

GAUSS DECOMPOSITION FOR CHEVALLEY GROUPS, REVISITED

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Communicated by Alireza Abdollahi

ABSTRACT. In the 1960's Noboru Iwahori and Hideya Matsumoto, Eiichi Abe and Kazuo Suzuki, and Michael Stein discovered that Chevalley groups $G = G(\Phi, R)$ over a semilocal ring admit remarkable Gauss decomposition $G = T U U^{-1} U$, where $T = T(\Phi, R)$ is a split maximal torus, whereas $U = U(\Phi, R)$ and $U^{-1} = U^{-1}(\Phi, R)$ are unipotent radicals of two opposite Borel subgroups $B = B(\Phi, R)$ and $B^{-1} = B^{-1}(\Phi, R)$ containing T . It follows from the classical work of Hyman Bass and Michael Stein that for classical groups Gauss decomposition holds under weaker assumptions such as $\text{sr}(R) = 1$ or $\text{asr}(R) = 1$. Later the third author noticed that condition $\text{sr}(R) = 1$ is necessary for Gauss decomposition. Here, we show that a slight variation of Tavgen's rank reduction theorem implies that for the elementary group $E = E(\Phi, R)$ condition $\text{sr}(R) = 1$ is also sufficient for Gauss decomposition. In other words, $E = H U U^{-1} U$, where $H = H(\Phi, R) = T \cap E$. This surprising result shows that stronger conditions on the ground ring, such as being semi-local, $\text{asr}(R) = 1$, $\text{sr}(R, \Lambda) = 1$, etc., were only needed to guarantee that for simply connected groups $G = E$, rather than to verify the Gauss decomposition itself.

1. Introduction

Let Φ be a reduced irreducible root system, R be a commutative ring with 1 and $G(\Phi, R)$ be a Chevalley group of type Φ over R . We fix a split maximal torus $T(\Phi, R)$ in $G(\Phi, R)$ and a pair $B(\Phi, R)$ and $B^{-1}(\Phi, R)$ of opposite Borel subgroups containing $T(\Phi, R)$. Further, let $U(\Phi, R)$ and $U^{-1}(\Phi, R)$ be the unipotent radicals of $B(\Phi, R)$ and $B^{-1}(\Phi, R)$, respectively.

The whole theory of Chevalley groups over semi-local rings rests upon the following analogue of Gauss decomposition established by Noboru Iwahori and Hideya Matsumoto [19], Eiichi Abe and

MSC(2010): Primary: 20G25; Secondary: 20F12.

Keywords: Chevalley groups, elementary Chevalley groups, triangular factorisations, rings of stable rank 1, parabolic subgroups, Gauss decomposition, commutator width.

Received: 24 September 2011, Accepted: 12 October 2011.

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Kazuo Suzuki [1, 2], and by Michael Stein [35]. In fact, it plays the same role in this case, as Bruhat decomposition does over fields. Let R be a semi-local ring. Then one has the following decomposition

$$G(\Phi, R) = T(\Phi, R)U(\Phi, R)U^-(\Phi, R)U(\Phi, R).$$

For the simply connected Chevalley group $G_{\text{sc}}(\Phi, R)$ its maximal torus $T_{\text{sc}}(\Phi, R)$ is contained in the elementary subgroup

$$E(\Phi, R) = \langle U(\Phi, R), U^-(\Phi, R) \rangle,$$

generated by $U(\Phi, R)$ and $U^-(\Phi, R)$. In particular, Gauss decomposition implies that for a simply connected group over a semi-local ring one has $G_{\text{sc}}(\Phi, R) = E_{\text{sc}}(\Phi, R)$. In other words, for semi-local rings

$$K_1(\Phi, R) = G_{\text{sc}}(\Phi, R)/E_{\text{sc}}(\Phi, R)$$

is trivial.

In general, when the group is not simply connected or the ring R is not semi-local, elementary subgroup $E(\Phi, R)$ can be strictly smaller than the Chevalley group $G(\Phi, R)$ itself. In fact, for a non simply connected group even the subgroup $H(\Phi, R)$ spanned by semi-simple root elements $h_\alpha(\varepsilon)$, $\alpha \in \Phi$, $\varepsilon \in R^*$, where R^* is the multiplicative group of the ring R . Clearly,

$$H(\Phi, R) = T(\Phi, R) \cap E(\Phi, R),$$

can be strictly smaller than the torus $T(\Phi, R)$ itself.

