More group-theoretic applications of geometric methods
B. Sury

This was meant to be a chapter in a publication of the proceedings of a workshop on geometric group theory held at the IIT Guwahati in December 2002. The idea of publication was shelved. In this article, we give some more applications to group theory coming from geometric methods including group actions on trees. To start with we sketch the beautiful proof of Gupta and Sidki's negative solution ($[GS]$) to the *general Burnside problem*.
Following that, a criterion of Culler & Vogtmann ($[CV]$) will be discussed. This will prove at one stroke the property (FA) of Serre for several groups. Finally, we discuss the proof of Formanek & Procesi ($[FP]$) asserting that the automorphism group of a free group of rank at least 3 is not linear. The relevance here is that it turns out that a certain HNN extension cannot have a faithful linear representation.

1 Gupta-Sidki group

In this section, we give a counterexample to the general Burnside problem which shows that a finitely generated group, all of whose elements have finite p-power order (for a fixed prime p), can be infinite. This beautiful construction is due to Narain Gupta and Said Sidki [4]. It should be noted that the orders of elements in this example are unbounded. As of now, known counterexample constructions to the bounded torsion version of the Burnside problem involves complicated van Kampen diagram techniques due to Olshanskii and others. We do not discuss them here.
Let p be a fixed odd prime and let X be the set of all finite strings of symbols from the set of alphabets $\{0, 1, \cdots, p-1\}$. Here the empty string is of length 0. For $r \geq 0$, we write 0^r to denote the string of length r consisting of r zeros. Whenever we add or subtract two symbols from the alphabet set, it should be read modulo p. Define two permutations t and z on X as follows. They fix the empty string and on nonempty strings, their actions are :

(*i*) t changes the first symbol i to $(i+1)$ and leaves the rest of the string unchanged.

(*ii*) For a string of the form $0^r ijw$ with $i \neq 0$ and $r \geq 0$,

$$(0^r ijw)^z = 0^r i(j+i)w.$$

Thus, z only changes the symbol j which follows the first nonzero symbol i (if any) to $j+i$.

Let G be the group of permutations of X generated by t and z. Note that both t and z leave the lengths of strings invariant. So all orbits of G are finite.

Theorem 1.1 *G is an infinite group and all its elements have finite, p-power order.*

Proof

Note that each of z and t is of order p. Set $S = \{s_h = t^{-h} z t^h : 0 \leq h < p\} \subset G$ and let H be the subgroup of G generated by S. Then each element of S has order p and H is a normal subgroup of G containing z. A key observation we shall shortly make is that H acts on the subset of X consisting of strings starting with 0, exactly as G does on the whole of X, and this is the fact would imply that G is infinite.

For $0 \leq k < p$, the subsets $X_k = \{kw : w \in X\}$ of strings in X starting with k, together with the subset of X consisting of the empty string, form a partition of X. A simple calculation shows that for $kw \in X_k$:

$$(kw)^{s_h} = \begin{cases} k(w)^z & if k = h \\ k(w)^{t^{k-h}} & if k \neq h \end{cases} . \tag{1}$$

The notation $k(w)^z$ above means the string starting with k followed by the string defined by the action of z on w.

The above observation shows that X_k are H-invariant. In particular, $t \notin H$ and so H is a proper subgroup of G. Since $G = \langle H, t \rangle$, G/H has order p. Now, (1) implies that the restriction of H to X_0 contains the permutations $0w \mapsto 0(w)^z$ and $0w \mapsto 0(w)^t$, and so contains a copy of G. Since H is a proper subgroup of G, this is possible only if G is infinite.

Next we prove that each element of G has p-power order. Using the identities $z^i t^j = t^j s_j^i$, each $x \in G$ can be written in the form

