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# AN ELEMENTARY PROOF OF THE HILBERT–MUMFORD CRITERION*

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**Abstract.** An elementary proof of the Hilbert–Mumford semistability criterion is given that is valid over $\mathbb{C}$. The proof of the criterion is deduced from an elementary lemma in linear algebra that may be of independent interest.

**Key words.** Semistability criterion, Algebraic one-parameter groups.

**AMS subject classifications.** 20G20, 14A25

**1. Introduction and lemma.** A classical result of geometric invariant theory is the Hilbert–Mumford semistability criterion. In one form, it deals with a linear action of a reductive algebraic group $G$ on a vector space over any field $k$. The references [1], [2], [3], [4] contain proofs over algebraically closed fields, and [5] contains a proof that works over algebraic number fields as well. Here, a transparent elementary proof is given that is valid over $\mathbb{C}$. An elementary positivity lemma in linear algebra is proved and used to deduce the proof of the criterion over $\mathbb{C}$. The lemma may be of independent interest.

We start with the following positivity lemma.

LEMMA 1.1. *Let $m_{ij}, 1 \leq i \leq r, 1 \leq j \leq n$, be integers satisfying the following property:*

*If $b_1, \ldots, b_r$ are real numbers (not all zero) such that*

$$b_1 m_{1j} + \cdots + b_r m_{rj} = 0 \ \forall \ j = 1, \ldots, n,$$

*then at least two of the $b_i$ must have opposite signs.*

*Then there are real numbers (and, therefore, also integers) $c_i$ such that*

$$m_{i1}c_1 + \cdots + m_{in}c_n > 0 \ \forall \ i \leq r.$$

*Proof.* The property above means that the kernel of the linear map $M$ from $\mathbb{R}^r$ to $\mathbb{R}^n$ given by

$$(b_1, \ldots, b_r) \mapsto \left( \sum_{i=1}^{r} b_i m_{i1}, \ldots, \sum_{i=1}^{r} b_i m_{in} \right)$$

intersects the "positive orthant" $\mathcal{O}$ in $\mathbb{R}^r$ only in zero. The assertion of the lemma amounts to the statement that the image of the transpose ${}^tM$ of $M$ intersects the

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interior of $\mathcal{O}$. Since $Ker(M)$ and the image of ${}^tM$ are orthogonal complements of each other, it suffices to show that $Ker(M)^\perp$ intersects the interior of $\mathcal{O}$. We show, more generally, that if $K$ is a subspace of $\mathbb{R}^r$ intersecting $\mathcal{O}$ only in zero, then $K^\perp$ intersects the interior of $\mathcal{O}$. We will first show that $K$ can be assumed to be of codimension 1. Suppose that $K$ has codimension $k \geq 2$. Now, $D$ denotes the image of $\mathcal{O}$ in $\mathbb{R}^r/K \equiv \mathbb{R}^k$. Since $\mathbb{R}^k \setminus \{0\}$ is connected, there is a vector $v \neq 0$ in $\mathbb{R}^k \setminus (D \cup -D)$. Hence the line $\mathbb{R}.v$ intersects $D$ only in 0. Pulling back to $\mathbb{R}^r$, we get a subspace $L$ containing $K$ in $\mathbb{R}^r$ of one more dimension such that $L \cap \mathcal{O} = \{0\}$. In the above argument, one could replace $\mathcal{O}$ more generally by a closed cone $C$ in $\mathbb{R}^r$ such that $C \cap -C = \{0\}$. We have used the fact that $D$ is again closed. Proceeding in this way, we can assume that $K$ has codimension 1. Now, let the equation of $K$ be $\sum_{i=1}^r \lambda_i X_i = 0$. Then $K \cap \mathcal{O} = \{0\}$, which evidently forces either all of the $\lambda_i > 0$ or all of the $\lambda_i < 0$. Suppose $\lambda_i > 0 \; \forall \, i$. Then $K^\perp$ is generated by the vector $(\lambda_1, \ldots, \lambda_r)$ and, obviously, $(\lambda_1, \ldots, \lambda_r)$ is in the interior of $\mathcal{O}$. This completes the proof. □

REMARK 1.2. Note that in the above, $\mathcal{O}$ can be replaced by any closed cone subtending an angle $\geq 90°$. The statement is false for cones of smaller angle.

**2. The proof of the semistability criterion.** Let us see how the lemma applies to the following statement, known as the semistability criterion.

