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CONTINUOUS FUNCTIONS AND THE GAUSS LEMMA

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Abstract. In Article 42 of his celebrated book 'Disquisitiones Arithmeticae', Gauss proved the following result:
If the coefficients $A, B, C, \cdots, N; a, b, c, \cdots n$ of two functions of the form

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \cdots + N \qquad (P)$$

$$x^\mu + ax^{\mu-1} + bx^{\mu-2} + cx^{\mu-3} + \cdots + n \qquad (Q)$$

are all rational and not all integers, and if the product of (P) and (Q)

$$= x^{m+\mu} + \mathfrak{A}x^{m+\mu-1} + \mathfrak{B}x^{m+\mu-2} + etc. + \mathfrak{Z}$$

then not all the coefficients $\mathfrak{A}, \mathfrak{B}, \cdots, \mathfrak{Z}$ can be integers.
This is the famous Gauss lemma which has been rephrased and generalized in several ways over 150 years. Some of the statements have only existential proofs while some have surprisingly explicit proofs. We discuss these aspects of the Gauss lemma and its generalizations.

1. Introduction

If f, g are polynomials in one variable over any commutative ring with unity, a lemma due (independently) to Dedekind and Mertens from 1892 generalizes the classical Gauss lemma and asserts that

$$c(f)^{deg(g)}c(fg) = c(f)^{deg(g)}c(f)c(g).$$

Here, for a polynomial f, one defines the content of f to be the ideal $c(f)$ generated by its coefficients. However, one thing that is true over ANY commutative ring with unity is that, for any f and g, the equality $c(fg) = c(f)c(g)$ holds *if $c(f), c(g)$ are unit ideals.* We start first by recalling that the statement "$c(fg) = c(f)c(g)$ if $c(f), c(g)$ are unit ideals" has a

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purely existential proof, and that indeed, no proof is known that is either constructive or, is accomplished by some algebraic manipulations. Following that, we provide a twist in the tale for a certain ring of functions where a Gauss-lemma-like proof does work. Finally, in the next few sections, we give a brief tour of some generalizations that the subject of Gauss's lemma has led to over the years.

2. Thus spake Gauss

In Article 42 of his celebrated book 'Disquisitiones Arithmeticae', Gauss proved the following result (here is an English translation of his statement):

If the coefficients $A, B, C, \cdots, N; a, b, c, \cdots n$ of two functions of the form

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \cdots + N \qquad (P)$$

$$x^\mu + ax^{\mu-1} + bx^{\mu-2} + cx^{\mu-3} + \cdots + n \qquad (Q)$$

are all rational and not all integers, and if the product of (P) and (Q)

$$= x^{m+\mu} + \mathfrak{A}x^{m+\mu-1} + \mathfrak{B}x^{m+\mu-2} + etc. + \mathfrak{Z}$$

then not all the coefficients $\mathfrak{A}, \mathfrak{B}, \cdots, \mathfrak{Z}$ can be integers.

This is the famous Gauss lemma which is often re-phrased in several ways, one of which is the following statement:
Over a unique factorization domain (abbreviated as UFD), the product of primitive polynomials is a primitive polynomial.
Here, the adjective 'primitive' refers to a polynomial whose coefficients have no common divisor in the UFD other than units. The Gauss lemma has been generalized over time. For instance, Kaplansky showed that the above statement holds over any integral domain in which any two elements admit a GCD (greatest common divisor) - these are now known as GCD domains and we discuss them in a later section here.
Note that over a UFD, any two non-zero non-units have a GCD which is unique up to multiplication by units. The Gauss lemma can also be thought of as the assertion that over a UFD, the product of the GCDs of polynomials f and g is the GCD of the polynomial fg (up to multiplication by units).
The main implication of Gauss's lemma is that for any UFD A, the polynomial ring $A[X]$ is also a UFD.

For a polynomial f, one may define the content of f to be the ideal $c(f)$ generated by its coefficients - this definition makes sense over any commutative ring with unity. It is evident that we have an inclusion $c(fg) \subseteq c(f)c(g)$ for polynomials f, g.