$$x = t^a s_{i_1} \cdots s_{i_m} \tag{2}$$

where $0 \leq a < p$. Here, the notation s_j^i stands for the i-th power of the permutation s_j. We choose an expression for x in the form (2) with smallest m. We use induction on m to prove that x has p-power order. This is clear if $m = 0$, since t has order p. Assume that $m > 0$ and that the result is true for all elements x of the form (2) with a product of fewer than m of the s_i.
Case (I): Suppose that $a = 0$. If the subscripts i_h in (2) are all equal, say i, then $x = s_i^m$ and so order of x divides p. So either x is the identity element or it has order p. Now, assume that i_h are not all equal. Since $x \in H$, each X_i is x-invariant. By (1), for each string $kw \in X_k$ we have $(kw)^x = k(w)^u$, where u has the form $t^b s_{j_1} \cdots s_{j_n}$ and $n = |\{i_h : i_h = k\}|$. Thus for each k, by induction, x acts on X_k as a permutation whose order is a power of p. So as a permutation of the whole of X, the order of x is a power of p.
Case (II): Suppose that $0 < a < p$. Set $y = s_{i_1} \cdots s_{i_m}$. Then

$$x^p = (t^a y)^p = (t^a y t^{-a})(t^{2a} y t^{-2a}) \cdots (y). \tag{3}$$

Now, $t^r y t^{-r} = (t^r s_{i_1} t^{-r})(t^r s_{i_2} t^{-r}) \cdots (t^r s_{i_m} t^{-r}) = s_{l_1} \cdots s_{l_m}$. So x^p can be written as a product of pm terms of the s_i $(0 \leq i < p)$. Since p does not divide a, the exponents $a, 2a, \cdots, 0$ in the expression (3) for x^p correspond to a full set of residue classes modulo p. So each s_i appears as a factor in x^p exactly m times. Applying (1) again, for each k we have $(kw)^{x^p} = k(w)^v$, where v (depending on k) is a product of pm factors consisting of either z or powers of t. Also, z occurs as a factor exactly m times and the total power to which t occurs is $b = m(1+2+\cdots+(p-1)) = m(p-1)p/2$. By using identities of the form $s_i t^r = t^r s_{i+r}$, v can be rewritten in the form $v = t^b s_{j_1} s_{j_2} \cdots s_{j_m}$. Since p is odd, p divides b and so $t^b = 1$. Now the argument in the second step of Case I can be applied to conclude that v acts on X_k as a permutation whose order is a power of p. Since this is true for each k, it follows that x^p has p-power order; so x has p-power order as well. This completes the proof.

Remark 1.2 *The construction above does not work - as it is - for $p = 2$. A corresponding theorem for $p = 2$ can be obtained with a small change. Take X to be the set of all finite strings over $\{0, 1, 2, 3\}$, define t as above and* *modify the the definition of z as follows. For any string of the form $0^r ijw$ with $i \neq 0$:*

$$(0^r ijw)^z = 0^r i(j+i)w \quad if \quad i = 1 \ \ or \ \ 3, \quad and$$

$$(0^r 2jw)^z = 0^r 2jw.$$

2 Criterion of Culler-Voigtmann

Serre's proof of property (FA) for $SL_3(\mathbb{Z})$ (see chapter 1) actually shows the property $(F\mathbb{R})$ viz. that every action on an $\mathbb{R}$-tree has a fixed point. One shows this in a manenr similar to the proof of (FA) by proving that any action without fixed points must have a line on which the group acts by a non-zero translation. Instead of discussing the details of such a proof, let us use the following definition introduced by Culler and Voigtmann ($[CV]$).

Definition 2.1 *A group G is said to have property $(A\mathbb{R})$ if every action without fixed points has an invariant line on which the action is by a non-zero translation.*

Note that the existence of such a line as above implies that there is a nontrivial homomorphism from G to $\mathbb{R}$. Thus, if $G/[G,G]$ is finite, the impossibility of existence of a nontrivial homomorphism from G to $\mathbb{R}$, would imply that G either has property $(F\mathbb{R})$ or it does not have property $(A\mathbb{R})$.
The criterion of Culler and Voigtmann to be discussed below is for a group to have property $(A\mathbb{R})$. It is easily verified to be true for many groups like $Aut\ F_n$; and $SL_n(\mathbb{Z}); n \geq 3$. Thus, we would have a uniform proof of property (FA) for all such groups. The Culler-Voigtmann criterion will be formulated in terms of a commuting graph defined by means of a finite set S of generators of G.