In [45] the third author observed that condition $\text{sr}(R) = 1$ is *necessary* for Gauss decomposition to hold for a Chevalley group $G(\Phi, R)$ over a ring R and made the following remark: “One might hope that the condition $\text{sr}(R) = 1$ is also *sufficient* to prove that Chevalley groups of all types over R admit Gauss decomposition (some experts believe that this is rather unlikely).”

In the present paper, which is a sequel to our paper [52], we show that a slight modification of the same argument by Oleg Tavgen [40], immediately gives the following surprising result, asserting that condition $\text{sr}(R) = 1$ is necessary and *sufficient* for the *elementary* Chevalley group $E(\Phi, R)$ to admit Gauss decomposition.

Theorem 1.1. *Let Φ be a reduced irreducible root system and R be a commutative ring such that $\text{sr}(R) = 1$. Then the elementary Chevalley group $E(\Phi, R)$ admits Gauss decomposition*

$$E(\Phi, R) = H(\Phi, R)U(\Phi, R)U^-(\Phi, R)U(\Phi, R).$$

Conversely, if Gauss decomposition holds for some [elementary] Chevalley group, then $\text{sr}(R) = 1$.

The proof of this result follows *exactly* the same lines as the proof of Theorem 1 in [52]¹. Now, we are interested not in unitriangular factorisations, but in triangular ones. Thus, we have to modify

¹The statement of Theorem 1 in the Russian original of [52] contains a very unfortunate misprint. The formula there reads as the unitriangular factorisation of length 4 for the simply connected Chevalley group $G(\Phi, R)$. Of course, the rest of that paper and the proofs there discuss such a factorisation for the *elementary* Chevalley group $E(\Phi, R)$. This inconsistency is corrected in the English version.

induction base, which now becomes even easier, and superficially the reduction step itself. It is a total mystery, why we failed to notice these obvious modifications when writing [52].

What is truly amazing here, is that as in [52] the usual *linear* stable rank condition works for groups of all types! Before, this decomposition was known for the Chevalley group $G(\Phi, R)$ itself, under the following [stronger!] assumptions on R .

- For $\Phi = A_l, C_l$ under $\text{sr}(R) = 1$.
- For $\Phi = B_l, D_l$ under $\text{asr}(R) = 1$, or under an appropriate unitary/form ring stable rank condition $\Lambda\text{sr}(R) = 1$, $\text{sr}(R, \Lambda) = 1$, etc.
- For exceptional groups, when R is semi-local.

These results, especially for classical groups, were immediately obvious, after the introduction of the corresponding stability conditions by Hyman Bass [7] and by Michael Stein [35]². Under these conditions Gauss decomposition for classical groups was [re]discovered dozens of times, and in the last section we provide assorted references.

Our Theorem 1.1 divorces existence of Gauss decomposition from the triviality of $K_1(\Phi, R)$. In fact, it shows that these stronger stability conditions are only needed to ensure that $G_{\text{sc}}(\Phi, R) = E_{\text{sc}}(\Phi, R)$, but are not necessary for the *elementary* Chevalley group $E(\Phi, R)$ to admit Gauss decomposition!

Let us state some immediate corollaries of Theorem 1.

Corollary 1.2. *Let Φ be a reduced irreducible root system and R be a commutative ring such that $\text{sr}(R) = 1$. Then any element g of the elementary Chevalley group $E(\Phi, R)$ is conjugate to an element of*

$$U(\Phi, R)H(\Phi, R)U^-(\Phi, R).$$

Corollary 1.3. *Let Φ be a reduced irreducible root system and R be a commutative ring such that $\text{sr}(R) = 1$. Then the elementary Chevalley group $E(\Phi, R)$ admits unitriangular factorisation*

$$E(\Phi, R) = U(\Phi, R)U^-(\Phi, R)U(\Phi, R)U^-(\Phi, R)U(\Phi, R)$$

of length 5.

Notice that this corollary is a very broad generalisation of results on unitriangular factorisations obtained by Martin Liebeck, Laszlo Pyber, Laszlo Babai and Nikolay Nikolov [23, 4], with a *terribly* much easier proof. Actually, Theorem 1 of [52], which is proven by essentially the same method, but starts with a slightly more precise induction base, asserts that under condition $\text{sr}(R) = 1$ the elementary Chevalley group $E(\Phi, R)$ admits unitriangular factorisation

$$G(\Phi, R) = U(\Phi, R)U^-(\Phi, R)U(\Phi, R)U^-(\Phi, R)$$

of length 4.

²Actually, absolute stable rank as such was first introduced by David Estes and Jack Ohm [17], but its role in the proof of stability results for orthogonal groups was noted only by Stein.

