}$. So by Schur-Zassenhaus theorem (see [14, Schur-Zassenhaus Theorem, Chapter 3]), H has a complement in G.

Let χ be a non-linear irreducible complex character of G. As we have seen before, $\chi = \phi^G$ for some irreducible complex character ϕ of H. Since ϕ is an irreducible complex character of the p-group H, then $\phi(1)$ is either 1 or a power of p. Then by [11, Lemma 10.8], the Schur index of χ with respect to $\mathbb{Q}$ divides $\phi(1)$. So the Schur index of χ with respect to $\mathbb{Q}$ is either 1 or a power of p. Again as $\chi = \phi^G$ and $G/H \cong C_{p-1}$, the Schur index of χ with respect to $\mathbb{Q}$ is a divisor of $p - 1$. As a consequence of that the Schur index of χ with respect to $\mathbb{Q}$ is equal to 1. This completes the proof. □

To find all the inequivalent irreducible rational representations of G, we divide the set Ω_G of all inequivalent irreducible $\mathbb{Q}$-representations of G into two parts:

  (1) The irreducible $\mathbb{Q}$-representations whose kernels contain G'.
  (2) The irreducible $\mathbb{Q}$-representations whose kernels do not contain G'.

The irreducible $\mathbb{Q}$-representations whose kernels contain G' can be obtained by the lifts of the irreducible $\mathbb{Q}$-representations of G/G'. Moreover, they are in a bijective correspondence with the Galois conjugacy classes of complex degree 1 representations. The irreducible $\mathbb{Q}$-representations whose kernels do contain G' is in a bijective correspondence with the Galois conjugacy classes of irreducible complex representations of degrees equal to 1.

Consider the orbit $\mathcal{O}_0$. As we have seen in Section 6, the corresponding irreducible complex representations are $\tilde{\rho}_{0,\omega}$, where ω varies over the set of all $(p^n - p^{n-1})$-th root of unity, and moreover they are all one dimensional. Two such representations $\widetilde{\rho}_{0,\omega}$ and $\widetilde{\rho}_{0,\omega'}$ are Galois conjugate over $\mathbb{Q}$ if and only if the complex numbers ω and ω' are Galois conjugate over $\mathbb{Q}$; that is, their orders are equal. The number of $\mathbb{Q}$-irreducible representations, which are obtained in this way, is equal to the number of divisors of $\phi(p^n)$, where ϕ is the Euler's phi function. Moreover they are determined by the factorization of $X^{\phi(p^n)} - 1$ into irreducible polynomials over $\mathbb{Q}$.

Consider the orbits $\mathcal{O}_k$, $k = 1, 2, \ldots, n$. For each orbit $\mathcal{O}_k$, and for any two p^{n-k}-th roots of unity ω, ω', the irreducible complex representations $\widetilde{\rho}_{k,\omega}$ and $\widetilde{\rho}_{k,\omega'}$ are Galois conjugate over $\mathbb{Q}$ if and only if the complex numbers ω and ω' are Galois conjugate over $\mathbb{Q}$. Let ω_i be a primitive p^i-th root of unity. Then for $k = 1, 2, \ldots, n$ and $0 \leq i \leq n - k$, let

$$\rho_{k,i} = \bigoplus_{\sigma} (\widetilde{\rho}_{k,\omega_i})^\sigma,$$

where σ runs over the Galois group of $\mathbb{Q}(\omega_i)$ over $\mathbb{Q}$. By Theorem 2.1 and Lemma 7.1, $\rho_{k,i}$ is irreducible over $\mathbb{Q}$.

Thus for each $k \in \{1, 2, \ldots, n\}$, the p^{n-k} irreducible complex representations $\widetilde{\rho}_{k,\omega}$ are partitioned into $(n - k) + 1$ Galois conjugacy classes over $\mathbb{Q}$, the (direct) sum of irreducible complex representations in each Galois conjugacy class gives an irreducible rational representation of G. Thus the number of rational irreducible representations of G which do not contain G' in their kernel is

$$\sum_{k=1}^{n} (n - k + 1) = n + (n - 1) + \cdots + 1 = \frac{n(n + 1)}{2}.$$

The above discussion can be summarised as the following theorem:

Theorem 7.2. *Let p be an odd prime and consider the group $G = \mathrm{Hol}(C_{p^n})$ as before, with the presentation:*

$$G = \langle a, b \mid a^{p^n} = 1, \ b^{p^{n-1}(p-1)} = 1, \ b^{-1}ab = a^r \rangle,$$

where r has multiplicative order $p^{n-1}(p - 1)$ modulo p^n. For $m \geq 1$, let $\Phi_m(X)$ denote m-th cyclotomic polynomial. Then we have the following results on its irreducible representations over $\mathbb{Q}$:

1. *The set of all inequivalent irreducible $\mathbb{Q}$-representations of G whose kernels contain G' is in a bijective correspondence with the Galois conjugacy classes of complex degree 1 representations, and also in a bijective correspondence with monic irreducible polynomials $\Phi_d(X)$, as d varies over divisors of $\phi(p^n)$.*

2. *The set of all inequivalent irreducible $\mathbb{Q}$-representations of G whose kernels do not contain G' is in a bijective correspondence with the set of all monic irreducible polynomials $\Phi_1(X), \Phi_p(X), \ldots, \Phi_{p^{n-k}}(X)$, where k runs over $\{1, 2, \ldots, n\}$.*

3. *The number of inequivalent irreducible $\mathbb{Q}$-representations of G is equal to $d + \frac{n(n+1)}{2}$, where d is the number of divisors of $\phi(p^n)$.*

8. Rational Wedderburn decomposition of Hol(C_{p^n}), p odd

As we have information on the Schur indices form the Lemma 7.1, and the description of irreducible representations over $\mathbb{Q}$ from the previous theorem, we may easily find the Wedderburn decomposition of the group algebra of $\mathrm{Hol}(C_{p^n})$, p odd, over $\mathbb{Q}$. With the same notations as in the previous section, we prove:

