Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121, No. 3, August 2011, pp. 245–247.


# On diophantine equations of the form $(x - a_1)(x - a_2) \dots (x - a_k) + r = y^n$

MANISHA KULKARNI[1] and B SURY[2]

[1]Poornaprajna Institute of Scientific Research, Devanahalli, Bangalore 562 110, India
[2]Statistics & Mathematics Unit, Indian Statistical Institute, 8th Mile, Mysore Road, Bangalore 560 059, India
E-mail: manisha@isibang.ac.in; sury@isibang.ac.in

MS received 11 June 2010; revised 20 April 2011

**Abstract.** Erdős and Selfridge [3] proved that a product of consecutive integers can never be a perfect power. That is, the equation $x(x+1)(x+2)...(x+(m-1)) = y^n$ has no solutions in positive integers $x, m, n$ where $m, n > 1$ and $y \in \mathbf{Q}$. We consider the equation

$$(x - a_1)(x - a_2) \dots (x - a_k) + r = y^n$$

where $0 \leq a_1 < a_2 < \cdots < a_k$ are integers and, with $r \in \mathbf{Q}$, $n \geq 3$ and we prove a finiteness theorem for the number of solutions $x$ in $\mathbf{Z}$, $y$ in $\mathbf{Q}$. Following that, we show that, more interestingly, for every nonzero integer $n > 2$ and for any nonzero integer $r$ which is not a perfect $n$-th power for which the equation admits solutions, $k$ is bounded by an effective bound.

**Keywords.** Diophantine equations; Erdős–Selfridge.

Erdős and Selfridge [3] proved that a product of consecutive integers can never be a perfect power. That is, the equation $x(x+1)(x+2)\dots(x+(m-1)) = y^n$ has no solutions in positive integers $x, y, m, n$ where $m, n > 1$. After this, a natural question is to study $x(x+1)(x+2)\dots(x+(m-1)) + r = y^n$ with a nonzero integral or rational parameter $r$. However, this equation is not symmetric like the Erdős–Selfridge equation and requires different methods. In [1], we have proved that in this case there are effective finiteness results for $x, m, n \in \mathbf{Z}$ and $y \in \mathbf{Q}$. We shall also prove finiteness results if we delete many terms from the product involving consecutive integers. We consider the equation

$$(x - a_1)(x - a_2) \dots (x - a_k) + r = y^n$$

where $0 \leq a_1 < a_2 < \cdots < a_k$ are integers and, with $r \in \mathbf{Q}$, $n \geq 3$. Our first aim is to prove a finiteness theorem for the number of solutions $x$ in $\mathbf{Z}$, $y$ in $\mathbf{Q}$. Following that, we show that, more interestingly, for every nonzero integer $n > 2$ and for any nonzero integer $r$ which is not a perfect $n$-th power for which the equation admits solutions, $k$ is bounded by an effective bound. We recall that the height $H(\alpha)$ of an algebraic number $\alpha$ is the maximum of the absolute values of the integer coefficients in its minimal defining polynomial. In particular, if $\alpha$ is a rational integer, then $H(\alpha) = |\alpha|$ and if $\alpha$ is a rational number $\frac{p}{q} \neq 0$, then $H(\alpha) = \max(|p|, |q|)$.

Our first result is as follows.

**Theorem 1.** *Let $r \in \mathbf{Q}$, let $0 \leq a_1 < a_2 < \cdots < a_k$ be integers where $k > 2$. Further, let $n > 2$ and assume that we are not in the case when $n = k = 4$. Then, there are only finitely many solutions $x \in \mathbf{Z}$, $y \in \mathbf{Q}$ to the equation*

$$(x - a_1)(x - a_2) \ldots (x - a_k) + r = y^n$$

*and, all the solutions satisfy*

$$\max\{H(x), H(y)\} < C,$$

*where $C$ is an effectively computable constant depending only on $n$, $r$ and the $a_i$'s.*

When $r$ is an integer and not a perfect $n$-th power, we bound $k$ in the following result.

**Theorem 2.** *Let $n$ be a fixed positive integer $> 2$ and let $r$ be a nonzero integer which is not a perfect $n$-th power. Let $\{t_m\}_m$ be a sequence of positive integers such that $m/t_m \to \infty$ as $m \to \infty$. There exists an effectively computable number $C$ depending only on $n$ and $r$ such that if $(x - a_1)(x - a_2) \ldots (x - a_{m-t_m}) + r = y^n$ with $0 \leq a_1 < a_2 < \cdots < a_{m-t_m}$ has a solution, then $m/(t_m + 1) < C$.*

To prove Theorem 1, we use a theorem of Brindza [2].

Let $K$ be an algebraic number field, $R \subset K$ be a finitely generated subring and $g \in R[X]$. Write $g = a \prod_{i=1}^{s} (X - \beta_i)^{r_i}$ over an extension of $K$, where $a \neq 0$ and $\beta_i \neq \beta_j$ for $i \neq j$. Let $R_1$ be the ring generated by $R$ along with the denominators of the $\beta_i$'s. For an integer $n > 1$, consider the equation $g(x) = y^n$ with $x, y \in R_1$. Then, Brindza's theorem [2] asserts:

**Theorem [2].** *With the above notations, put $t_i = \frac{n}{(n, r_i)}$, $i = 1, 2, \ldots, s$. Suppose that $(t_1, t_2, \ldots, t_s)$ is not a permutation of any of the $s$-tuples. Then*

(i) $(t, 1, 1, 1, \ldots, 1)$ *for some $t$, or*
(ii) $(2, 2, 1, 1, 1, \ldots, 1)$.

