Ramanujan's mathematics - some glimpses

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Talk on January 1, 2013
At the IIIT Bangalore, India
In celebration of the National Mathematics Day

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$$\frac{e^{-2\pi/5}}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+}\,\frac{e^{-6\pi}}{1+}\cdots=\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+1}{2}.$$

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This continued fraction appeared in Ramanujan's first letter to Hardy written on January 16, 1913. Of this and some other formulae in that letter, Hardy said in 1937:

"They defeated me completely. I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them."

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$$a_0 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \cdots}}}$$

A less cumbersome notation is

$$l = a_0 + \frac{1}{a_1+} \frac{1}{a_2+} \cdots := \lim_{n \to \infty} (a_0 + \frac{1}{a_1+} \frac{1}{a_2+} \cdots \frac{1}{a_n})$$

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$\sqrt{7} = 2 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{4+}\cdots$ where the $1, 1, 1, 4$ will keep repeating; one writes briefly as $[2; \overline{1, 1, 1, 4}, \cdots]$

The continued fraction quoted in the beginning can be proved using the so-called Rogers-Ramanujan identities which are, in turn, intimately connected to the theory of partitions to which Ramanujan made fundamental contributions.

A quick peek at partitions

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For instance, $p(200)$ is almost 4×10^{12}.
So, it would be impossible to enumerate big numbers like $p(200)$ actually.

Ramanujan first observed empirically, then conjectured and finally also proved the following amazing congruences:

$$p(5n+4) \equiv 0 \pmod{5}$$

$$p(7n+5) \equiv 0 \pmod{7}$$

$$p(11n+6) \equiv 0 \pmod{11}$$

Generating functions

The partition function has a nice generating function :

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The following wonderful identities have reformulation in terms of the partition functions.

Rogers-Ramanujan identities

These are:
If $|q| < 1$, then

$$1 + \sum_{n\geq 1} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

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$$1 + \sum_{n \geq 1} \frac{q^{n(n+1)}}{(1-q)\cdots(1-q^n)} = \prod_{n \geq 0} \frac{1}{(1-q^{5n+3})(1-q^{5n+4})}.$$

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These identities are equivalent forms of:

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Partition identities are intimately related to many subjects like statistical mechanics, representation theory, modular forms etc.

What is the exact number of partitions?

$$p(n) = \frac{1}{\sqrt{2}\pi} \sum_{q=1}^{\infty} A_q(n)\sqrt{q}[\frac{d}{dx} \frac{\sinh((\pi/q)(\frac{2(x-1/24)}{3})^{1/2})}{(x-1/24)^{1/2}}]_{x=n}$$

where $A_q(n) = \sum \omega_{p,q} e^{-2np\pi i/q}$, the last sum being over p's prime to q and less than it, $\omega_{p,q}$ is a certain $24q$-th root of unity.

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The exact formula above is due to Rademacher but it was based on an asymptotic formula of Hardy and Ramanujan. Here is an interesting aspect which may not be well-known!

Thus spake Selberg

25 years back, a conference was held in TIFR Bombay to celebrate Ramanujan's centenary, where my favourite mathematician Atle Selberg (who won the Fields medalis for his elementary proof of the prime number theorem) mentioned the following words:

“If we look at Ramanujan’s first letter to Hardy, there is a statement which has relation to his later work on the partition function. He claims an approximate expression for a certain coefficient of a reciprocal of a theta series. This is the exact analogue of the leading term in Rademacher’s formula.

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Ramanujan, in whatever way, had been led to the correct term. It must have been, in a way, Hardy who did not fully trust Ramanujan's insight and intuition when he chose another expression which they developed into an asymptotic formula.

If Hardy had trusted Ramanujan more, they would have inevitably ended with the Rademacher series."

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"One might speculate, although it may be somewhat futile, about what would have happened if Ramanujan had come in contact not with Hardy but with a great mathematician of more similar talents, someone who was more inclined in the algebraic directions, for instance, Erich Hecke in Germany. This might perhaps proved much more beneficial and brought out new things in Ramanujan that did not come to fruition by his contact with Hardy. But Hardy deserves greatest credit for recognizing Ramanujan's originality and assisting him and his work in the best way he could."