The present note is a by-product of our joint work on arithmetic problems of our cooperative Russian–Indian project “Higher composition laws, algebraic K-theory and exceptional groups” at the Saint Petersburg State University, Tata Institute of Fundamental Research (Mumbai) and Indian Statistical Institute (Bangalore).

In § 1 we recall some fundamentals concerning $\text{sr}(R) = 1$ and similar stability conditions. In §§ 2 and 3 we introduce basic notation related to Chevalley groups and their parabolic subgroups. In § 4 we prove another version of Tavgen’s rank reduction theorem, which immediately implies Theorem 1.1. Finally, in § 5 we discuss existing literature on the subject and state several unsolved problems.

The proofs in the present paper, as also in [52], are based on a slight variation of an idea by Oleg Tavgen. We finished this paper at the end of July 2011, and just as we planned to send it to Oleg, we were deeply shocked and grieved by the news of his sudden and untimely death. We dedicate this paper to his memory.

2. Stability Conditions

Recall that a ring R has **stable rank 1**, if for all $x, y \in R$, which generate R as a *right* ideal, there exists a $z \in R$ such that $x + yz$ is *right* invertible. In this case we write $\text{sr}(R) = 1$.

It is classically known that rings of stable rank 1 are actually *weakly finite* (Kaplansky–Lenstra theorem), so that in their definition one could from the very start require that $x + yz \in R^*$. Since for the linear case the result is well known, and Chevalley groups of other types only exist over commutative rings, from here on we assume that the ring R is commutative, in which case the proof below at the same time demonstrates that $\text{SL}(2, R) = E(2, R)$.

Special linear groups are simply connected Chevalley groups of type A_l . In particular, $\text{SL}(2, R)$ and $E(2, R)$ are the simply connected Chevalley group of type A_1 and its elementary subgroup. All other notations are modified accordingly. Thus,

$$U(2, R) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \quad U^-(2, R) = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$$

refers to the groups $U(A_1, R)$ and $U^-(A_1, R)$, under the above identification of $G_{\text{sc}}(A_1, R) = \text{SL}(2, R)$, etc. Similarly, $H(A_1, R) = T(A_1, R)$ is now identified with

$$H(2, R) = \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \mid \varepsilon \in R^* \right\},$$

in other words, with the group of diagonal matrices with determinant 1, usually denoted by $\text{SD}(2, R)$.

The proof of the following lemma is essentially contained already in [7] and was rediscovered *dozens* of times after that. The lemma itself, induction base, is the only step in the proof of Theorem 1 that invokes stability condition.

Lemma 2.1. *Let R be a commutative ring of stable rank 1. Then*

$$\text{SL}(2, R) = E(2, R) = U(2, R)H(2, R)U^-(2, R)U(2, R).$$

Proof 2.2. Consider an arbitrary matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, R)$. Since rows of an invertible matrix are unimodular, one has $cR + dR = R$. Since $\mathrm{sr}(R) = 1$, there exists such an $z \in R$, that $d + cz \in R^*$.

Thus,

$$\begin{pmatrix} 1 & -(b + az)(d + cz)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (d + cz)^{-1} & 0 \\ c & d + cz \end{pmatrix},$$

as claimed.

3. Chevalley Groups

Our notation pertaining to Chevalley groups are utterly standard and coincide with the ones used in [46, 49], where one can find many further references.

Let as above Φ be a reduced irreducible root system of rank l , $W = W(\Phi)$ be its Weyl group and \mathcal{P} be a weight lattice intermediate between the root lattice $\mathcal{Q}(\Phi)$ and the weight lattice $\mathcal{P}(\Phi)$. Further, we fix an order on Φ and denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$, Φ^+ and Φ^- the corresponding sets of fundamental, positive and negative roots, respectively. Our numbering of the fundamental roots follows Bourbaki. Finally, let R be a commutative ring with 1, as usual, R^* denotes its multiplicative group.

It is classically known that with these data one can associate the Chevalley group $G = G_{\mathcal{P}}(\Phi, R)$, which is the group of R -points of an affine groups scheme $G_{\mathcal{P}}(\Phi, -)$, known as the Chevalley—Demazure group scheme. In the case $\mathcal{P} = \mathcal{P}(\Phi)$ the group G is called simply connected and is denoted by $G_{\mathrm{sc}}(\Phi, R)$. In the opposite case $\mathcal{P} = \mathcal{Q}(\Phi)$ the group G is called adjoint and is denoted by $G_{\mathrm{ad}}(\Phi, R)$. Many results do not depend on the lattice \mathcal{P} and hold for all groups of a given type Φ . In such cases we omit any reference to \mathcal{P} in the notation and denote by $G(\Phi, R)$ any Chevalley group of type Φ over R .