Definition 2.2 *Let $S = \{s_1, \ldots, s_n\}$ be generators for a group G. For $i \neq j$, a word in $s_i, s_j, s_i^{-1}, s_j^{-1}$ is said to be minipotent, if it is of the form $s_i^{a_1} s_j^{b_1} \ldots s_j^{b_r}$ or of the form $s_j^{a_1} s_i^{b_1} \ldots s_i^{b_r}$ for some $a_i, b_i = \pm 1$.*

Before defining the graph in terms of which we formulate the group theoretic criterion for property (FA), we motivate it by means of the following observations. In what follows, G is any group acting on an $\mathbb{R}$-tree (X, d) by isometries.

Lemma 2.3 *Let $g, h \in G$ and let $w \in G$ be a minipotent word in $g^{\pm}, h^{\pm}$. If $C_g \cap C_h = \phi$, then w is hyperbolic, and its axis contains the bridge from C_g to C_h.*

Proof
Consider the bridge $[p, q]$ from C_g to C_h. Look at the tree P consisting of all $x \in X$ from which the geodesic from x to g contains p. Of course, $P \supseteq C_g$ by definition of the bridge. Similarly, the tree Q of all $y \in X$ for which the geodesic from y to P contains q, contains C_h. Also, evidently $P \cap Q = \phi$.
Further, if $x \in X \backslash P$, then $gx, g^{-1}x \in P$ and if $y \in X \backslash Q$, then $hy, h^{-1}y \in \mathbb{Q}$. Thus, any word of the form $g^{a_1}h^{b_1} \ldots g^{a_r}h^{b_r}$ (with $a_i, b_i = \pm 1$) takes $X \backslash Q$ to P. Similarly, any word of the form $h^{a_1}g^{b_1} \ldots h^{a_r}g^{b_r}$ (with $a_i, b_i = \pm 1$) takes $X \backslash P$ to Q. Note that w is a word of one of these forms. As a consequence, if r is an interior point of $[p, q]$, then $[w^{-1}r, r] \cap [r, w_r] = \{r\}$ and $[w^{-1}r, r] \cup [r, w_r] \supseteq [p, q]$. This means that w must be hyperbolic and its characteristic subtree on which it acts by a non-zero translation, contains the segment $[p, q]$.

Corollary 2.4 *Let $g, h \in G$ and let w be a minipotent word in $g^{\pm}, h^{\pm}$ which commutes either with g or with h. Then $C_g \cap C_h \neq \phi$. In particular, if both g, h are elliptic and there exists w as above, then gh is also elliptic.*

Proof
Suppose, if possible, $C_g \cap C_h = \phi$. By the lemma above, w must be hyperbolic and its axis C_w must contains the bridge between C_g and C_h. In particular, $C_w \not\subseteq C_g$ and $C_w \not\subseteq C_h$. Now, let $[w, g] = 1$ say (here, $[w, g]$ denotes the commutator $wgw^{-1}g^{-1}$).
Then $wC_g = C_{wgw^{-1}} = C_g$, i.e., C_g is a w-invariant tree. But, any w-invariant tree must contains its axis C_w as w is hyperbolic. Thus, we have a contradiction if $C_g \cap C_h = \phi$. Finally, if g, h are both elliptic, the $C_g = X^g, C_h = X^h$ and thus $X^g \cap X^h = X^{gh} \neq \phi$ shows that gh is also elliptic. This proves the corollary.

We now define a special class of minipotent words which will turn out to be very useful when discussing special groups like $SL_n(\mathbb{Z}), Aut(F_n)$ etc.
For $x, y \in G$, define

$$[x, y^{(0)}] = x, [x, y^{(1)}] = [[x, y^{(0)}], y] = [x, y],$$
$$[x, y^{(2)}] = [[x, y^{(1)}], y] = [[x, y], y];$$
$$[x, y^{(k+1)}] \stackrel{d}{=} [(x, y^{(k)}], y] \; \forall \; k \geq 0.$$

Lemma 4 *Let $g, h \in G$ with h hyperbolic. If $\exists \; r > 0$ such that $[g, h^{(r)}] = 1$, then $gC_h = C_h$. If, further, g is elliptic and $\exists \; s > 0$ such that $[h, g^{(s)}] = 1$, then g fixes C_h.*