The content ideal is the same as the ideal generated by the GCD when the ring is a PID (principal ideal domain); hence the above is an equality in this case.

However, it is interesting to observe the subtlety that the inclusion $c(fg) \subseteq c(f)c(g)$ could be proper for polynomials f, g over UFDs A.

For instance, if $A = K[X, Y]$ for a field A, the polynomials $f(t) = X + Yt$ and $g(t) = X - Yt$ have the property that $fg = X^2 - Y^2t^2$ and hence

$$c(f)c(g) = (X, Y)^2 = (X^2, XY, Y^2) \supset (X^2, Y^2) = c(fg)$$

where the inclusion is proper.

Another example is $A = \mathbb{Z}[X]$ where $f(t) = 2 + Xt, g(t) = 2 - Xt$ give

$$c(f)c(g) = (2, X)^2 = (4, 2X, X^2) \supset (4, X^2) = c(fg)$$

which is a strict inclusion.

3. Existential proofs - a twist in the tale

If f, g are polynomials in one variable over any commutative ring with unity, a lemma due (independently) to Dedekind and Mertens from 1892 which will be discussed in detail in the next section asserts that

$$c(f)^{deg(g)}c(fg) = c(f)^{deg(g)}c(f)c(g).$$

However, one thing that is true over ANY commutative ring with unity is that, for any f and g, the equality $c(fg) = c(f)c(g)$ holds *if* $c(f), c(g)$ *are unit ideals.*

Our purpose is to start first by recalling that the statement "$c(fg) = c(f)c(g)$ if $c(f), c(g)$ are unit ideals" has a purely existential proof, and that indeed, no proof is known that is either constructive or, is accomplished by some algebraic manipulations. This may be instructive to bring to the notice of the students. Following that, we provide a twist in the tale for a certain ring of functions where a Gauss-lemma-like proof does work. Finally, in the next two sections, we give a brief tour of some generalizations that the subject of Gauss's lemma has led to over the years.

First, we recall the existential argument alluded to:

Let R be a commutative ring with unity. Let $f = \sum_{i=0}^{n} a_i X^i, g = \sum_{j=0}^{m} b_j X^j \in R[X]$ be such that $c(f), c(g)$ are unit ideals; that is,

$$1 = \sum_{i=0}^{n} a_i A_i = \sum_{j=0}^{m} b_j B_j$$

for some $A_i, B_j \in R$. If $fg = \sum_{k=0}^{m+n} c_k X^k$, then $c(fg)$ is the unit ideal; that is, there exist $C_k \in R$ so that $\sum_{k=0}^{m+n} c_k C_k = 1$.

To prove this, suppose the ideal generated by $c_0, \cdots, c_{m+n}$ is a proper ideal, and let M be a maximal ideal containing it. Then, under the natural ring homomorphism from $R[X]$ to $(R/M)[X]$, the polynomial fg maps to zero. However, neither the image of f nor that of g maps to zero which contradicts the fact that $(R/M)[X]$ is an integral domain.

As mentioned above, the proof is purely existential. Having said this, we observe now that for a ring like $C[0,1]$, the ring of real-valued continuous functions on $[0,1]$, which is far from being even an integral domain, we provide a twist in the tale by showing that a proof akin to Gauss's lemma works.

Here is the result and a constructive proof.

Lemma. Let $R = C[0,1]$ with addition and multiplication of functions given in terms of their values. Let $F = \sum_{i=0}^{n} f_i X_i, G = \sum_{i=0}^{m} g_i X^i \in R[X]$. If $c(F) = c(G) = R$, then $c(FG) = R$; further, one can prove this constructively.