In what follows, we fix a split maximal torus $T(\Phi, -)$ of the group scheme $G(\Phi, -)$ and set $T = T(\Phi, R)$. As usual, X_α , $\alpha \in \Phi$, denotes a unipotent root subgroup in G , elementary with respect to T . We fix isomorphisms $x_\alpha : R \rightarrow X_\alpha$, so that $X_\alpha = \{x_\alpha(\xi) \mid \xi \in R\}$, which are interrelated by the Chevalley commutator formula, see [11, 37]. Further, $E(\Phi, R)$ denotes the elementary subgroup of $G(\Phi, R)$, generated by all root subgroups X_α , $\alpha \in \Phi$.

Elements $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, are called [elementary] unipotent root elements or, for short, simply root unipotents. Next, let $\alpha \in \Phi$ and $\varepsilon \in R^*$. As usual, we set $w_\alpha(\varepsilon) = x_\alpha(\varepsilon)x_{-\alpha}(-\varepsilon^{-1})x_\alpha(\varepsilon)$ and $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(1)^{-1}$. Elements $h_\alpha(\varepsilon)$ are called semisimple root elements. Define

$$H(\Phi, R) = \langle h_\alpha(\varepsilon), \alpha \in \Phi, \varepsilon \in R^* \rangle.$$

For a simply connected group one has

$$H_{\mathrm{sc}}(\Phi, R) = T_{\mathrm{sc}}(\Phi, R) = \mathrm{Hom}(\mathcal{P}(\Phi), R^*).$$

In general, though, $H(\Phi, R) = T(\Phi, R) \cap E(\Phi, R)$ can be — and even over a field usually is! — strictly smaller, than $T(\Phi, R)$.

Finally, let $N = N(\Phi, R)$ be the algebraic normaliser of the torus $T = T(\Phi, R)$, i. e. the subgroup, generated by $T = T(\Phi, R)$ and all elements $w_\alpha(1)$, $\alpha \in \Phi$. The factor-group N/T is canonically isomorphic to the Weyl group W , and for each $w \in W$ we fix its preimage $n_w \in N$. Clearly, such a preimage can be taken in $E(\Phi, R)$. Indeed, for a root reflection w_α one can take $w_\alpha(1) \in E(\Phi, R)$ as its preimage, any element w of the Weyl group can be expressed as a product of root reflections. In particular, we get the following classical result.

Lemma 3.1. *The elementary Chevalley group $E(\Phi, R)$ is generated by unipotent root elements $x_\alpha(\xi)$, $\alpha \in \pm\Pi$, $\xi \in R$, corresponding to the fundamental and negative fundamental roots.*

Further, let $B = B(\Phi, R)$ and $B^- = B^-(\Phi, R)$ be a pair of opposite Borel subgroups containing $T = T(\Phi, R)$, standard with respect to the given order. Recall that B and B^- are semidirect products $B = T \ltimes U$ and $B^- = T \ltimes U^-$, of the torus T and their unipotent radicals

$$U = U(\Phi, R) = \langle x_\alpha(\xi), \alpha \in \Phi^+, \xi \in R \rangle,$$

$$U^- = U^-(\Phi, R) = \langle x_\alpha(\xi), \alpha \in \Phi^-, \xi \in R \rangle.$$

Here, as usual, for a subset X of a group G one denotes by $\langle X \rangle$ the subgroup in G generated by X . Semidirect product decomposition of B amounts to saying that $B = TU = UT$, and at that $U \trianglelefteq B$ and $T \cap U = 1$. Similar facts hold with B and U replaced by B^- and U^- . Sometimes, to speak of both subgroups U and U^- simultaneously, we denote $U = U(\Phi, R)$ by $U^+ = U^+(\Phi, R)$.

In general, one can associate a subgroup $E(S) = E(S, R)$ to any closed set $S \subseteq \Phi$. Recall that a subset S of Φ is called *closed*, if for any two roots $\alpha, \beta \in S$ the fact that $\alpha + \beta \in \Phi$, implies that already $\alpha + \beta \in S$. Now, we define $E(S) = E(S, R)$ as the subgroup generated by all elementary root unipotent subgroups X_α , $\alpha \in S$:

$$E(S, R) = \langle x_\alpha(\xi), \alpha \in S, \xi \in R \rangle.$$

In this notation, U and U^- coincide with $E(\Phi^+, R)$ and $E(\Phi^-, R)$, respectively. The groups $E(S, R)$ are particularly important in the case where S is a *special* ($=$ *unipotent*) set of roots; in other words, where $S \cap (-S) = \emptyset$. In this case $E(S, R)$ coincides with the *product* of root subgroups X_α , $\alpha \in S$, in some/any fixed order.