Theorem 8.1. *Let $G = Hol(C_{p^n})$, p odd. For $m \geq 1$, let ζ_m denote a primitive m-th root of unity. The Wedderburn decomposition of $\mathbb{Q}[G]$ is*

$$\mathbb{Q}[G] = \bigoplus_{d | \phi(p^n)} \mathbb{Q}(\zeta_d) \bigoplus_{l=0}^{n-k} (\bigoplus_{k=1}^{n} M_{\phi(p^k)}(\mathbb{Q}(\zeta_{p^l}))).$$

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible $\mathbb{Q}$-representation of G. Let the minimal 2-sided ideal of $\mathbb{Q}[G]$ corresponding to ρ be abstractly isomorphic to $M_n(D)$, for suitable n, and a division ring D. Let Z be the center of D. By Lemma 7.1, the Schur index of any irreducible complex representation of G with respect to $\mathbb{Q}$ is equal to 1. Therefore the Schur index of D (or $M_n(D)$) over $\mathbb{Q}$ is equal to 1, and consequently $D = Z$. By Theorem 2.1, ρ is completely reducible over $\mathbb{C}$ into a certain number inequivalent irreducible $\mathbb{C}$-representations $\widetilde{\rho} = \widetilde{\rho}_1, \widetilde{\rho}_2, \ldots, \widetilde{\rho}_\delta$ with the same multiplicity 1, and they are Galois conjugates with respect to $\mathbb{Q}$. Let $\widetilde{\chi}_i$ is the character of $\widetilde{\rho}_i$. Then by [10, Theorem 3], $Z \cong F(\widetilde{\chi}_i)$ and $\delta = [F(\widetilde{\chi}_i), F]$. Again as the Schur index of any irreducible complex representation of G with respect to $\mathbb{Q}$ is equal to 1, then n is equal to the common degree of the representations $\widetilde{\rho} = \widetilde{\rho}_1, \widetilde{\rho}_2, \ldots, \widetilde{\rho}_\delta$.

If ρ is the irreducible $\mathbb{Q}$-representation whose kernel contain G', then the minimal 2-sided ideal of $\mathbb{Q}[G]$ corresponding to ρ be abstractly isomorphic to $\mathbb{Q}(\zeta_d)$, where d is a divisor of $\phi(p^n)$.

If ρ is an irreducible $\mathbb{Q}$-representation whose kernel does not contain G', then for some $k \in \{1, 2, \ldots, n\}$ and $0 \leq l \leq n - k$,

$$\rho = \rho_{k,l} = \bigoplus_{\sigma} (\widetilde{\rho}_{k,\zeta_{p^l}})^{\sigma},$$

where ζ_{p^l} is a primitive p^l-th root of unity and σ runs over the Galois group of $\mathbb{Q}(\zeta_{p^l})$ over $\mathbb{Q}$. Note that the degree of $\widetilde{\rho}_{k,\zeta_{p^l}}$ is $\phi(p^k)$ and the character field of $\widetilde{\rho}_{k,\zeta_{p^l}}$ over $\mathbb{Q}$ is $\mathbb{Q}(\zeta_{p^l})$. Therefore the minimal 2-sided ideal of $\mathbb{Q}[G]$ corresponding to ρ is abstractly isomorphic to $M_{\phi(p^k)}(\mathbb{Q}(\zeta_{p^l}))$. This completes the proof. $\square$

Let us illustrate Theorem 7.2 and the previous theorem through an example.

Example 8.2. *Let $G = Hol(C_9)$. G has a presentation of the form:*

$$G = \langle a, b \mid a^9 = b^6 = 1, b^{-1}ab = a^2 \rangle.$$

We have seen in Example 6.2 that corresponding to the orbit $\mathcal{O}_0 = \{1\}$, there are six irreducible representations of degree 1.

Over $\mathbb{Q}$,

$$X^6 - 1 = \phi_1(X)\phi_2(X)\phi_3(X)\phi_6(X) = (X - 1)(X + 1)(X^2 + X + 1)(X^2 - X + 1)$$

is the factorization into monic irreducible polynomials. Therefore, there are four $\mathbb{Q}$-irreducible representations whose kernels contain G', and they can be realized as

$$\rho_{0,1}(a) \mapsto 1, \rho_{0,1}(b) \mapsto 1;$$

$$\rho_{0,-1}(a) \mapsto 1, \rho_{0,-1}(b) \mapsto -1;$$

$$[\rho_{0,\omega}(a)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, [\rho_{0,\omega}(b)] = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix};$$

$$[\rho_{0,\zeta_6}(a)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, [\rho_{0,\zeta_6}(b)] = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},$$

where ω is a primitive cube root of unity and ζ_6 is a primitive 6-th root of unity.

In general, we get companion matrices of the irreducible polynomial factors.

We have seen in Example 6.2 that corresponding to the orbit $\mathcal{O}_1$, there are three irreducible representations of degree 2.

The representation $\widetilde{\rho}_{1,1}$ is realizable over $\mathbb{Q}$, and therefore irreducible.

The representations $\widetilde{\rho}_{1,\omega}$ and $\widetilde{\rho}_{1,\omega^2}$ are Galois conjugate over $\mathbb{Q}$, and hence $\widetilde{\rho}_{1,\omega} \oplus \widetilde{\rho}_{1,\omega^2}$ irreducible over $\mathbb{Q}$.

The representation $\widetilde{\rho}_{2,1}$ is realizable over $\mathbb{Q}$, and hence irreducible over $\mathbb{Q}$.

The Wedderburn decomposition of $\mathbb{Q}[G]$ is:

$$\mathbb{Q}[Hol(C_9)] = \bigoplus_{d|6} \mathbb{Q}(\zeta_d) \bigoplus M_2(\mathbb{Q}) \bigoplus M_2(\mathbb{Q}(\omega)) \bigoplus M_6(\mathbb{Q}).$$

9. Representations of $\mathbf{Hol}(C_{p^n})$, $p = 2$

In this section, we consider the case $p = 2$. That is, we find the set of all inequivalent irreducible representations of $\mathrm{Hol}(C_{p^n})$, $p = 2$, over $\mathbb{C}$ and $\mathbb{Q}$. We will see that the end results are analogous to the odd case but the intermediary descriptions are different and more complicated as the group is not metacyclic when $n > 2$.