Note that if $FG = \sum_{i=0}^{m+n} h_i X^i$, then $c(FG) = R$ if, and only if, $h_0, \cdots, h_{m+n}$ have no common zero in $[0,1]$. This is because if h_i's have no common zero, the elements $H_i = \frac{h_i}{\sum_i h_i^2} \in R$ satisfy $\sum_i h_i H_i = 1$, the constant function 1, which is the unity of R. Therefore, the assumptions $c(F) = c(G) = R$ imply that the f_i's have no common zero and the g_j's have no common zero as well. Consider an arbitrary $a \in [0,1]$. Then we would have a smallest r with $0 \leq r \leq n$ for which $f_r(a) \neq 0$; similarly, we would have a smallest s with $0 \leq s \leq m$ so that $g_s(a) \neq 0$. Evidently $h_{r+s}(a) = f_r(a) g_s(a) \neq 0$, which means all the h_i's cannot have a common zero. Hence $c(FG) = R$. This proof is just like the Gauss-lemma proof for $\mathbb{Z}$.

4. Dedekind-Mertens

As mentioned in the previous section, if f, g are polynomials in one variable over any commutative ring with unity, Dedekind and Mertens independently, proved the so-called (by Krull) Dedekind-Mertens Lemma. It has been generalized by Prüfer and many others in diverse directions. The readers can refer to [6] for a recent description of some beautiful generalizations. The paper [4] which defines and studies something called the Dedekind-Mertens number mentions the interesting history of Dedekind and Mertens's works. One form of the original lemma asserts that

$$c(f)^{deg(g)}c(fg) = c(f)^{deg(g)}c(f)c(g).$$

Here is a lovely, simple Gauss-lemma-like proof due to Coquand - who champions the cause of constructive mathematics.

Coquand's Proof of Dedekind-Mertens

Let A be a commutative ring with unity. Suppose $f = \sum_{i=0}^{n} f_i X^i, g = \sum_{j=0}^{m} g_j X^j$ and $h = fg = \sum_{r=0}^{m+n} h_r X^r$ in $A[X]$. Write the content ideals $c(f) = (f_0, \cdots, f_n), c(g) = (g_), \cdots, g_m)$ and $c(h) = (h_0, \cdots, h_{m+n})$. We may take the ring A to be $\mathbb{Z}[f_0, \cdots, f_n, g_0, \cdots, g_m]$ where the f_i's and g_j's can be regarded as indeterminates. We wish to prove $c(f)^{m+1}c(g) \subseteq c(f)^m c(h)$ because the reverse inclusion is evident. Let F, G, H denote, respectively, the abelian subgroup of A generated by the coefficients of f, g, h. We wish to prove:

$$F^{m+1}G \subseteq F^m H.$$

This will be proved by induction on m where it is obvious when $m = 0$. Assume $m > 0$ and let G_m denote the additive subgroup of G generated by $g_0, g_1, \cdots, g_{m-1}$. As usual, a symbol f_k for $k < 0$ or $k > n$ stands for 0. Note

$$h_r = f_{r-m}g_m + \sum_{s<m} f_{r-s}g_s.$$

Therefore,

$$\sum_{s<m} f_{r-s}g_s = h_r - f_{r-m}g_m \in H + Fg_m$$

which gives, by the definition of G_m that

$$FG_m \subseteq H + Fg_m.$$

So, inductively, $F^2G_m \subseteq FH + F^2g_m$,

$$F^3G_m \subseteq F^2H + F^3g_m$$

etc. Inductively, we obtain

$$F^mG_m \subseteq F^{m-1}H + F^mg_m.$$

Therefore, for $0 \leq i \leq n$, we have

$$f_iF^mG_m \subseteq f_iF^{m-1}H + f_iF^mg_m \subseteq F^mH + f_iF^mg_m.$$

On the other hand, since

$$f_ig_m = h_{i+m} - \sum_{s<m} f_{i+m-s}g_s \in H + f_{i+1}G_m + \cdots + f_nG_m.$$

This implies that for all $0 \leq i \leq n$,

$$f_iF^mG_m \subseteq F^mH + f_{i+1}F^mG_m + \cdots + f_nF^mG_m.$$

Taking respectively $i = n, n-1, \cdots$ etc., we have for all $0 \leq i \leq n$ that

$$f_iF^mG_m \subseteq F^mH.$$

Hence $F^{m+1}G_m \subseteq F^mH$ which proves the assertion.