Ramanujan's Tau function

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We discussed the partition function $p(n)$ which has the generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{r=1}^{\infty} \frac{1}{1-q^r}$$

Related to $p(n)$ is the function

$$\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

where $q = e^{2i\pi z}$ and $z = x + iy$ with $y > 0$.

It is outside the scope of this talk to give a good motivation of why this function is studied. Suffice it to say that this function helps in determining the number of ways of writing a given positive integer N as a sum of $2r$ squares for any $r > 1$ etc.

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$\Delta(z)$ has strong transformation properties under the transformations $z \mapsto z+1$ and $z \mapsto -1/z$; indeed $\Delta(z+1) = \Delta(z)$.

So $\Delta(z)$ has a Fourier expansion in powers of $q = e^{2i\pi z}$:

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n$$

where $\tau(n)$ is now known as Ramanujan's tau function.

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Ramanujan's tau function takes integer values and he conjectured:
$\tau(mn) = \tau(m)\tau(n)$ if m, n are coprime;
$\tau(p^{r+1}) = \tau(p^r)\tau(p) - p^{11}\tau(p^{r-1})$ for $r > 0$ and p prime;
$|\tau(p)| \leq 2p^{11/2}$ for prime p.

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The first two conjectures were proved by Mordell not very long after they were made but the third one was proved by Pierre Deligne who won a Fields medal for that work in 1974.

An elementary (but perhaps bizarre-looking) implication is that for any natural number n, the value $\tau(n)$ differs from $\sigma_{11}(n)$ by a multiple of the prime 691.
Here $\sigma_{11}(n)$ denotes the sum of the 11-th powers of the divisors of n!

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A famous unsolved conjecture of D.H.Lehmer from 1947 asserts that Ramanujan's tau function never vanishes! In fact, even the question whether p divides $\tau(p)$ for infinitely many primes p is open.

Ramanujan primes

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Ramanujan went further and analyzed the number of primes between n and $2n$ - this increases with n.

Indeed, for each r, if n_r is the smallest positive integer such that there are at least r primes between $N/2$ and N for any $N \geq n_r$, then clearly n_r is itself a prime - called the r-th Ramanujan prime.

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It is a consequence of the prime number theorem (PNT) that the n-th Ramanujan prime is between the $2n$-th prime and the $4n$-th prime for every n.

We mention in passing that the PNT is the statement that the number of primes up to x is asymptotic to $x/\log(x)$; equivalently, the n-th prime is asymptotic to $n \log(n)$.

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There are interesting open conjectures like there are arbitrarily long strings of primes which consist entirely of Ramanujan primes etc.!

Ramanujan and denesting

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Ramanujan had posed similar, more complicated problems of "*de-nesting radicals*". In this regard, he proved the following beautiful theorem :

If m, n are arbitrary, then

$$\sqrt{m\sqrt[3]{4m-8n}+n\sqrt[3]{4m+n}} =$$

$$\pm\frac{1}{3}\left(\sqrt[3]{(4m+n)^2}+\sqrt[3]{4m-8n)(4m+n)}-\sqrt[3]{2(m-2n)^2}\right).$$

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Actually, this is easy to verify simply by squaring both sides ! However, it is neither clear how this formula was arrived at nor how general it is. Are there more general formulae?

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Let $\alpha, \beta \in \mathbf{Q}^*$ *such that* α/β *is not a perfect cube in* $\mathbf{Q}$. *Then,* $\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$ *can be denested if and only if there are integers* m, n *such that* $\frac{\alpha}{\beta} = \frac{(4m-8n)m^3}{(4m+n)n^3}$.

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For instance, it follows by this theorem that $\sqrt{\sqrt[3]{3} + \sqrt[3]{2}}$ cannot be denested.

In fact, the proof of theorems like the one above use deep mathematics - for those in the know, the above theorem uses Kummer theory of Galois extensions.

Ramanujan probably did not know this theory but then he had this uncanny ability to unearth a special result which turns out each time to be the only one of its kind!

Taxicabs and Door numbers

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In the Strand magazine, Mahalanobis had seen the following problem which he mentioned to Ramanujan:

Imagine that you are on a street with houses marked 1 through n. There is a house in between such that the sum of the house numbers to the left of it equals the sum of the house numbers to its right. If n is between 50 and 500, what are n and the house number?