9.1. **Complex Irreducible Represntations of $\mathbf{Hol}(C_{p^n})$, $p = 2$.** We have seen that for $n \geq 3$, $\mathrm{Hol}(C_{2^n})$ has a presentation of the form:

$$G = \mathrm{Hol}(C_{2^n}) = \langle a, b, c \, | \, a^{2^n} = b^{2^{n-2}} = c^2 = 1, b^{-1}ab = a^5, c^{-1}ac = a^{-1}, c^{-1}bc = b \rangle.$$

Let $N = \langle a \rangle \cong C_{2^n}$. Let $\widehat{N}$ denote the set of all inequivalent irreducible representations of N over $\mathbb{C}$. The set of all irreducible representations of N are given by

$$\widetilde{\chi}_i(a) = \zeta^i, i = 0, 1, 2, \ldots, 2^n - 1$$

where ζ is a primitive 2^n-th root of unity. Now G is acting on $\widehat{N}$ by conjugation, N is acting trivially, which induces an action of G/N on $\widehat{N}$. There are $n + 1$ distinct orbits. Let $\mathcal{O}_0$ denote the orbit of $\widetilde{\chi}_0$. Let $\mathcal{O}_k$ denote the orbit of $\widetilde{\chi}_{2^{n-k}}$, $k = 1, 2, \ldots, n$. In fact, $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_n$ are all the distinct $n+1$ orbits. One can see that orbits $\mathcal{O}_0$ and $\mathcal{O}_1$ are singleton sets, and so the corresponding stabilizer subgroup in G is $G = \langle a, b, c \rangle$ itself. Let $\mathrm{Stab}(\widetilde{\chi}_{2^{n-k}})$ denote the stabilizer subgroup of $\widetilde{\chi}_{2^{n-k}}$, $k = 2, \ldots, n$. One can see that

$$\mathrm{Stab}(\widetilde{\chi}_{2^{n-k}}) = \langle a, b^{2^{(n-2)-(n-k)}} \rangle, k = 2, \ldots, n.$$

So for each $k = 2, \ldots, n$, $\mathrm{Stab}(\widetilde{\chi}_{2^{n-k}})$ is of order 2^{2n-k}. Corresponding to the orbits $\mathcal{O}_0$ and $\mathcal{O}_1$, there are 2^{n-1} many irreducible complex representations of degree 1. Corresponding to each orbit $\mathcal{O}_k, k = 2, \ldots, n$, there are $\frac{\mathrm{Stab}(\widetilde{\chi}_{2^{n-k}})}{|N|} = 2^{n-k}$ many distinct irreducible complex representations of G, and of degree $|\mathcal{O}_k| = \phi(2^k) = 2^{k-1}$.

Corresponding to the orbit $\mathcal{O}_0$, the irreducible complex representations of G are given by

$$\widetilde{\rho}_{0,\omega,1}(a) = 1, \widetilde{\rho}_{0,\omega,1}(b) = \omega, \widetilde{\rho}_{0,\omega,1}(c) = 1$$

and

$$\widetilde{\rho}_{0,\omega,-1}(a) = 1, \widetilde{\rho}_{0,\omega,-1}(b) = \omega, \widetilde{\rho}_{0,\omega,-1}(c) = -1,$$

where ω is any 2^{n-2}-th root of unity. In this way, we obtain 2^{n-1} distinct irreducible complex representations of degree 1.

Corresponding to the orbit $\mathcal{O}_1$, the irreducible complex representation of G are given by

$$\widetilde{\rho}_{1,\omega,1}(a) = -1, \widetilde{\rho}_{1,\omega,1}(b) = \omega, \widetilde{\rho}_{1,\omega,1}(c) = 1$$

and

$$\widetilde{\rho}_{1,\omega,-1}(a) = -1, \widetilde{\rho}_{1,\omega,-1}(b) = \omega, \widetilde{\rho}_{1,\omega,-1}(c) = -1,$$

where ω is a primitive 2^{n-2}-th root of unity. In this way, we obtain 2^{n-1} distinct irreducible complex representations of degree 1.

Thus we have obtained 2^n number of degree 1 irreducible complex representations of G; these exhaust all the degree 1 complex representations of G.

Corresponding to the orbit $\mathcal{O}_k, k = 2, \ldots, n$, the irreducible complex representation of G are given explicitly as:

$$\widetilde{\rho}_{k,\omega}(a) = \left[\begin{array}{cccc|cccc} \zeta^{2^{n-k}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \zeta^{5 \cdot 2^{n-k}} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta^{5^{2^{(n-2)-(n-k)}-1} \cdot 2^{n-k}} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & \zeta^{-2^{n-k}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \zeta^{-5 \cdot 2^{n-k}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \zeta^{-5^{2^{(n-2)-(n-k)}-1} \cdot 2^{n-k}} \end{array}\right],$$

$$\widetilde{\rho}_{k,\omega}(b) = \left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & \omega & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \omega \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right], \widetilde{\rho}_{k,\omega}(c) = \left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ \hline 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{array}\right].$$

where ζ is a primitive 2^n-th root of unity and ω is any 2^{n-k}-th root of unity.

In the above discussion, we have obtained $(2^{n-1} + 2^{n-1}) + \sum_{k=2}^{n} 2^{n-k} = (1 + 2 + \cdots + 2^n) - 2^{n-1}$ distinct complex irreducible representations of G. Noting that

$$2^{n-1}(1) + 2^{n-1}(1) + 2^{n-2}(2)^2 + \cdots + 2(2^{n-2})^2 + (2^{n-1})^2$$

$$= 2^n + 2^n + 2^{n+1} + \cdots + 2^{2n-3} + 2^{2n-2}$$

$$= 2^n + 2^n(1 + 2 + 2^2 + \cdots + 2^{n-2})$$

$$= 2^n + 2^n(2^{n-1} - 1) = 2^{2n-1} = |G|.$$

It follows that these exhaust all the inequivalent irreducible complex representations of G. Summarising the above discussion, we obtain the following theorem.