5. GCD Domains

As we saw, some versions of Gauss's lemma involve the GCD of elements. The notions of GCD and LCM can be generalized to any integral domain D in an obvious manner but they do not always exist for two given elements and there are also some surprises. Before starting a discussion, recall that the GCD and LCM of a set of integers is defined only up to sign; so, in reality, one should call it "A" GCD (but understand that it is unique up to multiplication by a unit).

In an integral domain D, define "a" GCD of two non-zero elements $a \neq b$ in D to be an element d such that $d|a, d|b$ and any c dividing both a and b divides d also. It is clear that if c, d are two GCDs of a and b, then they are associates as we are in a domain. A similar definition of "a" LCM is easily given. The first fact which may not be all that surprising is that two elements may not have a GCD at all (because there is no reason to expect they should). But, a fact that is surprising is that two elements may have a GCD but may not an LCM. Moreover, the opposite implication is not true.

For instance, it is a little exercise to check that in the domain $K[X^2, X^3]$ for a field K, the elements X^2, X^3 have GCD 1 (and its associates) but do not have any LCM. We discuss these aspects in some detail now. The readers are invited to read the beautiful exposition by D D Anderson in [1]. For other interesting exercises on GCD domains, readers may refer to Kaplansky's book [5].

We shall use the symbol (a, b) for the ideal generated by a and b and write the qualifiers GCD, LCM etc. explicitly. Anderson uses the symbols $[a, b]$ and $]a, b[$ for GCD and LCM respectively but these are not so common. We first state the following obvious lemma:

Lemma. Let D be an integral domain, and let $0 \neq a, b \in D$. Then,
$GCD(a, b)$ exists if, and only if, the ideal $\cap\{(c) : (c) \supset (a, b)\}$ is principal;
$LCM(a, b)$ exists if, and only if, the ideal $\sum\{(c) : (c) \subset (a) \cap (b)\}$ is principal.
In the respective cases, a generator of the corresponding principal ideal is, respectively, a GCD and an LCM of a and b.
The statements generalize to a finite number of elements.

Proposition. Let D be an integral domain and let $0 \neq a, b \in D$.
(i) If $LCM(a, b)$ exists, then $GCD(a, b)$ also exists and they satisfy

$$GCD(a, b)LCM(a, b) = ab$$

up to units.
(ii) If $c \in D$, and if $GCD(ca, cb)$ exists, then $GCD(a, b)$ exists and

$$c.GCD(a, b) = GCD(ca, cb).$$

Consequently, if $GCD(a, b)$ exists, say d, then the GCD of a/d and b/d exists, and equals 1.
(iii) $LCM(a, b)$ exists if, and only if, $GCD(ca, cb)$ exists for all $c \in D$.
(iv) $GCD(a, b)$ exists for all $0 \neq a, b \in D$ if, and only if, $LCM(c, d)$ exists for all $0 \neq c, d \in D$.
Proof. We prove (i) first.
Suppose $LCM(a, b)$ exists; say ℓ. We want to show that $d := ab/\ell$ equals $GCD(a, b)$. As $a = d\ell/b$ and $b = d\ell/a$, it follows that d divides both a and b. Now suppose that h is a common divisor of a and b. Now as a, b both divide ab/h, ℓ divides ab/h which implies that h divides $ab/\ell = d$. Thus,

we have proved (i).
The proof of (ii) is obvious, and we skip it.
Now, we prove (iii). We first show that if $LCM(a,b)$ exists, then so does $LCM(ca,cb)$ for all $c \in D$. Note that both ca, cb divide $cLCM(a,b)$. Now suppose m is a common multiple of ca, cb. Then c divides m and both a, b divide m/c. Thus $LCM(a,b)$ divides m/c and so $cLCM(a,b)$ divides m. Thus $LCM(ca,cb)$ exists, and equals $cLCM(a,b)$. In particular, by (i), $GCD(ca,cb)$ exists for every c.
Now, we claim that if $GCD(ca,cb)$ exists for every c, then $LCM(a,b)$ exists and equals $ab/GCD(a,b)$. Clearly both a, b divide $ab/GCD(a,b)$. Now, suppose both a, b divide m. Then ab is a common divisor of ma and mb and so ab divides $GCD(ma,mb) = mGCD(a,b)$ by (ii) above. This implies that $ab/GCD(a,b)$ divides m. Thus (iii) follows.
Finally, (iv) is an immediate consequence of (i),(ii),(iii).

