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Sum of the Reciprocals of the Binomial Coefficients

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1. INTRODUCTION

The sum in the title will be shown to satisfy the following identities.
For any natural number n,

$$\sum_{k=0}^{n} ({}^{n}C_k)^{-1} = (n+1)/2^n \sum_{i=0}^{n} 2^i/(i+1)$$

$$= (n+1)/2^n \sum_{j \text{ odd}, j \leqslant n+1} {}^{n+1}C_j 1/j. \qquad \spadesuit$$

The sum on the right-hand side of the first equality involves n only at the upper limit. Thus, we are able to find a recurrence for the sum of the reciprocals of the binomial coefficients.

The proof of ♠ is extremely easy: we will give it and make a few remarks on some consequences.

Proof of ♠:

$$\sum_{k=0}^{n} ({}^{n}C_k)^{-1} = \sum_{k=0}^{n} \Gamma(k+1)\Gamma(n-k+1)/\Gamma(n+1)$$

$$= (n+1) \sum_{k=0}^{n} \beta(k+1, n-k+1)$$

$$= (n+1) \int_0^1 \left(\sum_{k=0}^{n} t^k (1-t)^{n-k} \right) dt$$

$$= (n+1) \int_0^1 (t^{n+1} - (1-t)^{n+1}\, dt/((t-(1-t))$$

$$= (n+1)/2^{n+1} \int_0^1 ((s+1)^{n+1} - (1-s)^{n+1})\, ds/s$$

$$= (n+1)/2^{n+1} \int_0^1 \sum_{j=0}^{n} {}^{n+1}C_j (1-(-1)^j) s^j\, ds/s$$

$$= (n+1)/2^n \sum_{j \text{ odd}, j \leqslant n+1} {}^{n+1}C_j 1/j.$$

This proves one of the equalities.

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Moreover, the intermediary step

$$(n+1)/2^{n+1}\int_0^1((s+1)^{n+1}-(1-s)^{n+1})\,ds/s$$
$$=(n+1)/2^{n+1}\left(\int_0^1((s+1)^{n+1}-1)\,ds/s+\int_0^1(1-(1-s)^{n+1})\,ds/s\right)$$
$$=(n+1)/2^{n+1}\sum_{i=0}^{n}\left(\int_0^1(1+s)^i\,ds+\int_0^1(1-s)^i\,ds\right)$$
$$=(n+1)/2^{n+1}\sum_{i=0}^{n}((2^{i+1}-1)/(i+1)+1/(i+1))$$
$$=(n+1)/2^n\sum_{i=0}^{n}2^i/(i+1),$$

which proves ♠.

2. Remarks

Let S_n be the sum $\sum_{k=0}^n ({}^nC_k)^{-1})$ and let $\sigma_n=\sum_{i=0}^n 2^i/(i+1)=2^n/(n+1)+\sigma_{n-1}$. Then, $S_n=(n+1)\sigma_n/2^n$ and, consequently,

$$S_n=S_{n-1}(n+1)/2n+1. \tag{1}$$

This shows at once that

$$\lim_{n\to\infty}S_n=2. \tag{2}$$

If we set $T_n=n!\,S_n$ so that T_n is a positive integer, then we find that the recurrence is

$$T_n=T_{n-1}(n+1)/2+n! \tag{3}$$

We are thus enabled to calculate the following modest table of T_n:

n	T_n
1	2
2	5
3	16
4	64
5	312
6	1812
7	12288
8	95616
9	840960
10	8254080
11	89441280
12	1060369920
13	13784808960
14	191094543360
15	28240773120000

The recurrence (3) gives a number of theorems of the following type:

Theorem 1. *If p is a prime, $S_{p-1}\equiv 1 \bmod p$.*

Theorem 2. *If p is a prime, $T_{p-1}\equiv -1 \bmod p$.*

Theorem 3. *If p is a prime, p divides T_n for $n\geqslant 2p-1$.*

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