Theorem 9.1. *Consider the group $G = Hol(C_{2^n})$ which has a presentation:*

$$G = Hol(C_{2^n}) = \langle a, b, c \,|\, a^{2^n} = b^{2^{n-2}} = c^2 = 1, b^{-1}ab = a^5, c^{-1}ac = a^{-1}, c^{-1}bc = b \rangle$$

for $n \geq 3$. Then, we have the following.

1. *G has $2\phi(2^n)(= 2^n)$ many degree 1 complex representations, and they are: $\widetilde{\rho}_{0,\omega,\pm 1}$ and $\widetilde{\rho}_{1,\omega,\pm 1}$, where ω varies over the set of 2^{n-2}-th roots of unity.*

2. *Corresponding to orbit $\mathcal{O}_k$, $k = 2, \ldots, n$, G has 2^{n-k} number of irreducible complex representations of degree $\phi(2^k)$ and they are: $\widetilde{\rho}_{k,\omega}$, where ω varies over the set of 2^{n-k}-th roots of unity.*

3. *G has $(1 + 2 + \cdots + 2^n) - 2^{n-1}$ many inequivalent irreducible complex representations.*

4. *Corresponding to the orbit $\mathcal{O}_n$, there is only one irreducible complex representation, which is of degree is $\phi(2^n)$. In fact, this is the unique faithful irreducible complex representation of G.*

As mentioned in the beginning of this section, the form of the above theorem is exactly similar to that of the analogous theorem when p is odd, but the descriptions of the complex irreducible representations was more involved in the case $p = 2$.

Here is an example illustrating the Theorem 9.1. We use the notations as in the above.

Example 9.2. *Let $G = Hol(C_{2^5})$; it has a presentation*

$$\langle a, b, c \,|\, a^{2^5} = b^{2^3} = c^2 = 1, b^{-1}ab = a^5, c^{-1}ac = c^{-1}, c^{-1}bc = b \rangle.$$

Let $N = C_{2^5} = \langle a \rangle$. Let $\widehat{N}$ denote the set of all inequivalent irreducible representations of N over the field of complex numbers. The set of all inequivalent irreducible complex representations of N are given by

$$\widetilde{\chi}_i(a) = \zeta^{2^5 - i}, i = 0, 1, 2, \ldots, 2^5 - 1.$$

G is acting on $\widehat{N}$ by conjugation, and N is acting trivially. This induces an action of G/N on $\widehat{N}$. Let $\mathcal{O}_0$ denote the orbit of $\widetilde{\chi}_0$ and $\mathcal{O}_k$ denote the orbit of $\widetilde{\chi}_{2^{5-k}}, k = 1, 2, 3, 4, 5$. Corresponding to the orbit $\mathcal{O}_0$, the complex irreducible representations of G are given by

$$\widetilde{\rho}_{0,\omega,1}(a) = 1, \widetilde{\rho}_{0,\omega,1}(b) = \omega, \widetilde{\rho}_{0,\omega,1}(c) = 1$$

and

$$\widetilde{\rho}_{0,\omega,-1}(a) = 1, \widetilde{\rho}_{0,\omega,-1}(b) = \omega, \widetilde{\rho}_{0,\omega,-1}(c) = 1,$$

where ω is any 2^3-th root of unity. In this way, we obtain 2^4 distinct irreducible complex representations of degree 1.

Corresponding to the orbit $\mathcal{O}_1$, the complex irreducible representation of G are given by

$$\widetilde{\rho}_{1,\omega,1}(a) = 1, \widetilde{\rho}_{1,\omega,1}(b) = \omega, \widetilde{\rho}_{1,\omega,1}(c) = 1$$

and

$$\widetilde{\rho}_{1,\omega,-1}(a) = 1, \widetilde{\rho}_{1,\omega,-1}(b) = \omega, \widetilde{\rho}_{1,\omega,-1}(c) = -1,$$

where ω is a primitive 2^3-th root of unity. In this way, we obtain 2^4 distinct irreducible complex representations of degree 1.

Corresponding to the orbit $\mathcal{O}_k, k = 2, 3, 4, 5$, the complex irreducible representation of G are given by

$$\widetilde{\rho}_{k,\omega}(a) = \left[\begin{array}{cccc|cccc} \zeta^{2^{5-k}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \zeta^{5.2^{5-k}} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta^{5^{2^{(5-2)-(5-k)}-1}.2^{5-k}} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & \zeta^{-2^{5-k}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \zeta^{-5.2^{5-k}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \zeta^{-5^{2^{(5-2)-(5-k)}-1}.2^{5-k}} \end{array} \right],$$

$$\widetilde{\rho}_{k,\omega}(b) = \left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & \omega & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \omega \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array} \right], \widetilde{\rho}_{k,\omega}(c) = \left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ \hline 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{array} \right].$$

where ζ is a primitive 2^5-th root of unity and ω is any 2^{5-k}-th root of unity.

Thus G has $(1 + 2 + \cdots + 2^5) - 2^4$ distinct irreducible complex representations.

9.2. Rational Representations of $\mathrm{Hol}(C_{p^n})$, $p = 2$.

Now, we turn to a description of the set of inequivalent irreducible $\mathbb{Q}$-representations of $\mathrm{Hol}(C_{2^n})$. This is again more complicated than the odd p case but the form of the end result is the same. We use the same notations as in the previous section.

Lemma 9.3. *The Schur index of any irreducible complex representation of $\mathrm{Hol}(C_{2^n})$ with respect to $\mathbb{Q}$ is equal to 1.*

Proof. Let $G = \mathrm{Hol}(C_{2^n})$. We have seen that for $n \geq 3$, $\mathrm{Hol}(C_{2^n})$ has a presentation of the form:

$$G = \mathrm{Hol}(C_{2^n}) = \langle a, b, c \, | \, a^{2^n} = b^{2^{n-2}} = c^2 = 1, b^{-1}ab = a^5, c^{-1}ac = a^{-1}, c^{-1}bc = b \rangle.$$

Each irreducible complex representation in the orbit $\mathcal{O}_0$ and $\mathcal{O}_1$ extends $\phi(2^n)$ distinct ways to G, and they are of degree 1. So the Schur index of each such irreducible complex representation with respect to $\mathbb{Q}$ is equal to 1.