Definition. A GCD-domain is an integral domain D such that the equivalent properties in (iv) of the proposition holds; that is, each pair of non-zero elements has a GCD as well as an LCM. The nomenclature is due to I. Kaplansky.

Remarks.
(a) In a commutative ring that is not an integral domain, there is no relation between the existence of an LCM of two elements and the existence of a GCD. For example, in the ring $K[X^2, X^3]/(X^9, X^{10}]$, an LCM of X^5 and X^6 is X^8 whereas these elements do not have a GCD.
(b) In contrast with the polynomial ring over a UFD, it is known that there exist UFDs D such that $D[[X]]$ is not a UFD. It is a fact that these power series rings cannot be GCD-domains also. The proof of this needs other characterizations of GCD domains that we do not go into here, and refer to Anderson's article.

Since UFDs are GCD domains, one can show certain domains such as $\mathbb{Z}[\sqrt{-d}]$ ($d \geq 3$), are not GCD domains and hence not UFDs by exhibiting two elements which do not have an LCM. In a proposition below, we will observe that GCD domains are integrally closed; 'one-third' of the domains in the corollary below (namely, when $-d \equiv 1 \bmod 4$) are not even integrally closed.

Corollary. In each of the domains $\mathbb{Z}[\sqrt{-d}]$ $(d \geq 3)$ with d square-free, there exist two elements a, b such that $GCD(a, b)$ exists but $LCM(a, b)$ does not exist. In particular, $\mathbb{Z}[\sqrt{-d}]$ $(d \geq 3)$, is not a GCD domain and hence, is not a UFD.

Proof. Here $\mathbb{Z}[\sqrt{-d}] = \{a + b\sqrt{-d} : a, b \in \mathbb{Z}\}$.
Firstly, suppose that $d + 1$ is not a prime number. Let $d + 1 = pk$, where p is a prime and $k \geq 2$. Clearly $a^2 + db^2 \neq p$ for any $a, b \in \mathbb{Z}$ because the left hand side is bigger than p if $b \neq 0$. If $p = (a + b\sqrt{-d})(u + v\sqrt{-d})$ in $\mathbb{Z}[\sqrt{-d}]$, then taking complex conjugates we see that $u = a, v = -b$. Thus, $p = a^2 + db^2$, which is impossible as observed above. Therefore, p is an irreducible element in $\mathbb{Z}[\sqrt{-d}]$. Also p does not divide $1 + \sqrt{-d}$ because $p(a + b\sqrt{-d}) = 1 + \sqrt{-d}$ gives $pa = 1$ which is impossible. Thus, $GCD(p, 1 + \sqrt{-d})$ exists, and equals 1.
We shall show that $GCD(pk, (1 + \sqrt{-d})k)$ does not exist. If it did, then by the proposition, $GCD(pk, (1+\sqrt{-d})k) = k$. As $1+\sqrt{-d}$ divides $pk = 1+d$, both $(1 + \sqrt{-d})k$, $1 + \sqrt{-d}$ divide k. Let $k = (1 + \sqrt{-d})(a + b\sqrt{-d}) = (a - bd) + (a + b)\sqrt{-d}$. This gives $a = -b$ and $a - bd = a + ad = k$. Thus $apk = a(1 + d) = k$ which is a contradiction. In view of the proposition, it follows that $LCM(p, 1 + \sqrt{-d})$ does not exist.

Suppose now that $d \geq 3$ and $d + 1$ is a prime. Then d is. Let $d + 4 = 2k$, for some $k > 1$. As above, one easily checks that 2 is irreducible and 2 does not divide $2 + \sqrt{-d}$. Thus $GCD(2, 2 + \sqrt{-d})$ exists and equals 1. We show that $GCD(2k, (2+\sqrt{-d})k)$ does not exist. If it did, then as above, $2+\sqrt{-d}$ divides k and which in turn implies that $4 + d$ divides $k = (4 + d)/2$ in $\mathbb{Z}$, a contradiction which shows that $LCM(2, 2 + \sqrt{-d})$ does not exist.