Proof
Since $1 = [g, h^{(r)}] = [[g, h^{(r-1)}], h]$, we have $hC_{[g,h^{(r-1)}]} = C_{[g,h^{(r-1)}]}$. Therefore, as h is hyperbolic, each h-invariant tree contains the axis C_h, so $C_h \subseteq C_{[g,h^{(r-1)}]}$. Now C_h is invariant under $[g, h^{(r-1)}] = [[g, h^{(r-2)}], h]$. Write $[g, h^{(r-2)}] = g_{r-2}$ for simplicity. Now

$$C_h = (g_{r-2}hg_{r-2}^{-1}h^{-1})C_h = (g_{r-2}hg_{r-2}^{-1})C_h.$$

This implies that C_h is the axis $C_{g_{r-2}hg_{r-2}^{-1}}$ for the hyperbolic element $g_{r-2}hg_{r-2}^{-1}$.
But $C_{g_{r-2}}hg_{r-2}^{-1} = g_{r-2}c_h$; thus $g_{r-2}C_h = C_h$.
So C_h is invariant under $g_{r-2} = [g, h^{(r-2)}]$. Proceeding inductively in this manner, we get that C_h is invariant under $[g, h^{(0)}] = g$. This proves the first assertion.
Now, if g is elliptic, and $gC_h \neq C_h$, we shall now prove that $[h, g^{(s)}] \neq 1 \; \forall \; x > 0$. We have under this condition that g acts as a reflection on C_h. Then g, h generate an infinite dihedral group where h acts by a translation on C_h and g acts by a reflection. Clearly $[h, g^{(s)}]$ acts as a translation on C_h and g by a reflection. Clearly $[h, g^{(s)}]$ acts as a translation by $2^s|h|$ on C_h and, therefore, $[h, g^{(s)}] \neq 1 \; \forall \; s > 0$. This proves the lemma.

Now, we can define the graph in terms of which Culler-Voigtmann's criterion will be formulated.

Definition 2.5 *Let G be any group and $S = \{g_1, \ldots, g_n\} \subseteq G$, a set of generators. Define $\Delta(G, S)$ to be the graph with vertex set S and a geometric edge between s_i and s_j if, and only if, there is a minipotent word in $s_i^{\pm}, s_j^{\pm}$ which commutes either with s_i or with s_j.*

In particular, if $(s_i, s_j) = 1$, then there is an edge. A reason to define this graph is already provided by the next proposition. First, note that the proof of property (FA) in the first chapter carries over verbation to give:

Lemma 5 *Let G act on an $\mathbb{R}$-tree X. Suppose $G = \langle s_1, \ldots, s_n \rangle$ with each s_i and each $s_i s_j$ elliptic. Then $X^G \neq \phi$.*

Using this and the above observations, we have:

Proposition 2.6 *Suppose G acts on an $\mathbb{R}$-tree X. Let $S = \{s_1, \ldots, s_n\}$ and $G = \langle S \rangle$, where each s_i is elliptic. If the graph $\Delta(G, S)$ is complete then $X^G \neq \phi$.*

Proof

As $\Delta(G, S)$ is complete, s_i and s_j are connected by an edge. That is, $\forall\, i, j,\ \exists$ a minipotent word in $s_i^{\pm}, s_j^{\pm}$ which commutes either with s_i or with s_j. By the previous corollary, since s_i and s_j are elliptic, this means $s_i s_j$ is also elliptic. So, the last lemma implies the result.

Now, we can state the main characterisation theorem of Culler and Voigtmann in a weaker form which suffices for our purpose. One final definition we need is the following. Given generators $s_1, \ldots, s_n$ of G, one calls an edge from s_i to s_j distinguished if $[s_i, s_j^{(k)}]$ commutes with s_j for some $k > 0$. Note that the opposite edge of a distinguished edge may not be distinguished - in fact, this happens in the example of $\mathrm{Aut}(F_n)$ that we will discuss. From the subgraph $\Delta'(G, S)$ consisting of the same set S of vertices but whose edges are either distinguished edges in $\Delta(G, S)$ or their opposites.

Theorem 2.7 *Suppose all generators in S are conjugate in G. If $\Delta(G, S)$ is connected, then G has property $(A\mathbb{R})$. If, in addition, $G/[G, G]$ is finite, then G has property $(F\mathbb{R})$.*

Proof

First, note the simple fact that if g, h are hyperbolic and $[g, h]$ commutes with h, then $C_g = C_h$. The reason is that $C_h = [g, h]C_h = ghg^{-1}C_h$ means C_h is the axis of the hyperbolic element ghg^{-1}. So $C_h = C_{ghg^{-1}} = gC_h$. But, the only line invariant under a hyperbolic element is its axis; so $C_g = C_h$.