As we have seen in the previous section that for each $k, (2 \leq k \leq n)$, there are 2^{n-k} many irreducible complex representations of G, and they are of degree $\phi(2^k) = 2^{k-1}$. Each of them is induced from an one dimensional representation of the subgroup $\langle a, b^{2^{(n-2)-(n-k)}} \rangle$, which can be obtained by extension of the irreducible representation $\widetilde{\chi}_{2^{n-k}}$ of the subgroup $\langle a \rangle$ to $\langle a, b^{2^{(n-2)-(n-k)}} \rangle$. We have seen that $\widetilde{\chi}_{2^{n-k}}$ extends 2^{n-k} many distinct ways to $\langle a, b^{2^{(n-2)-(n-k)}} \rangle$. We denote all the 2^{n-k} extensions of $\widetilde{\chi}_{2^{n-k}}$ by $\widetilde{\chi}_{2^{n-k},\omega}$, where ω runs over the set of all the 2^{n-k}-th roots of unity. They are defined by $\widetilde{\chi}_{2^{n-k},\omega}(a) = \zeta^{2^{n-k}}$, where ζ is a primitive 2^n-th root of unity, and $\widetilde{\chi}_{2^{n-k},\omega}(b^{2^{(n-2)-(n-k)}}) = \omega$, where ω runs over the set of all the 2^{n-k}-th roots of unity. By Clifford's theorem (see [11, Theorem 6.2]), $\widetilde{\rho}_{k,\omega} = \widetilde{\chi}^G_{2^{n-k},\omega}$ is irreducible. It is easy to see that the character field of each $\widetilde{\chi}_{2^{n-k},\omega}$ over $\mathbb{Q}$ is $\mathbb{Q}(\zeta^{2^{n-k}}, \omega)$ and the character field of $\widetilde{\rho}_{k,\omega}$ is equal to $\mathbb{Q}(\omega)$. If the character field of $\widetilde{\rho}_{k,\omega}$ contains $\sqrt{-1}$ then by Roquette's theorem (see [11, Theorem 10.14]), the Schur index of $\widetilde{\rho}_{k,\omega}$ with respect to $\mathbb{Q}$ is equal to 1. For each $k, (2 \leq k \leq n-1)$, there are two irreducible complex representations of G, namely $\widetilde{\rho}_{k,1}$ and $\widetilde{\rho}_{k,-1}$, with character field $\mathbb{Q}$. It can be easily shown that the Schur index of $\widetilde{\rho}_{k,1}, \widetilde{\rho}_{k,-1}$ with respect to $\mathbb{Q}$ is equal to 1.

For $k = n$, there is only one irreducible complex representation of G of degree $\phi(2^n) = 2^{n-1}$, which is induced representation of $\widetilde{\chi}_1$ to G. Since $N = \langle a \rangle$ has a complement in G, by Lemma 10.8 in [11], the Schur index of $\widetilde{\chi}_1^G$ with respect to $\mathbb{Q}$ is equal to 1. This completes the proof. □

Now, thanks to this Lemma, we may find all the inequivalent irreducible rational representations of G as we did in the odd p case.

Once again, the set Ω_G of all inequivalent irreducible $\mathbb{Q}$-representations of G is comprised of two parts:

(1) The irreducible $\mathbb{Q}$-representations whose kernels contain G'.
(2) The irreducible $\mathbb{Q}$-representations whose kernels do not contain G'.

The irreducible $\mathbb{Q}$-representations whose kernels contain G' can be obtained by the lifts of the irreducible $\mathbb{Q}$-representations of G/G'. Moreover, they are in a bijective correspondence with the Galois conjugacy classes of complex degree 1 representations. The irreducible $\mathbb{Q}$-representations whose kernels do contain G' is in a bijective correspondence with the Galois conjugacy classes of irreducible complex representations of degrees equal to 1.

Consider the orbit $\mathcal{O}_0$. The corresponding irreducible complex representations are $\widetilde{\rho}_{0,\omega,\pm 1}$, where ω is a 2^{n-2}-th root of unity, are all of degree 1. For any two 2^{n-2}-th roots of unity ω, ω', two representations $\widetilde{\rho}_{0,\omega,1}$ and $\widetilde{\rho}_{0,\omega',1}$ are Galois conjugate over $\mathbb{Q}$ if and only if the complex numbers ω and ω' are Galois conjugate over $\mathbb{Q}$. Similarly, two representations $\widetilde{\rho}_{0,\omega,-1}$ and $\widetilde{\rho}_{0,\omega',-1}$ are Galois conjugate over $\mathbb{Q}$ if and only if the complex numbers ω and ω' are Galois conjugate over $\mathbb{Q}$. So the total number of irreducible $\mathbb{Q}$-representations, which are obtained in this way, is equal to $2(n-1)$.

Consider the orbit $\mathcal{O}_1$. The corresponding complex irreducible representations are $\widetilde{\rho}_{1,\omega,\pm 1}$, where ω is a 2^{n-2}-th root of unity. For any two 2^{n-2}-th roots of unity ω, ω', two representations $\widetilde{\rho}_{1,\omega,1}$ and $\widetilde{\rho}_{1,\omega',1}$ are Galois conjugate over $\mathbb{Q}$ if and only if the complex numbers ω and ω' are Galois conjugate over $\mathbb{Q}$. Similarly, two representations $\widetilde{\rho}_{1,\omega,-1}$ and $\widetilde{\rho}_{1,\omega',-1}$ are Galois conjugate over $\mathbb{Q}$ if and only if the complex numbers ω and ω' are Galois conjugate over $\mathbb{Q}$. So the total number of irreducible $\mathbb{Q}$-representations, which are obtained in this way, is equal to $2(n-1)$.

Thus in this way, we obtain $4(n-1)$ many irreducible $\mathbb{Q}$-representations of G.