Let again $S \subseteq \Phi$ be a closed set of roots. Then S can be decomposed into a disjoint union of its *reductive* ($=$ *symmetric*) part S^r , consisting of those $\alpha \in S$, for which $-\alpha \in S$, and its *unipotent* part S^u , consisting of those $\alpha \in S$, for which $-\alpha \notin S$. The set S^r is a closed root subsystem, whereas the set S^u is special. Moreover, S^u is an *ideal* of S , in other words, if $\alpha \in S$, $\beta \in S^u$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in S^u$. *Levi decomposition* asserts that the group $E(S, R)$ decomposes into semidirect product $E(S, R) = E(S^r, R) \ltimes E(S^u, R)$ of its *Levi subgroup* $E(S^r, R)$ and its *unipotent radical* $E(S^u, R)$.

4. Elementary Parabolic Subgroups

The main role in the proof of Theorem 1.1 is played by Levi decomposition for elementary parabolic subgroups. Denote by $m_k(\alpha)$ the coefficient of α_k in the expansion of α with respect to the fundamental

roots:

$$\alpha = \sum m_k(\alpha)\alpha_k, \quad 1 \leq k \leq l.$$

Now, fix an $r = 1, \dots, l$ – in fact, in the reduction to smaller rank it suffices to employ only terminal parabolic subgroups, even only the ones corresponding to the first and the last fundamental roots, $r = 1, l$. Denote by

$$S = S_r = \{\alpha \in \Phi, m_r(\alpha) \geq 0\}$$

the r -th standard parabolic subset in Φ . As usual, the reductive part $\Delta = \Delta_r$ and the special part $\Sigma = \Sigma_r$ of the set $S = S_r$ are defined as

$$\Delta = \{\alpha \in \Phi, m_r(\alpha) = 0\}, \quad \Sigma = \{\alpha \in \Phi, m_r(\alpha) > 0\}.$$

The opposite parabolic subset and its special part are defined similarly

$$S^- = S_r^- = \{\alpha \in \Phi, m_r(\alpha) \leq 0\}, \quad \Sigma^- = \{\alpha \in \Phi, m_r(\alpha) < 0\}.$$

Obviously, the reductive part S_r^- equals Δ .

Denote by P_r the *elementary* maximal parabolic subgroup of the elementary group $E(\Phi, R)$. By definition,

$$P_r = E(S_r, R) = \langle x_\alpha(\xi), \alpha \in S_r, \xi \in R \rangle.$$

Now Levi decomposition asserts that the group P_r can be represented as the semidirect product

$$P_r = L_r \ltimes U_r = E(\Delta, R) \ltimes E(\Sigma, R)$$

of the elementary Levi subgroup $L_r = E(\Delta, R)$ and the unipotent radical $U_r = E(\Sigma, R)$. Recall that

$$L_r = E(\Delta, R) = \langle x_\alpha(\xi), \alpha \in \Delta, \xi \in R \rangle,$$

Whereas

$$U_r = E(\Sigma, R) = \langle x_\alpha(\xi), \alpha \in \Sigma, \xi \in R \rangle.$$

A similar decomposition holds for the opposite parabolic subgroup P_r^- , whereby the Levi subgroup is the same as for P_r , but the unipotent radical U_r is replaced by the opposite unipotent radical $U_r^- = E(-\Sigma, R)$

As a matter of fact, we use Levi decomposition in the following form. It will be convenient to slightly change the notation and write $U(\Sigma, R) = E(\Sigma, R)$ and $U^-(\Sigma, R) = E(-\Sigma, R)$.