To proceed with the proof, if S consists entirely of elliptic elements, we have proved that $X^G \neq \phi$ i.e., G acts trivially on X. Therefore, let us assume that S consists of some hyperbolic elements as well; hence all its elements (being conjugate) are hyperbolic. Now, since $\Delta'(G,S)$ is connected, $\forall\ i,j$ we have either a distinguished edge from s_i to s_j or one from s_j to s_i. Thus, either $[s_i, s_j^{(k)}]$ commutes with s_j for some $k > 0$ or $[s_j, s_i^{(k)}]$ commutes with s_i for some $k > 0$. Using the simple fact observed in the beginning of this proof, $C_{s_i} = C_{s_j}\ \forall\ i,j$. Thus, we have a common line invariant under each s_i, and, hence invariant under G. Thus, G has property $(A\mathbb{R})$. Of course, this gives a nontrivial homomorphism from G to $\mathbb{R}$. If $G/[G,G]$ were finite, then this would be impossible thereby proving that S must consist of elliptic elements and G must fix a vertex.

Corollary 2.8 *$Aut(F_n), n \geq 3$ has property $(F\mathbb{R})$.*
In particular, $Out(F_n), GL(n,\mathbb{Z}), SL(n,\mathbb{Z})$ for $n \geq 3$ have property $(F\mathbb{R})$.

Proof
Consider the subgroup of special automorphisms $SAut(F_n) \stackrel{d}{=} \pi^{-1}(SL_n(\mathbb{Z}))$, where $\pi : Aut(F_n) \to Aut(\mathbb{Z}^n)$ is the obvious map. Write $\{x_1, \ldots, x_n\}$ for a basis of F_n.
Let $S = \{\lambda_{ij}, \rho_{ij}; i \neq j\}$, where $\lambda_{ij} : x_i \mapsto x_j x_i, \rho_{ij} : x_i \mapsto x_i x_j$ and $\lambda_{ij}(x_k) = x_k = \rho_{ij}(x_k)\ \forall\ k \neq i$. It is known (see [$LS$]) that $SAut(F_n) = \langle S \rangle$. In fact, since $[\rho_{ij}, \rho_{jk}] = \rho_{ik}$ and $[\lambda_{ij}, \lambda_{jk}] = \lambda_{ik}$ for i,j,k distinct, it follows that we may take $S = \{\rho_{i,i+1}, \lambda_{i+i+1}; 1 \leq i \leq n\}$ where we mean $x_{n+1} = x_1$. So, $SAut(F_n)$ contains A_n and also the automorphisms which send exactly two of the generators to their inverses. Clearly, the elements of S are conjugate under such automorphisms.
Note that $[\rho_{j,j+1}, \rho_{i,i+1}]$ and $[\lambda_{j,j+1}, \rho_{i,i+1}]$ commute with $\rho_{i,i+1}$ unless $j = i + 1\ mod\ n$. Similarly, $[\rho_{j,j+1}, \lambda_{i,i+1}]$ and $[\lambda_{j,j+1}, \lambda_{i,i+1}]$ commute with $\lambda_{i,i+1}$ unless $j \equiv i + 1\ mod\ n$.
Therefore, $\Delta'(G,S)$ is already a complete graph; however, not all its edges are distinguished. Thus, by the theorem, $SAut(F_n); n \geq 3$ has property $(A\mathbb{R})$. Since each element of S is a commutator, the abelianisation $SAut(F_n)/[SAut(F_n), SAut(F_n)]$ is finite. Thus, $SAut(F_n)$ has property $(F\mathbb{R})$. As it is of finite index in $Aut(F_n)$, the group $Aut(F_n), n \geq 3$ itself has $(F\mathbb{R})$. Being a quotient (respectively, a finite extension) of $Aut(F_n)$, the groups $GL(n,\mathbb{Z}), n \geq 3$ (respectively, $\mathrm{Out}(F_n), n \geq 3$) have property $(F\mathbb{R})$.