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Evidently, Ramanujan wanted to have some fun instead of directly giving the answer! So, what is behind this?

If the house number is r, then we have

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Multiplying by 8 and adding 1, we have $8r^2 + 1 = (2n+1)^2$.

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Their solution - the 'CHAKRAVALA' method - can be expressed using continued fractions as follows.

For a positive integer N which is not a perfect square, the continued fraction expansion of $\sqrt{N}$ looks like
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Further, the penultimate convergent $[b_0; b_1, b_2, \cdots, b_r]$; in fact, each of the convergents

$$[b_0; b_1, b_2, \cdots, b_r, 2b_0, b_1, b_2, \cdots, b_r]$$

etc. gives a solution of $x^2 - Ny^2 = -1$ or of $x^2 - Ny^2 = 1$ according as to whether the period $r + 1$ above is odd or even.

As $\sqrt{8} = [2; \overline{1, 4}]$, the convergents $\frac{3}{1}, \frac{17}{6}, \frac{99}{35}, \frac{577}{204}, \frac{3363}{1189}, \cdots$ give solutions of $x^2 - 8y^2 = 1$.

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In the above problem, $(2n+1)^2 - 8r^2 = 1$ means $(n, r) = (1, 1), (8, 6), (288, 204), (1681, 1189), \cdots$.

The unique solution for n between 50 and 500 is $n = 288$ and the house number is $r = 204$ but there are infinitely many solutions all given by the continued fraction of $\sqrt{8}$ as Ramanujan dictated!

Ramanujan and fast convergents to Pi

Ramanujan wrote a paper, 'Modular equations and approximations to π' where one of his amazing formulae reads

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Each successive term adds roughly eight more correct digits. The Borweins bettered Ramanujan's result in 1987. In an article in Scientific American of 1988, they say:

"Iterative algorithms (where the output of one cycle is taken as the input for the next) which rapidly converge to pi were, in many respects, anticipated by Ramanujan, although he knew nothing of computer programming.
Indeed, computers not only have made it possible to apply Ramanujan's work but have also helped to unravel it.
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Indeed, computers not only have made it possible to apply Ramanujan's work but have also helped to unravel it.
Sophisticated algebraic manipulations software has allowed further exploration of the road Ramanujan travelled alone and unaided 75 years ago."

A sense of incredulity prevails on reading these words when one pictures Ramanujan sitting and writing on a slate and erasing with his elbow !

Highly composite numbers

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Erdös recalls that he got access to a manuscript of Ramanujan on this topic which was not completely published because "*during the first world war, paper was expensive*"!

Probabilistic number theory

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It is now well-documented that this was the key result that inspired Erdös and Mark Kac to come up with a new subject now called probabilistic number theory.

Ramanujan sums

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He proved several nice properties of certain finite sums which are now known as Ramanujan sums. Even though Dirichlet and Dedekind had already considered these sums in the 1860's, according to G.H.Hardy, *Ramanujan was the first to appreciate the importance of the sum and to use it systematically.*

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These sums have numerous other applications in combinatorics, graph theory and even in physics; they have applications in the processing of low-frequency noise and in the study of quantum phase locking - subjects about which Ramanujan had no remarkable knowledge! So, what are these sums?

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The first remarkable property the Ramanujan sums have is that they are integers!

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They often yield very pretty elementary number-theoretic identities. Recently, mathematicians have used the theory of group representations of the permutation groups (the so-called supercharacter theory) to re-prove the old identities in a quick way and also discover new identities.

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$$\sigma_r(k) = k^r \zeta(r+1) \sum_{n=1}^{\infty} \frac{c_n(k)}{n^{r+1}}$$

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The expansion for the divisor function $d(k) = \sigma_0(k)$ can also be deduced from the above as

$$d(k) = \sum_{n=1}^{\infty} -c_n(k) \frac{\log(n)}{n}$$

For any $m \geq 1$, a generalization of the Euler's totient function is $\phi_m(k) = k^m \prod_{p|k}(1 - p^{-s})$ where the product on the right is over all the prime divisors of k; ϕ_1 is the phi function.