Lemma 4.1. *The group $\langle U^\sigma(\Delta, R), U^\rho(\Sigma, R) \rangle$, where $\sigma, \rho = \pm 1$, is the semidirect product of its normal subgroup $U^\rho(\Sigma, R)$ and the complementary subgroup $U^\sigma(\Delta, R)$.*

In other words, it is asserted here that the subgroups $U^\pm(\Delta, R)$ normalise each of the groups $U^\pm(\Sigma, R)$, so that, in particular, one has the following four equalities for products

$$U^\pm(\Delta, R)U^\pm(\Sigma, R) = U^\pm(\Sigma, R)U^\pm(\Delta, R),$$

and, furthermore, the following four obvious equalities for intersections hold:

$$U^\pm(\Delta, R) \cap U^\pm(\Sigma, R) = 1.$$

In particular, one has the following decompositions:

$$U(\Phi, R) = U(\Delta, R) \ltimes U(\Sigma, R), \quad U^-(\Phi, R) = U^-(\Delta, R) \ltimes U^-(\Sigma, R).$$

5. Proof of Theorem 1

The following result, like Theorem 3 of [52], is another minor elaboration of Proposition 1 from the paper by Oleg Tavgen [40]. Tavgen considered *unitriangular factorisations*, in other words, expressions of $E(\Phi, R)$ as products of $U(\Phi, R)$ and $U^-(\Phi, R)$,

$$E(\Phi, R) = U(\Phi, R)U^-(\Phi, R) \dots U^\pm(\Phi, R).$$

Here, we are interested in *triangular factorisations*, in other words expressions of $E(\Phi, R)$ as products of

$$B(\Phi, R) \cap E(\Phi, R) = H(\Phi, R)U(\Phi, R)$$

and

$$B^-(\Phi, R) \cap E(\Phi, R) = H(\Phi, R)U^-(\Phi, R).$$

However, since $T(\Phi, R)$ — and, a fortiori, $H(\Phi, R)$ — normalises $U(\Phi, R)$ and $U^-(\Phi, R)$, we can collect all toral factors together, and consider factorisations of the form

$$E(\Phi, R) = H(\Phi, R)U(\Phi, R)U^-(\Phi, R) \dots U^\pm(\Phi, R).$$

The length of such a decomposition is the number of distinct *triangular* factors, in other words, the number of $U^\pm(\Phi, R)$ occurring in this product.

Theorem 5.1. *Let Φ be a reduced irreducible root system of rank $l \geq 2$, and R be a commutative ring. Suppose that for the two subsystems $\Delta = \Delta_1, \Delta_l$, the elementary Chevalley group $E(\Delta, R)$ admits a triangular factorisation*

$$E(\Delta, R) = H(\Delta, R)U(\Delta, R)U^-(\Delta, R) \dots U^\pm(\Delta, R)$$

of length L . Then the elementary Chevalley group $E(\Phi, R)$ admits triangular factorisation

$$E(\Phi, R) = H(\Phi, R)U(\Phi, R)U^-(\Phi, R) \dots U^\pm(\Phi, R)$$

of the same length L .

The leading idea of Tavgen's proof is very general and beautiful, and works in many other related situations. It relies on the fact that for systems of rank ≥ 2 every fundamental root falls into the subsystem of smaller rank obtained by dropping either the first or the last fundamental root. Similar consideration was used by Eiichi Abe and Kazuo Suzuki [1] and [2] to extract root unipotents in their description of normal subgroups. Compare also the simplified proof of Gauss decomposition with prescribed semisimple part by Vladimir Chernousov, Erich Ellers, and Nikolai Gordeev [13].

Remark. As was pointed out by the referee, the argument below applies without any modification in a much more general setting. Namely, it suffices to assume that the required decomposition holds

for elementary Chevalley groups $E(\Delta, R)$ for *some* subsystems $\Delta \leq \Phi$, whose union contains all fundamental roots. These subsystems do not have to be terminal, or even irreducible, for that matter. However, we do not see any immediate application of this more general form of Theorem 5.1, since the use of terminal subsystems invariably gives better results depending on weaker stability conditions.

Let us reproduce the details of the argument. By definition

$$Y = H(\Phi, R)U(\Phi, R)U^-(\Phi, R) \dots U^\pm(\Phi, R)$$

is a *subset* in $E(\Phi, R)$. Usually, the easiest way to prove that a subset $Y \subseteq G$ coincides with the whole group G consists in the following.

Lemma 5.2. *Assume that $Y \subseteq G$, $Y \neq \emptyset$, and $X \subseteq G$ be a symmetric generating set. If $XY \subseteq Y$, then $Y = G$.*

Proof 5.3 (Proof of Theorem 5.1). *By Lemma 3.1 the group G is generated by the fundamental root elements*

$$X = \{x_\alpha(\xi) \mid \alpha \in \pm\Pi, \xi \in R\}.$$

Thus, by Lemma 5.2 it suffices to prove that $XY \subseteq Y$.