3 Unfaithfulness of Aut (F)

Now, we discuss an important result proved by E.Formanek and C.Procesi ([FP]) which asserts that the automorphism group of a free group of rank at least 3 does not admit a faithful linear representation over any field. This uses some basic representation theory of algebraic groups and also gives a more general result that Aut $(G*\mathbb{Z})$ can admit a faithful linear representation only if G is virtually solvable.

Let G be any group. There are three natural embeddings α, β and Δ of G in $G \times G$. These are the first and second inclusions and the diagonal embedding respectively. Consider the HNN extension

$$\mathcal{H}(G) = \langle G \times G, t | t \cdot \Delta(G) t^{-1} = \beta(g) \ \forall \ g \in G \rangle.$$

The relation can be reinterpreted as

$$t\alpha(g)\beta(g)t^{-1} = \beta(g)$$

$$t\alpha(g) = \beta(g)t\beta(g)^{-1}.$$

Notice that $\alpha(g)\beta(g') = \beta(g')\alpha(g)$. The basic theorem proved by Formanek and Procesi shows that the HNN extension $\mathcal{H}(G)$ can be linear only if G is virtually solvable (i.e., has a solvable subgroup of finite index).
The group generated by $\alpha(g)$ and t is normalized by the whole group i.e. $\mathcal{H}(G) = \beta(G) \propto (\alpha(G) * \langle t \rangle)$, with the above action.
So $\mathcal{H}(G) = F\beta(G)$ where we have called the group $\alpha(G) * \langle t \rangle$ as F.
Therefore, we have $\mathcal{H}(G) \to Aut\ F$.
If $G \neq \{1\}$, then this map is injective since $\beta(g)$ does not coincide with any inner conjugation from F.

Theorem 3.1 *If $\mathcal{H}(G)$ is linear then G is virtually solvable.*

Corollary 3.2 *If G is not virtually solvable, the $Aut(G * \mathbb{Z})$ is not linear. Therefore, $Aut\ F_n$ is non-linear $\forall\ n \geq 3$.*

It is interesting to note that recently D.Krammer has proved ([K]) that the group Aut F_2 does have a faithful 5-dimensional representation. To prove the above theorem, we will need some facts from the representation theory of algebraic groups.

Facts from algebraic group theory ([H])

For a (complex) representation $\rho : G \to GL(V)$, let V_0 denote the same vector space V but with trivial G-action $\rho_0 : G \to GL(V_0); g \mapsto Id$.
All the definitions and results remain valid over any algebraically closed field of character zero in place of $\mathbf{C}$.
We shall also denote by $\bar{V}$, the semi-simplification of V; i.e., if $V \supset V_1 \supset \ldots \supset V_r = (0)$ is a Jordan-Holder series, then $\bar{V} = \oplus V_i/V_{i+1}$.
Call $\bar{\rho} : G \to GL(\bar{V})$. We have, therefore, an exact sequence

$$1 \to N \to \rho(G) \to \bar{\rho}(G) \to 1$$

where N is a nilpotent group because the matrices in $\rho(G) \leq GL(V)$ with respect to a basis of V which restricts to bases of V_i for each i, are upper triangular with 1's on the diagonal.
One also thinks of a representation of G as a left module under the group algebra $\mathbf{C}[G]$ and simply calls it a G-module.
When G is an algebraic group, then representations considered are algebraic ones and then the module is over the co-ordinate ring of G.
If G, H are groups then a simple $G \times H$-module is of the form $V \otimes W$ where V, W are simple modules over G and H respectively.
Suppose that G is a semisimple algebraic group over an algebraically closed field K. Associated with G and a choice of a Borel subgroup B of G is a free abelian group Λ with basis $\lambda_1, \cdots, \lambda_t$, called the lattice of abstract weights of G. The generators λ_i are called the fundamental dominant weights and a weight $\lambda = \sum_{i=1}^{t} c_i \lambda_i$ is called dominant if all $c_i \geq 0$. The dominant weights are the algebraic characters of B. We partially order Λ by specifying that $\lambda \geq \mu$ if $\lambda - \mu$ is a dominant weight. With these notations, one knows :
(a) Any irreducible G-module V has a unique B-stable one-dimensional subspace spanned by a vector v of some dominant weight $\lambda(v)$ (called the highest weight of V).
(b) If V, V' are irreducible G-modules, then they are isomorphic if, and only if, they have the same highest weight. One usually writes $V(\lambda)$ for the irre-

ducible G-module with the highest weight λ.
(c) If V, W are irreducible G-modules, then some composition factor of $V \otimes W$ has highest weight $\lambda(V) + \lambda(W)$. In fact,

$$\overline{V(\lambda) \otimes V(\mu)} = V(\lambda + \mu) + \oplus_{w < \lambda + \mu} V(w).$$