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Let $r_m(k) = |\{(a,b) : a, b \in \mathbf{Z}, a^m + b^m = k\}|$, the number of ways to write k as a sum of two m-th powers.
Ramanujan obtained expressions for r_2, r_4, r_6, r_8 and a few other related arithmetic functions.

For $r_2(k)$, this is:

$$r_2(k) = \pi\left(\frac{c_1(k)}{1} - \frac{c_3(k)}{3} + \frac{c_5(k)}{5} - \frac{c_7(k)}{7} + \cdots\right)$$

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Here is a curiosity: a form of the famous prime number theorem is the assertion that $\sum_n \frac{\mu(n)}{n} = 0$ and this is also equivalent to the assertion that $\sum_{n\geq 1} \frac{c_n(k)}{n} = 0$ for all k !

Ramanujan sums and cyclotomic polynomials

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Recall the Möbius function defined by
$\mu(n) = 1$,if $n = 1$;
$\mu(n) = (-1)^k$ if $n = p_1 \cdots p_k$, a product of k distinct primes;
$\mu(n) = 0$ otherwise.

Properties of $c_k(n)$

(i) $c_n(k) = c_n(-k) = c_n(n-k)$;
(ii) $c_n(0) = \phi(n)$ and $c_n(1) = \mu(n)$;
(iii) $c_n(ks) = c_n(k)$ if $(s,n) = 1$;
in particular, $c_n(s) = \mu(n)$ if $(s,n) = 1$;
(iv) $c_n(k) = c_n(k')$ if $(k,n) = (k',n)$;
in particular, $c_n(k) \equiv c_n(k')$ mod n if $k \equiv k'$ mod n;
(v) $\sum_{k=0}^{n-1} c_n(k) = 0$;
(vi) $\sum_{d|n} c_d(k) = \delta_{n|k} n$ and
$c_n(k) = \sum_{d|n} d\mu(n/d)\delta_{d|k} = \sum_{d|(n,k)} d\mu(n/d)$;
(vii) $c_{mn}(k) = c_m(k)c_n(k)$ if $(m,n) = 1$.

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The equality $c_n(k) = \sum_{d|n} d\mu(n/d)\delta_{d|k}$ is very useful; even computationally the defining sum for $c_n(k)$ requires approximately n operations where as the other sum requires roughly $\log(n)$ operations.

Recall that $c_n((k,n)) = c_n(k)$; thus, for each fixed n, one may say that the function $k \mapsto c_n(k)$ is "*even modulo n*". This is in analogy with even functions which are 'even modulo 2'. The following beautiful general theorem holds good.

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Let n be a fixed positive integer and let f be any arithmetic function which is even modulo n. Then, there exists unique numbers a_d for each $d|n$ which satisfy

$$f(k) = \sum_{d|n} a_d c_d(k)$$

In fact, for each $d|n$, we have

$$a_d = \frac{1}{n} \sum_{e|n} f(n/e) c_e(n/d)$$

Orthogonality relations:

• $\sum_{r|n} \phi(r) c_d(n/r) c_e(n/r) = n\phi(d)$ or 0 according as to whether $d = e$ or not.

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- $c_d(n/e)\phi(e) = c_e(n/d)\phi(d)$ if d, e are divisors of n.
- $\sum_{d,e|n} c_d(n/e) c_e(n/d) = nd(n)$ for divisors d, e of n.

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• For a divisor d of n, we have $\sum_{r|n} c_d(n/r)\mu(r) = n$ or 0 according as to whether $d = n$ or not.

Mock theta functions

There are several other topics like Ramanujan graphs and the circle method which we have not even alluded to due to shortage of time. We just look at one other topic - mock theta functions - which Ramanujan mentioned in his last letter to Hardy 3 months before his death and which is proving to be of deep interest today in conformal field theory, the theory of black holes and quantum invariants of some special 3-dimensional manifolds.

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In this last letter, Ramanujan talks excitedly about some functions called 'mock theta functions'. He does not define these functions but gives 17 examples and observes a certain key property they possess.