Let us fix a fundamental root unipotent $x_\alpha(\xi)$. Since $\text{rk}(\Phi) \geq 2$, the root α belongs to at least one of the subsystems $\Delta = \Delta_r$, where $r = 1$ or $r = l$, generated by all fundamental roots, except for the first or the last one, respectively. Set $\Sigma = \Sigma_r$ and express $U^\pm(\Phi, R)$ in the form

$$U(\Phi, R) = U(\Delta, R)U(\Sigma, R), \quad U^-(\Phi, R) = U^-(\Delta, R)U^-(\Sigma, R).$$

Using Lemma 4.1 we see that

$$Y = H(\Phi, R)U(\Delta, R)U^-(\Delta, R) \dots U^\pm(\Delta, R).$$

$$U(\Sigma, R)U^-(\Sigma, R) \dots U^\pm(\Sigma, R).$$

Since $\alpha \in \Delta$, one has $x_\alpha(\xi) \in E(\Delta, R)$, so that the inclusion $x_\alpha(\xi)Y \subseteq Y$ immediately follows from the assumption.

6. Final Remarks

A major application of Theorem 1.1 we have in mind, is the commutator width of elementary Chevalley group.

One of the major recent advances was the positive solution of Ore's conjecture, asserting that any element of $E_{\text{ad}}(\Phi, K)$ over a field K is a single commutator, whenever this group is simple [as an abstract group]. For large fields, say, all fields containing ≥ 8 elements, this was proven by Erich Ellers and Nikolai Gordeev [16], using their remarkable results on Gauss decomposition with prescribed semi-simple part, see [13] and references there. For small fields, this result was obtained by Martin Liebeck, Eamonn O'Brien, Aner Shalev and Pham Huu Tiep [22], using very delicate character estimates. It

is essential that the groups are *adjoint*. Beware, that in general one may need *two* commutators to express some elements of $E_{\text{sc}}(\Phi, K)$.

We believe that solution of the following two problems is now at hand. Compare the works of Arlinghaus, Leonid Vaserstein, Ethel Wheland and You Hong [43, 44, 53, 3], where this is essentially done for classical groups, over rings subject to $\text{sr}(R) = 1$ or some stronger stability conditions, and the work by Nikolai Gordeev and Jan Saxl [18], where this is essentially done over local rings.

Problem 6.1. *Under assumption $\text{sr}(R) = 1$ prove that any element of $E_{\text{ad}}(\Phi, R)$ is a product of ≤ 2 commutators in $G_{\text{ad}}(\Phi, R)$.*

Problem 6.2. *Under assumption $\text{sr}(R) = 1$ prove that any element of $E(\Phi, R)$ is a product of ≤ 3 commutators in $E(\Phi, R)$.*

It may well be that under this assumption the commutator width of $E(\Phi, R)$ is always ≤ 2 , but so far we were unable to control details concerning semisimple factors.

It seems, that one can apply the same argument to higher stable ranks. Solution of the following problem would be a generalisation of [14], Theorem 4.

Problem 6.3. *If the stable rank $\text{sr}(R)$ of R is finite, and for some $m \geq 2$ the elementary linear group $E(m, R) = E_{\text{sc}}(A_{m-1}, R)$ has bounded word length with respect to elementary generators, then for all Φ of sufficiently large rank one has*

$$E(\Phi, R) = (U(\Phi, R)U^-(\Phi, R))^3.$$

In particular, it would follow that in this case any element of $E(\Phi, R)$ is a product of ≤ 6 commutators. In fact, we expect a much better result.

Problem 6.4. *If the stable rank $\text{sr}(R)$ of R is finite, and for some $m \geq 2$ the elementary linear group $E(m, R)$ has bounded word length with respect to elementary generators, then for all Φ of sufficiently large rank any element of $E(\Phi, R)$ is a product of ≤ 4 commutators in $E(\Phi, R)$.*

Theorem 2 of [52] is the first step towards construction of short triangular factorisations of Chevalley groups over Dedekind rings of arithmetic type. At present, sharp bounds depend on the Generalised Artin Conjecture, which in turn depends on the Generalised Riemann Hypothesis, but with [27] there is some hope to divorce these bounds from GRH and we are presently working on that. This would then be a crucial advance in the direction of the following result.

