A Heuristic Argument for Hua’s Identity Using Geometric Series

In a ring, Hua observed the identity

\[ x - xyx = (x^{-1} + (y^{-1} - x)^{-1})^{-1} \]

when the inverses above exist. He used this to deduce that an additive homomorphism \( \theta \) of a division ring which satisfies \( \theta(1) = 1, \theta(x^{-1}) = \theta(x)^{-1} \), is either a ring homomorphism or an anti-homomorphism. A folklore trick is to use an analogy with geometric series to deduce in a ring that if \( a, b \) are such that \( 1 - ba \) is a unit, then so is \( 1 - ab \) and

\[ (1 - ab)^{-1} = 1 + a(1 - ba)^{-1}b. \]

Indeed, the way to discover this is to write informally

\[
(1 - ab)^{-1} = 1 + ab + abab + \cdots \\
= 1 + a(1 + ba + baba + \cdots) b \\
= 1 + a(1 - ba)^{-1}b.
\]

This heuristic argument enables us to discover the equality of the expressions \((1 - ab)^{-1}\) and \(1 + a(1 - ba)^{-1}b\). A formal proof is seen by simple verification!

Hua’s identity follows using this as:

\[
(x - xyx)^{-1} = ((1 - xy)x) = x^{-1}(1 - xy)^{-1} \\
= x^{-1}(1 + x(1 - xy)^{-1}y) = x^{-1} + (1 - yx)^{-1}y \\
= x^{-1} + (y^{-1}(1 - yx))^{-1} = x^{-1} + (y^{-1} - x)^{-1}.
\]

—Submitted by B. Sury, Indian Statistical Institute

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