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Lest it sound incredible, let me hasten to add that most of the time, modular forms are present somewhat below the surface making things work! Properties of arithmetic nature like the analysis of the number of divisors of an integer, the number of partitions or the number of expressions of a number as a sum of squares of integers are ‘ruled’ by modular forms.

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A classical example is Jacobi's theta function $\theta(x) = \sum_{n \in \mathbf{Z}} e^{i\pi n^2 x}$; it transforms nicely under $x \mapsto -1/x$ and $x \mapsto x + 1$. It is effective in determining the number of expressions of a positive integer as a sum of 4 squares.

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Ramanujan gave examples of functions which were not modular forms (which he called mock theta functions) but which asymptotically behaved like theta functions when the argument approached a root of unity.

Since Ramanujan's death, several mathematician have studied his examples but there was no unified theory behind them. Almost 82 years later in 2002, Zwegers, in his Bonn Ph.D. work done under the supervision of the versatile mathematician Don Zagier, uncovered such a theory.

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It is outside the scope here to explain the theory but we can definitely give some elementary consequences.

Recall Ramanujan's three congruences for partitions:

$$p(5n+4) \equiv 0 \pmod{5}$$

$$p(7n+5) \equiv 0 \pmod{7}$$

$$p(11n+6) \equiv 0 \pmod{11}$$

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In order to understand these, the physicist Freeman Dyson came up with the following conjecture which was proved to be correct for the first 2 congruences by Atkin and Swinnerton-Dyer:

Call the rank of a partition to be the largest part minus the number of parts.

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Dyson's (conjectural) refinement of the fact that $p(5n+4)$ is a multiple of 5 is that the number of partitions of $5n+4$ falls into 5 equal classes - the partitions whose rank is a given a residue mod 5. The same explanation works for $p(7n+6)$ being a multiple of 7. It does not work for the 3rd congruence and Dyson later defined something called the 'crank' which we don't go into here.

The generating function for the rank is:

$$\sum_{n\geq 1}\Big(\sum_{\lambda\in P(n)} w^{rank(\lambda)}\Big)q^n = \sum_{n\geq 1}\frac{q^{n^2}}{\prod_{m\leq n}(1-wq^m)(1-w^{-1}q^m)}$$

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In fact, if one multiplies the above expression (for $w=-1$) by $q^{1/24}$, the resulting function behaves like a modular form of weight $1/2$.

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Let t be a positive integer and Q be a prime power which is co-prime to 6. Then, there exists a positive integer A and a residue class B modulo A such that for any residue class r modulo t, and any positive integer $n \equiv B$ modulo A, the number $N(r, t, n)$ of partitions of n which have rank congruent to r modulo t is a multiple of Q.

A myth

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It turns out that

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This implies

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which, implies $e^{\pi\sqrt{163}}$ is "*almost*" an integer.

The Nobel Laureate S.Chandrasekhar wrote 25 years ago: It must have been a day in April 1920, when I was not quite ten years old, when my mother told me of an item in the newspaper of the day that a famous Indian mathematician, Ramanujan by name, had died the preceding day; and she told me further that Ramanujan had gone to England some years earlier, had collaborated with some famous English mathematicians and that he had returned only very recently, and was well-known internationally for what he had achieved.

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"The fact that Ramanujan's early years were spent in a scientifically sterile atmosphere, that his life in India was not without hardships, that under circumstances that appeared to most Indians as nothing short of miraculous, he had gone to Cambridge, supported by eminent mathematicians, and had returned to India with every assurance that he would be considered, in time, as one of the most original mathematicians of the century these facts were enough, more than enough, for aspiring young Indian students to break their bonds of intellectual confinement and perhaps soar the way that Ramanujan did."

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In his short life, Ramanujan had such a wealth of ideas as to transform 20th century mathematics. These ideas continue to shape mathematics of the 21st century.

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The theta functions he called 'mock'

are subject-matter of many a talk.

He died very young - yes, he too !

He was only thirty-two !

His name prefixes the function tau.

I would describe Ramanujan's inspiring mathematics poetically as:

Ramanujan did mathematics somehow;

we still can't figure out even now.

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He died very young - yes, he too !

He was only thirty-two !

His name prefixes the function tau.

Truly, that was his last bow !

THANK YOU FOR LISTENING!