Consider the orbits $\mathcal{O}_k$, $2 \leq k \leq n$. For each orbit $\mathcal{O}_k$ (of size $\phi(2^k)$), and for any two 2^{n-k}-th roots of unity ω, ω', the irreducible complex representations $\widetilde{\rho}_{k,\omega}$ and $\widetilde{\rho}_{k,\omega'}$ are Galois conjugates over $\mathbb{Q}$ if and only if the complex numbers ω and ω' are Galois conjugates over $\mathbb{Q}$. Let ω_j denote a primitive 2^j-th root of unity.

Then for $2 \leq k \leq n$ and $0 \leq i \leq n-k$, let

$$\rho_{k,i} = \bigoplus_\sigma (\widetilde{\rho}_{k,\omega_i})^\sigma,$$

where σ runs over the Galois group of $\mathbb{Q}(\omega_i)$ over $\mathbb{Q}$.

Clearly the character corresponding to $\rho_{k,i}$ is rational valued. By Lemma 9.3, as the Schur index of $\widetilde{\rho}_{k,\omega_i}$ with respect to $\mathbb{Q}$ is equal to 1, then $\rho_{k,i}$ is an irreducible representation of G over $\mathbb{Q}$. Thus for each k $(2 \leq k \leq n)$, 2^{n-k} many irreducible complex representations $\widetilde{\rho}_{k,\omega}$ are partitioned into $(n-k)+1$ Galois conjugacy classes over $\mathbb{Q}$, the (direct) sum of irreducible complex representations in each Galois conjugacy class gives an irreducible rational representation of G.

Hence, the number of irreducible $\mathbb{Q}$-representations of G which do not contain G' in their kernel equals

$$\sum_{i=2}^n (n-k+1) = (n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}.$$

So, the total number of inequivalent irreducible $\mathbb{Q}$-representations of G equals $4(n-1) + n(n-1)/2$. We may summarise the above discussion in the following theorem:

Theorem 9.4. *Consider the holomorph $G = Hol(C_{2^n})$ for $n \geq 3$. For $m \geq 1$, let $\Phi_m(X)$ denote m-th cyclotomic polynomial. Then, we have the following description of its inequivalent, irreducible representations over $\mathbb{Q}$:*

1. *The set of all inequivalent irreducible $\mathbb{Q}$-representations of G whose kernels contain G' is in a bijective correspondence with the Galois conjugacy classes of complex degree 1 representations. The number of such irreducible $\mathbb{Q}$-representations is equal to $4(n-1)$.*

2. *The set of all inequivalent irreducible $\mathbb{Q}$-representations of G whose kernels do not contain G' is in a bijective correspondence with the set of all monic irreducible polynomials $\Phi_1(X), \Phi_2(X), \ldots, \Phi_{2^{n-k}}(X)$, where k runs over $\{2, \ldots, n\}$.*

3. *The total number of inequivalent irreducible $\mathbb{Q}$-representations of G equals $4(n-1) + \frac{n(n+1)}{2}$.*

9.3. **Rational Wedderburn decomposition of $\mathrm{Hol}(C_{2^n})$.** We may use the previous results and the information on Schur indices to find the Wedderburn decomposition of $\mathrm{Hol}(C_{2^n})$ over $\mathbb{Q}$. The result asserts:

Theorem 9.5. *Let $G = Hol(C_{2^n})$. For $m \geq 1$, let ζ_m denote a primitive m-th root of unity. The Wedderburn decomposition of $\mathbb{Q}[G]$ is*

$$\mathbb{Q}[G] = \bigoplus_{d|2^{n-2}} (\mathbb{Q}(\zeta_d))^{(2)} \bigoplus_{d|2^{n-2}} (\mathbb{Q}(\zeta_d))^{(2)} \bigoplus_{l=0}^{n-k} (\bigoplus_{k=2}^{n} M_{2^{k-1}}(\mathbb{Q}(\zeta_{2^l}))).$$

The proof is exactly the same as for the odd case Theorem 8.1 on using the counterparts for $p = 2$ proved above.

We illustrate the descriptions of irreducible rational representations and the Wedderburn decomposition through the following example when $p = 2$.

Example 9.6. *Let $G = Hol(C_{2^5})$. G has a presentation of the form:*

$$\langle a, b, c \, | \, a^{2^5} = b^{2^3} = c^2 = 1, b^{-1}ab = a^5, c^{-1}ac = c^{-1}, c^{-1}bc = b \rangle.$$

We have seen in Example 9.2 that corresponding to the orbit $\mathcal{O}_0$, there are sixteen irreducible representations of degree 1.

Over $\mathbb{Q}$, $X^{2^3} - 1 = \phi_1(X)\phi_2(X)\phi_{2^2}(X)\phi_{2^3}(X) = (X-1)(X+1)(X^2+1)(X^4+1)$ is the factorization into monic irreducible polynomials. As a consequence, there are 8 irreducible $\mathbb{Q}$-representations whose kernels contain G'.

We have also seen in Example 9.2 that corresponding to the orbit $\mathcal{O}_1$, there are sixteen irreducible representations of degree 1.

Over $\mathbb{Q}$, $X^{2^3} - 1 = \phi_1(X)\phi_2(X)\phi_{2^2}(X)\phi_{2^3}(X) = (X-1)(X+1)(X^2+1)(X^4+1)$ is the factorization into monic irreducible polynomials. As a consequence of that there are 8 irreducible $\mathbb{Q}$-representations whose kernels contain G'.

So the total number of irreducible $\mathbb{Q}$-representations whose kernels contain G' is equal to 16.

From Example 9.2, corresponding to the orbit $\mathcal{O}_2$, there are eight irreducible complex representations of degree 2. From those irreducible complex representations we obtain 4 irreducible representations over $\mathbb{Q}$, and their degrees are 2, 2, 4 and 8 respectively. They are in a bijective correspondence with the set of all monic irreducible factors of $X^{2^3} - 1$, over $\mathbb{Q}$.