*Then, all the solutions of the equation $g(x) = y^n$ with $x, y \in R_1$ satisfy*

$$\max\{H(x), H(y)\} < C,$$

*where $C$ is an effectively computable constant depending on $K$, $n$ and $g$.*

Let us prove Theorem 1 using this now.

*Proof of Theorem* 1. Let us write $f$ for the polynomial $(X - a_1)(X - a_2) \ldots (X - a_k)$. Suppose $f + r = a \prod_{i=1}^{s} (X - \beta_i)^{r_i}$ with $a \neq 0$ and $\beta_i \neq \beta_j$ for $i \neq j$ algebraic integers. We take $R$ to be the subring $\mathbf{Z}[r]$ of $\mathbf{Q}$ and $K = Q(\beta_1, \beta_2, \ldots, \beta_s)$. We consider solutions $x, y \in O_K[r]$. We show that $r_i = 1$ or 2 and then use Brindza's theorem to get the result.

*Claim.* The multiplicity of a root of $f(x) + r$ is at most 2.

*Proof.* Note that $f + r$ is a polynomial of degree $k$ and hence, its derivative $f'$ is a polynomial of degree $k - 1$. Now, by Rolle's theorem, it has zeroes in the intervals $(a_1, a_2), (a_2, a_3), \ldots, (a_{k-1}, a_k)$. Thus, the roots of $f'$ are distinct. Therefore, if $f + r$ has a multiple root then its multiplicity can be at most two which proves the claim.

Thus,

$$f + r = a \prod_{i=1}^{s} (X - \alpha_i)^{r_i},$$

where each $r_i = 1$ or 2. Also note that $s > 1$ since $k > 2$.

Let $t_i = \frac{n}{(n,r_i)} = \frac{n}{(n,1)}$ or $\frac{n}{(n,2)}$. This implies $t_i = n$ or $\frac{n}{2}$. As $n > 2$, note that $t_i > 1$ for each $i$. So, the $s$-tuple $(t_1, t_2, \ldots, t_s)$ can never look like $(t, 1, 1, \ldots, 1)$ for any $t$. If this $s$-tuple looks like $(2, 2, 1, \ldots, 1)$, then it must be $(2, 2)$ which gives $k = 4 = n$ which is excluded by assumption. So, by Brindza's theorem we get the result.

*Proof of Theorem* 2. Since $r$ is not a perfect $n$-th power we can write $r$ as $r = p_1^{h_1 n + r_1} p_2^{h_2 n + r_2} \ldots p_t^{h_t n + r_t}$, where $p_i$'s are primes in $Z$ and $r_i$'s are such that not all of them are zeroes. Choose the smallest $p_i$ for which $r_i$ is not zero; so, the exact power of $p_i$ dividing $r$ is $h_i n + r_i$. Take $C = (h_i n + r_i + 1)p_i$ and suppose, if possible, $m/(t_m + 1) \geq C$. Let us write $k := m - t_m$ for simplicity. Then we claim that $(x - a_1)(x - a_2) \ldots (x - a_k)$ is divisible by $p_i^{h_i n + r_i + 1}$. Indeed, look at the number of terms of the product $(x - 1)(x - 2) \ldots (x - m)$ which are missing in the product $(x - a_1)(x - a_2) \ldots (x - a_k)$; this number is $m - k = t_m$. We claim that there is a string of consecutive integers of length at least $(h_i n + r_i + 1)p_i$ in the product $(x - a_1)(x - a_2) \ldots (x - a_k)$. Indeed, if each consecutive string of integers occurring in the last product is of length at most $(h_i n + r_i + 1)p_i - 1$, then we would have $k = m - t_m < (t_m + 1)((h_i n + r_i + 1)p_i - 1)$ which means $m < (t_m + 1)(h_i n + r_i + 1)p_i$. Thus, $m/(t_m + 1) < C$. In other words, if $m$ is so large that $m/(t_m + 1) \geq C$, then there is a string of consecutive integers of length at least $(h_i n + r_i + 1)p_i$ in the product $(x - a_1)(x - a_2) \ldots (x - a_k)$. Hence the power of $p_i$ in $(x - a_1)(x - a_2) \ldots (x - a_k)$ is at least $h_i n + r_i + 1$. Thus the power of $p_i$ in $(x - a_1)(x - a_2) \ldots (x - a_k) + r$ is exactly $h_i n + r_i \not\equiv 0 (\mathrm{mod}\ n)$ since $0 < r_i < n$. This is a contradiction to the equation under consideration.

**Acknowledgments**

We thank Professor R Tijdeman who asked the first author a question which is addressed in Theorem 2 here and we also thank him and Professor T N Shorey for showing interest in this work. We thank the referee for some valuable suggestions.

**References**

[1] Bilu Y, Kulkarni M and Sury B, On the diophantine equation $x(x + 1) \ldots (x + m - 1) + r = y^n$, *Acta Arith.* **113**(4) (2004) 303–308

[2] Brindza B, On the equation $f(x) = y^m$ over finitely generated domains, *Acta Math. Hung.* **53(3–4)** (1989) 377–383

[3] Erdős P and Selfridge J L, Product of consecutive integers is never a power, *Ill. J. Math.* **19** (1975) 292–301