THEOREM 2.1. *Let $G = GL(n, \mathbb{C})$ act linearly on a vector space $V$. Let $v \in V$ be a point that is not semistable, i.e., the closure $\overline{G.v}$ of the orbit $G.v$ (in the classical topology) contains $0$. Then there exists an algebraic one-parameter subgroup $A \cong GL_1$ of $G$ such that $0 \in \overline{A.v}$.*

*Proof.* We have the (Cartan) decomposition $G = KTK$, where $K$ is $U(n)$ and $T$ is the maximal diagonal torus—this can be easily deduced from the spectral theorem for Hermitian operators. From this decomposition, it immediately follows that $0 \in \overline{T.kv}$ for some $k \in K$. It is enough to get a multiplicative one-parameter subgroup $A$, as in the theorem, for the vector $kv$, since the group $k^{-1}Ak$ works for $v$ then. So, we rename $kv$ as $v$ and work with it without any loss of generality. Write $v = \sum_{i=1}^r v_{\chi_i}$, where

$$v_{\chi_i} \in V_{\chi_i} := \{w \in V : t.w = \chi_i(t)w \;\forall\; t \in T\}$$

for some algebraic characters $\chi_i : T \to \mathbb{C}^*$. Let $\chi_i = \sum_{j=1}^n m_{ij}\lambda_j$, where $\lambda_j : T \to \mathbb{C}^*$ is the character $\mathrm{diag}(t_1, \ldots, t_n) \mapsto t_j$; here $m_{ij}$ are integers. So, we have

$$t.v = \sum_{i=1}^r t_1^{m_{i1}} \cdots t_n^{m_{in}} v_{\chi_i}$$

for any $t = \mathrm{diag}(t_1, \ldots, t_n) \in T$.

*Claim.* If $b_1, \ldots, b_n$ are real numbers (not all zero) such that

$$b_1 m_{1j} + \cdots + b_r m_{rj} = 0 \;\forall\; j = 1, \ldots, n,$$

then at least two of the $b_i$'s are of opposite signs.

To prove the claim, we suppose, on the contrary, that there are $b_i$ (not all zero) all of the same sign such that

$$b_1 m_{1j} + \cdots + b_r m_{rj} = 0 \ \forall \ j = 1, \ldots, n.$$

Let $t^{(k)} = \mathrm{diag}(t_1^{(k)}, \ldots, t_n^{(k)}) \in T$ be a sequence such that $t^{(k)}.v \to 0$ as $k \to \infty$. Therefore, $\forall \ i \leq r$,

$$(1) \qquad (t_1^{(k)})^{m_{i1}} \cdots (t_n^{(k)})^{m_{in}} \to 0 \ \text{ as } \ k \to \infty \ .$$

Suppose, now, that $b_1 \neq 0$. Then

$$-m_{1j} = \frac{b_2}{b_1} m_{2j} + \cdots + \frac{b_r}{b_1} m_{rj} \ \forall \ j \leq n$$

so that

$$(2) \qquad t_1^{-m_{11}} \cdots t_n^{-m_{1n}} = (t_1^{m_{21}} \cdots t_n^{m_{2n}})^{\frac{b_2}{b_1}} \cdots (t_1^{m_{r1}} \cdots t_n^{m_{rn}})^{\frac{b_r}{b_1}} \ .$$

Since $\frac{b_i}{b_1} \geq 0 \ \forall \ i \geq 2$, and not all of them are zero, the right-hand side of (2) tends to 0 as $(t_1, \ldots, t_n)$ runs over the sequence $(t_1^{(k)}, \ldots, t_n^{(k)})$. Looking at the left-hand side of (2), we have a contradiction of (1). This proves the claim.

Let us continue with the proof of the theorem. First, an application of the lemma ensures the existence of integers $c_i$ such that

$$(3) \qquad m_{i1} c_1 + \cdots + m_{in} c_n > 0 \ \forall \ i \leq r.$$

Consider the algebraic one-parameter subgroup $GL_1$ in $T$ given by the homomorphism

$$\theta : GL_1 \to T, \ t \mapsto diag(t^{c_1}, \ldots, t^{c_n}).$$

Note that $\theta(t).v = \sum_{i=1}^r t^{m_{i1}c_1 + \cdots + m_{in}c_n} v_{\chi_i}$. By (3), it is clear that $0 \in \overline{\theta(GL_1).v}$. □

The following is a corollary of the proof.

COROLLARY 2.2. *Let* $(,)$ *denote the nondegenerate pairing*

$$X_*(T) \times X^*(T) \to \mathbb{Z},$$

*where* $X_*(T) = Hom(GL_1, T)$ *is the group of multiplicative one-parameter subgroups and* $X^*(T) = Hom(T, GL_1)$ *is the character group of* $T$. *Then* $\theta \in X_*(T)$ *satisfies* $0 \in \overline{\theta(GL_1).v}$ *if and only if* $(\theta, \chi_i) > 0 \ \forall \ i \leq r$.

REMARK 2.3. For the other classical groups over $\mathbb{C}$, the proof is completely similar.

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