From the same Example 9.2, corresponding to the orbit $\mathcal{O}_3$, there are four irreducible complex representations of degree 4. From those representations we obtain 3 irreducible representations over $\mathbb{Q}$, and their degrees are 4, 4 and 8 respectively. They are in a bijective correspondence with the set of all monic irreducible factors of $X^{2^2} - 1$, over $\mathbb{Q}$.

From Example 9.2, corresponding to the orbit $\mathcal{O}_4$, there are two irreducible complex representations of degree 8. From those representations we obtain 2 irreducible representations over $\mathbb{Q}$, and their degrees are 8 and 8 respectively. They are in a bijective correspondence with the set of all monic irreducible factors of $X^2 - 1$, over $\mathbb{Q}$.

By Example 9.2, *corresponding to the orbit* $\mathcal{O}_5$, *there is only one irreducible complex representation of degree* 16, *and is realizable over* $\mathbb{Q}$.

Finally, the Wedderburn decomposition of the group algebra $\mathbb{Q}[Hol(C_{2^5})]$ *is*

$$\mathbb{Q}[G] \cong \oplus_{d|2^3} (\mathbb{Q}(\zeta_d))^{(2)} \oplus_{d|2^3} (\mathbb{Q}(\zeta_d))^{(2)} \oplus M_2(\mathbb{Q}) \oplus M_2(\mathbb{Q}) \oplus M_2(\mathbb{Q}(i)) \oplus M_2(\mathbb{Q}(\zeta_8))$$

$$\oplus M_{2^2}(\mathbb{Q}) \oplus M_{2^2}(\mathbb{Q}) \oplus M_{2^2}(\mathbb{Q}(i)) \oplus M_{2^3}(\mathbb{Q}) \oplus M_{2^3}(\mathbb{Q}) \oplus M_{2^4}(\mathbb{Q}).$$

10. **For Hol**(C_{p^n})**,** p **a Prime,** $c(G)$**,** $q(G)$ **and** $p(G)$

In this final section, we compute $c(G), q(G)$ and $p(G)$ for $G = \mathrm{Hol}(C_{p^n})$, p any prime. We shall use the earlier notations. The first useful observation below is valid for all primes but the proofs are different for the odd and even cases. In some of the results below, we allow the cases $\mathrm{Hol}(C_2)$ and $\mathrm{Hol}(C_4)$ as well.

Lemma 10.1. *G has a unique minimal normal subgroup and is of order p.*

Proof. We consider the two cases p odd and $p = 2$ separately.

Case p odd.

Clearly G has a unique Sylow-p subgroup, and we denote it by P. Note that $P = \langle a, b^{p-1} \rangle \cong C_{p^n} \rtimes C_{p^{n-1}}$. Let $N = \langle a \rangle \cong C_{p^n}$ and H be a minimal normal subgroup of G. If H contains an element z of order coprime to p then $z^{-1}az = a^i$ where $i \not\equiv 1 \mod p$ (since the automorphisms of $N \cong C_{p^n}$ of the form $a \mapsto a^{1+kp}$ are of order some power of p, and so z can not act by such an automorphism). Therefore $a^{-1}z^{-1}az = a^{i-1}$ where $i - 1 \not\equiv 0 \pmod{p}$. As H is normal, then $a^{-1}z^{-1}az \in H$. It follows that $a^{i-1} \in H$ where $(i-1, p) = 1$. It follows that $\langle a^{i-1} \rangle = \langle a \rangle \subseteq H$. Also this containment is proper since H contains the element z of order coprime to p, whereas $|\langle a \rangle| = p^n$. But every subgroup of $N = \langle a \rangle = C_{p^n}$ is normal in G. This contradicts to H being a minimal normal subgroup of G. Therefore H contains elements of order some power of p. This follows that any minimal normal subgroup of G is contained in the unique Sylow-p subgroup P of G. It can be seen easily that the center of P has order p. As P is a p-group, then the center of P is the only minimal normal subgroup of P. Hence the center of P is the unique minimal normal subgroup of G

Case $p = 2$.

We have seen that $\mathrm{Aut}(C_{2^n})$ is an abelian 2-group of order 2^{n-1} (of course for $n = 1$, $\mathrm{Aut}(C_{2^n})$ is trivial). Now we assume that $n > 1$. Then there are exactly two fixed points of $\mathrm{Aut}(C_{2^n})$ in $N = \langle a \rangle \cong C_{2^n}$, and they are the identity element and the unique element $a^{2^{n-1}}$ of order 2. So the center of G is $\langle a^{2^{n-1}} \rangle \cong C_2$. As G is a 2-group and the center of G is of order 2, it follows that the center of G is the unique minimal normal subgroup of G. This completes the proof. $\square$

Lemma 10.2. *G has a unique faithful irreducible representation over $\mathbb{C}$ and is of degree $\phi(p^n)$.*

Proof. As the center of G is cyclic, it is well known that G has a faithful irreducible representation over $\mathbb{C}$. Let ρ be a faithful irreducible representation of G over $\mathbb{C}$. Let N be the cyclic normal subgroup $\langle a \rangle \cong C_{p^n}$.

In the case, $p = 2$ and $n = 1$, $|G| = 2$ and the assertion is true.

Now we assume that p is a prime and $n > 1$. As the restriction of ρ to N is faithful, then it decomposes into faithful irreducible representations. By Clifford's theorem (see [11, Theorem 6.2]), $\rho \downarrow_N \cong l(\rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k)$, where $\rho_1, \rho_2, \ldots, \rho_k$ are G-conjugates and l is a positive integer. Since the faithful irreducible representations of N over $\mathbb{C}$ correspond to the generators of N, and for any two generators, there is an automorphism of N taking one generator to the other. So all the faithful irreducible representations of N over $\mathbb{C}$ are G-conjugates. Thus $\rho \downarrow_N \cong l(\rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k)$, where $\rho_1, \rho_2, \ldots, \rho_k$ are all the faithful irreducible representations of N over $\mathbb{C}$. By Frobenius-reciprocity (see [11, Lemma 5.2]), ρ occurs as an irreducible component of $\rho_i \uparrow^G$, for all i. As the action of $\operatorname{Aut}(C_{p^n})$ on C_{p^n} is faithful, then the centralizer of N in G is N itself. Consequently, the inertia subgroup of ρ_i is N, and therefore by Clifford's theorem (see [11, Theorem 6.2]), $\rho_i \uparrow^G$ is an irreducible representation. So $\rho \cong \rho_1 \uparrow^G \cong \cdots \cong \rho_k \uparrow^G$ and $\rho \downarrow_N \cong (\rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k)$, where $\rho_1, \rho_2, \ldots, \rho_k$ are all the faithful irreducible representations of N over $\mathbb{C}$. Hence ρ is the unique faithful irreducible representation of G over $\mathbb{C}$. This compeltes the proof. $\square$