Remark. In the above proof, note that when $d+1 = pk$, p divides $d+1 = (1 + \sqrt{-d})(1 - \sqrt{-d})$ but p clearly does not divide either of $1 + \sqrt{-d}$ and $1 - \sqrt{-d}$, showing that p, which is irreducible, is not prime. Similarly in the second part of the proof, 2 divides $d + 4 = (2 + \sqrt{-d})(2 - \sqrt{-d})$ but does not divide either of them, which shows that 2 is not prime. This also proves that $\mathbb{Z}[\sqrt{-d}]$ $(d \geq 3)$, is not a UFD.

Proposition. *GCD domains are integrally closed.*

Proof. Let D be a GCD domain, with quotient field K. Let $a/b \in K$

satisfy

$$(a/b)^n + a_{n-1}(a/b)^{n-1} + \cdots + a_0 = 0$$

where $a_i \in D$, $a_0 \neq 0$ and $GCD(a,b) = 1$. The last condition can be assumed without loss of generality because we have by a proposition above $GCD(a/d, b/d) = 1$ if $GCD(a,b) = d$ in a GCD domain. So, we get

$$a^n + a_{n-1}a^{n-1}b + a_{n-2}a^{n-2}b^2 + \cdots + a_0 b^n = 0.$$

Then $b|a^n$. But $GCD(a,b) = 1$ implies $GCD(a^m, b) = 1$ for all $m \geq 1$ by induction on m; indeed, if this is true for m, then any common divisor c of a^{m+1} and b divides a^{m+1} and ab but $GCD(a^{m+1}, ab) = aGCD(a^m, b) = a$. This shows that $b|1$; that is, it is a unit. Hence $a/b \in D$.

5.1. **Gauss Lemma in GCD domains.** In any GCD domain D, Gauss's lemma is valid. Indeed, if we define $f \in D[X]$ to be primitive if GCD of its coefficients is 1, then over a GCD domain D, the polynomial $fg \in D[X]$ is primitive if f, g are. This is an easy exercise - the usual proof for UFDs can be adapted here. But, now we mention another version of Gauss's lemma that is valid over integrally closed domains. This version is the closest in spirit to what Gauss actually stated in his article 42 - albeit, in the case of $\mathbb{Z}$ and $\mathbb{Q}$. The proof is an easy exercise (indeed, it is Ex.8, P.42 of [5]).

Gauss Lemma for Integrally closed domains. *If D is an integrally closed domain with quotient field K, and if $f \in D[X]$ is a monic polynomial such that $f = gh$ with $g, h \in K[X]$ monic, then $g, h \in A[X]$.*

6. Kaplansky's conjecture

Over any commutative ring A with unity, one defines a polynomial $f \in A[X]$ to be Gaussian if $c(f)c(g) = c(fg)$ holds for all polynomials $g \in A[X]$. One calls A a *Gaussian ring* if every polynomial $f \in A[X]$ is Gaussian. Several papers in the last six decades have been written on possible characterizations of Gaussian rings or Gaussian polynomials. It is known that being Gaussian is a local property. In particular, it was known for a long time that if $c(f)$ is locally principal, then f is Gaussian. Similarly, over a domain, it was known that if $c(f)$ is an invertible ideal, then f is Gaussian. Kaplansky conjectured that the converse holds:

Kaplansky's Conjecture If A is a commutative ring with unity and $f \in A[X]$ is Gaussian, then the ideal $c(f)$ is either invertible or locally principal.

The authors of [3] mention that this was a question one of them heard in the 1960's from Kaplansky. In fact, this conjecture also appeared in the PhD thesis of Kaplansky's student H. Tsang in 1965 but has not appeared in print. Many cases of the conjecture have been proved by Sarah Glaz and others but it is not completely proved yet, along with other questions raised by Glaz and others.

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