Comment on:

H. N. Ramaswamy & C. R. Veena,
"On the energy of the unitary Cayley graph"
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Theorems 3.1, 3.7 of the paper assert that the energy of the unitary Cayley graph Cay$(\mathbf{Z}_n, \mathbf{Z}_n^*)$ is $2^{\omega(n)}\phi(n)$. We give an essentially one-sentence proof of these theorems; we do not address other results in the paper.
The Cayley graph $C_n =$ Cay $(\mathbf{Z}_n, \mathbf{Z}_n^*)$ is a connected $\phi(n)$-regular graph and the eigenvalues of its adjacency matrix are the Ramanujan sums $c(r,n) = \phi(n)\frac{\mu(n/(r,n))}{\phi(n/(r,n))}$ for $1 \leq r \leq n$. The energy of the graph is defined to be the sum of the absolute values of the eigenvalues. Let us prove that the energy of C_n is $2^{\omega(n)}\phi(n)$ by proving the identity: $\sum_{r=1}^{n} |\frac{\mu(n/(r,n))}{\phi(n/(r,n))}| = 2^{\omega(n)}$.
For each divisor d of n, call $S_d := \{r \leq n : (r,n) = d\}$. Note that $|S_d| = \phi(n/d)$ as $\{r \leq n : (r,n) = d\} = \{dR \leq n : (R, n/d) = 1\}$. Now, writing $n = p_1^{a_1} \cdots p_r^{a_r}$, we have $|\mu(n/d)| = 1$ if and only if n/d is square-free, which is so if and only if $d = p_1^{b_1} \cdots p_r^{b_r}$ with each $b_i = a_i$ or $a_i - 1$. Write T for such divisors; clearly $|T| = 2^r$. Therefore,

$$\sum_{r=1}^{n} |\frac{\mu(n/(r,n))}{\phi(n/(r,n))}| = \sum_{d|n} \frac{|S_d||\mu(n/d)|}{\phi(n/d)} = \sum_{d \in T} 1 = |T| = 2^r.$$

This completes the proof.