Lemma 10.3. *The unique faithful irreducible representation of G over $\mathbb{C}$ is realizable over $\mathbb{Q}$.*

Proof. Let ρ be the unique faithful irreducible representation of G over $\mathbb{C}$ with the character χ. As ρ is the unique faithful irreducible representation, then χ is a rational valued character. Consequently, the Galois conjugacy class of χ is χ itself. By Lemma 7.1 and Lemma 9.3, the Schur index of ρ with respect to $\mathbb{Q}$ is equal to 1. So ρ is realizable over $\mathbb{Q}$. This completes the proof. $\square$

We note the immediate corollary which is striking.

Corollary 10.4. *G has a unique faithful irreducible representation over $\mathbb{Q}$.*

10.1. **Computing $c(G)$ and $q(G)$.**

Lemma 10.5. *Let η be a faithful quasi-permutation representation of G over $\mathbb{C}$. Then η contains the unique faithful irreducible representation of G over $\mathbb{C}$ as an irreducible constituent.*

Proof. Let $\eta \cong \eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_r$ be the direct sum decomposition into irreducible $\mathbb{C}$-representations of G. If no η_i is faithful, then its kernel will contain the unique minimal normal subgroup of G. It follows that the unique minimal normal subgroup of G is contained in the kernel of η, and so η is not faithful, a contradiction. Thus some η_i is faithful, and by Lemma 10.2, η_i is equivalent to the unique faithful irreducible representation of G over $\mathbb{C}$. This completes the proof. $\square$

Corollary 10.6. *Let η be a faithful quasi-permutation representation of G over $\mathbb{Q}$. Then η contains the unique faithful irreducible representation of G over $\mathbb{Q}$ as an irreducible constituent.*

Lemma 10.7. *The character of the unique faithful irreducible representation of G over $\mathbb{C}$ takes values $\phi(p^n)$, $-p^{n-1}$ and 0, where ϕ is the Euler's phi function.*

Proof. Let ρ be the faithful irreducible representation of G over $\mathbb{C}$, and χ be its corresponding character. As we have seen in the proof of Lemma 10.2, ρ is induced by a faithful irreducible $\mathbb{C}$-representation of

the normal subgroup $N = \langle a \rangle \cong C_{p^n}$. So χ takes the value 0 outside H. The restriction of ρ to N is direct sum of all the faithful irreducible $\mathbb{C}$-representations of N. So $\chi(a^s)$ is the sum of s-th powers of primitive p^n-th roots of unity. If $(s,p) = 1$ then it is clear that $\chi(a) = \chi(a^s)$. So the values of χ on $N = \langle a \rangle$ are $\chi(1), \chi(a), \chi(a^p), \ldots, \chi(a^{p^{n-1}})$.

If $n = 1$, then $\chi(1) = \phi(p) = p - 1$, $\chi(a) = $ the sum of primitive p-th roots of unity $= -1$. Note that whenever $(s,p) = 1$, $\chi(a^s) = -1$.

Now we assume that $n > 1$. It is easy to see that for $i > 2$, the sum of primitive p^i-th roots of unity is equal to 0. Notice that $\chi(a^{p^{n-i}})$ is the sum of primitive p^i-th roots of unity (with multiplicity p^{n-i}). So for $i \geq 2$, $\chi(a^{p^{n-i}}) = 0$. Consider the case $i = 0$ and $i = 1$. If $i = 0$, then $\chi(a^{p^n}) = \chi(1) = \phi(p^n)$, where ϕ is the Euler's ϕ function. If $n = 1$, then $\chi(a^{p^{n-1}})$ is the sum of primitive p-th roots of unity with multiplicity p^{n-1}, and $\chi(a^{p^{n-1}}) = -p^{n-1}$. This completes the proof. $\square$

Lemma 10.8. $c(G) = q(G) = p^n$.

Proof. By Lemma 10.2, Lemma 7.1 and Lemma 9.3, G has a unique faithful irreducible representation over $\mathbb{C}$, and is of Schur index 1 over $\mathbb{Q}$. By Lemma 10.7, the minimum character value of that representation is $-p^{n-1}$. By Lemma 3.7, we get $c(G) = q(G) = p^n$. $\square$

10.2. **Computing $p(G)$.**

Lemma 10.9. $p(G) \leq p^n$.

Proof. We show that the subgroup $\text{Aut}(C_{p^n})$ is core-free in G, i.e., it does not contain any non-trivial normal subgroup of G. Suppose that $K \leq \text{Aut}(C_{p^n})$ is a non-trivial normal subgroup of G. Then K is normalized by $N = \langle a \rangle \cong C_{p^n}$. As N is normalized by K and $K \cap N = 1$, then K centralizes N. So every automorphism in K fixes every element of $\langle x \rangle$; it follows that $K = 1$. Consider the permutation representation of G on the cosets of $\text{Aut}(C_{p^n})$; there are p^n such cosets. Notice that the kernel of this permutation representation is the core of $\text{Aut}(C_{p^n})$, and which is 1. So this representation is faithful. Hence the minimum degree of a faithful permutation representation of G is not more than p^n. This completes the proof. $\square$

Finally, we have the following Theorem.

Theorem 10.10. $c(G) = q(G) = p(G) = p^n$.

Proof. As $c(G) \leq q(G) \leq p(G)$, then by Lemma 3.4 and Lemma 3.7, we get $c(G) = q(G) = p(G) = p^n$. $\square$

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