Proposition 3.3 *Suppose $\rho : \mathcal{H}(G) \to GL(U)$ is a representation and write ρ_G for the restriction $\rho|_{G \times G}$. As $G \times G$-modules, if we write $\bar{U} = \oplus(V_i \otimes W_i)$ where V_i, W_i are simple $\alpha(G)$-modules and $\beta(G)$-modules respectively, then, as a $\Delta(G)$-module, $\oplus V_i \otimes W_i \cong \oplus(V_i)_0 \otimes W_i$.*

Proof
Now, as $\Delta(g)$ and $\beta(g)$ are conjugate in $\mathcal{H}(G)$, it follows that $\rho_G \circ \Delta$ is conjugate to the representation $\rho_G \circ \beta$.
Therefore, passing to the associated graded, we have $\bar{\rho}_G \circ \Delta \cong \bar{\rho}_G \circ \beta$. But $\bar{\rho}_G \circ \Delta$ corresponds to $\oplus(V_i \otimes W_i)$, whereas $\bar{\rho}_G \circ \beta$ corresponds to $\oplus((V_i)_0 \otimes W_i)$. This completes the proof.

Now, we can state the following result which will finish the proof of the theorem on linearity of $\mathcal{H}(G)$.

Theorem 3.4 *Let G be a group and V_i, W_i be semi-simple G-modules. Assume that $\oplus(V_i \otimes W_i) \cong \oplus((V_i)_0 \otimes W_i)$ as $\Delta(G)$-modules. Then, $Im(G \to GL((\oplus V_i))$ is virtually Abelian.*

Proof
Let H denote the Zariski closure of the image of G in $GL(\oplus(V_i \otimes W_i))$. It suffices to prove the theorem for H. As the conclusion is 'virtual', we may assume that H is connected. Now, $\oplus(V_i \otimes W_i)$ is a faithful, semisimple H-module. But, for any nontrivial, irreducible H-module V, the fixed subspace V^U under the unipotent radical U of H is a G-submodule. Therefore, V^U must be trivial and thus U must be trivial; i.e., H is reductive. Then $H = T \cdot S$, with T the central torus, and $S = [H, H]$ connected semisimple. Hence, to prove that H is virtually abelian, it suffices to show that S acts trivially on each V_i. This follows from the following general result on semisimple algebraic groups :

Proposition 3.5 *Let S be a connected, semisimple algebraic group, and let V_i, W_i be semisimple S-modules. If $\oplus(V_i \otimes W_i) \cong \oplus((V_i)_0 \otimes W_i)$, then $V_i = (V_i)_0$ for all i.*

Proof
Expand V_i, W_i as sums of simple S-modules and remove all the V_i, W_i for which $V_i = (V_i)_0$. Let d_i denote dim V_i. Then,

$$\oplus(V(\lambda_i) \otimes V(\mu_i)) \cong \oplus d_i V(\mu_i).$$

Now, $\lambda_i > 0$ as $V_i \neq (V_i)_0$ for each i. Choose i_0 such that μ_{i_0} is maximal among the μ_i's. But, $V(\lambda_{i_0} + \mu_{i_0})$ occurs on the left hand side above. The resultant contradiction proves the proposition.

Acknowledgement.
I am indebted to Ms.Ashalata of our office here who toiled hard with me to make a cohesive volume out of the seven chapters written by different people. Although the idea of publication of the volume was shelved, her efforts were invaluable.

References

[1] M.Culler and K.Voigtmann, Proc. AMS 124 , 1996.

[2] E.Formanek & C.Procesi, Jour. of Algebra

[3] N.Gupta, American Math. Monthly, 1989.

[4] N. D. Gupta and S. Sidki, 'On the Burnside problem for periodic groups', Math. Z. 182 (1983), 385-388.

[5] J.Humphreys, Linear algebraic groups

[6] D.Krammer, Invent. Math.

[7] R.Lyndon & P.Schupp, Combinatorial group theory