Problem 6.5. *Let R be a Dedekind ring of arithmetic type with infinite multiplicative group. Prove that any element of $E_{\text{ad}}(\Phi, R)$ is a product of ≤ 3 commutators in $G_{\text{ad}}(\Phi, R)$.*

Some of our colleagues expressed belief that any element of $\text{SL}(n, \mathbb{Z})$, $n \geq 3$, is a product of ≤ 2 commutators. However, for Dedekind rings with *finite* multiplicative groups, such as \mathbb{Z} , at present we do not envisage any *obvious* possibility to improve the generic bound ≤ 4 even for large values of n . Expressing elements of $\text{SL}(n, \mathbb{Z})$ as products of 2 commutators, if it can be done at all, should require a lot of specific case by case analysis.

Triangular factorisations, such as Gauss decomposition considered in this paper, are the simplest instance of *parabolic* factorisations. Recently, Sergei Sinchuk and the third author obtained analogues of Dennis—Vaserstein decomposition for arbitrary pairs of maximal parabolic subgroups (P_r, P_s) , $r < s$, in classical groups and pairs of *terminal* parabolic subgroups in exceptional groups, see [33, 50, 51]. In some cases stronger conditions than $\text{sr}(R) < s - r$ were imposed. Now, it seems, that these stronger conditions were only used to ensure surjective stability of K_1 and are not needed to get decompositions of the elementary group itself. Here, $U_{rs}^- = U_r^- \cap U_s^-$.

Problem 6.6. *Prove Dennis—Vaserstein type decomposition*

$$G = P_r U_{rs}^- P_s$$

for elementary Chevalley groups $E(\Phi, R)$, under restrictions on the usual stable rank $\text{sr}(R)$.

A similar problem for unitary groups was recently solved by Sergei Sinchuk [32], as part of his efforts to improve stability results for unitary K_1 , see [5, 6].

Results of the present paper are also closely related to Bass—Kolster type decompositions. The classical Bass—Kolster decomposition for the group $\text{SL}(n, R)$ has the form

$$G = L_r U_r U_r^- U_r U_r^- = P_r U_r^- U_r U_r^-.$$

For groups of other types, one has to vary one of the unipotent radicals and gets decompositions of the form $G = L_r U_r U_r^- U_s U_s^-$. In [24] Ottmar Loos addresses the problem, whether one can shorten this decomposition. He uses language of Jordan pairs, and his results apply to some non-split reductive groups. In our context the problem he studies amounts to asking, whether

$$G = L_r U_r U_r^- U_r = P_r U_r^- U_r,$$

for parabolic subgroups with Abelian unipotent radical. He proves that for $\text{SL}(n, R)$ existence of such a decomposition is equivalent to $\text{sr}(R) = 1$, [24], Corollary 3.9, but this follows already from [7]. To establish similar shorter Bass—Kolster type decompositions for groups of other types, he imposes further conditions, similar in spirit to von Neumann regularity. However, our Theorem 1.1 suggests that no such conditions are necessary.

Problem 6.7. *Prove Bass—Kolster type decompositions for elementary Chevalley groups $E(\Phi, R)$, under restrictions on the usual stable rank $\text{sr}(R)$.*

Acknowledgments

The authors would like to thank Alireza Abdollahi, Nikolai Gordeev and You Hong for inspiring discussions of commutator width and related problems, and Anastasia Stavrova, Alexei Stepanov and Maxim Vsemirnov for some very pertinent remarks. The third author thanks Francesco Catino, Francesco de Giovanni and Carlo Scoppola for an invitation to give a talk on commutator width and unitriangular factorisations at the conference “Advances in Group Theory and Applications” (Porto Cesareo – June 2011), which helped him to focus thoughts in this direction.

The work of the first and the third author was supported by the State Financed task project 6.38.74.2011 at the Saint Petersburg State University “Structure theory and geometry of algebraic groups, and their applications in representation theory and algebraic K-theory”. The work of the second and the third author was conducted in the framework of a joint Russian–Indian project RFFI 10-01-92651 “Higher composition laws, algebraic K-theory and exceptional groups” (SPbGU). The second author is grateful to the Saint Petersburg Department of Steklov Mathematical Institute and to the Saint Petersburg State University for the invitation to visit Saint Petersburg in May–June 2011. The research of the third author was carried out in the framework of the RFFI projects 09-01-00784, “Efficient generation in groups of Lie type” (PDMI RAS), 09-01-00878 “Overgroups of reductive groups in algebraic groups over rings” (SPbGU), 10-01-90016 “The study of structure of forms of reductive groups and behaviour of small unipotent elements in representations of algebraic groups” (SPbGU). Apart from that, he was partially supported by the RFFI projects 09-01-00762 (Siberian Federal University), 09-01-91333 (POMI RAS) and 11-01-00756 (RGPU).

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