The Binomial Inequality

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Comprising its four cases, the well-known binomial inequality [1, p. 36] and [3, p. 39]

$$\begin{aligned}\mu a^{\mu-1}(a-b) &> a^\mu - b^\mu \\ &> \mu b^{\mu-1}(a-b) \quad \text{for} \quad \mu < 0 \quad \text{or} \quad \mu > 1, \\ \mu a^{\mu-1}(a-b) &< a^\mu - b^\mu \\ &< \mu b^{\mu-1}(a-b) \quad \text{for} \quad 0 < \mu < 1\end{aligned}$$

with $a > 0, b > 0$ and asserts $a \neq b$.

It is the recasted form of the inequality which compares $f(\mu)$ and $f(1)$ for $x = a/b$ and $x = b/a$ by using the

Theorem. *Given* $0 < x \neq 1$, *the function*

$$\mu \longmapsto f(\mu) = \frac{x^\mu - 1}{\mu}$$

defined on $\mathbb{R} - \{0\}$ *is increasing.*

Proof. For the major part of the theorem, we prove the

Lemma. *If* $a_n \in \mathbb{R}$ *and* $0 < b_n < b_{n+1}$, *then*

$$\frac{a_1}{b_1} < \frac{a_2 - a_1}{b_2 - b_1} < \frac{a_3 - a_2}{b_3 - b_2} < \cdots \Rightarrow \frac{a_1}{b_1} < \frac{a_2}{b_2} < \frac{a_3}{b_3} < \cdots .$$

We have only to note, with $\Delta x_n = x_{n+1} - x_n$, that the implications

$$\begin{aligned}\frac{a_n}{b_n} < \frac{\Delta a_n}{\Delta b_n} \Rightarrow \frac{a_n}{b_n} &< \frac{a_n + \Delta a_n}{b_n + \Delta b_n} \\ &< \frac{\Delta a_n}{\Delta b_n} \Rightarrow \frac{a_n}{b_n} < \frac{a_{n+1}}{b_{n+1}} < \frac{\Delta a_{n+1}}{\Delta b_{n+1}}\end{aligned}$$

hold for $n = 1$ and so prove the lemma by induction.

Let $a_n = x^n - 1$ $(0 < x \neq 1)$ and $b_n = n$. The sequence

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = a_n - a_{n-1}, \quad n = 1, 2, 3, \ldots$$

is increasing by the simple facts

$$a_{n+1} - a_n = x^n f(1), a_n - a_{n-1} = x^n f(-1),$$
$$f(1) - f(-1) = f(1) f(-1) > 0.$$

Hence, by the lemma, the sequence

$$f(n) = \frac{x^n - 1}{n}, \quad n = 1, 2, 3, \ldots$$

is increasing: f is increasing on $\mathbb{N}$.

As $0 < x^{-1} \neq 1$, f with x^{-1} in place of x is increasing on $\mathbb{N}$, so that

$$f(-n) = \frac{x^{-n} - 1}{-n}, \quad n = 1, 2, 3, \ldots$$

is decreasing sequence. We have proved that

$$\begin{aligned}\cdots < f(-2) < f(-1) < f(1) \\ < f(2) < \cdots : f \text{ is increasing on } \mathbb{Z} - \{0\}.\end{aligned}$$

Let $r/s \in \mathbb{Q} - \{0\}$ with $r < s$. Since $\ell r, \ell s \in \mathbb{Z} - \{0\}$ for a properly chosen $\ell \in \mathbb{N}$ and, by $0 < x^{1/\ell} \neq 1$, f with $x^{1/\ell}$ in place of x is increasing on $\mathbb{Z} - \{0\}$,

$$\frac{(x^{1/\ell})^{\ell r} - 1}{\ell r} < \frac{(x^{1/\ell})^{\ell s} - 1}{\ell s},$$

$$f(r) < f(s) : f \text{ is increasing in } \mathbb{Q} - \{0\}.$$

Let $\varrho/\sigma \in \mathbb{R} - \mathbb{Q}$ with $\varrho < \sigma$. The expansions of ϱ and σ as simple continued fractions provide infinite sequences (r_n) and (s_n) in $\mathbb{Q} - \{0\}$ such that

$$r_n < s_n, \quad r_n \downarrow \varrho, \quad s_n \uparrow \sigma.$$

Since f is increasing on $\mathbb{Q} - \{0\}$ and continuous on $\mathbb{R} - \{0\}$ (with $t \mapsto e^t$),

$$f(r_n) < f(s_n), f(r_n) \downarrow f(\varrho), f(s_n) \uparrow f(\sigma),$$

Prof. Nanjundiah passed away recently (on 19th March at the age of 93).

$f(\varrho) < f(\sigma)$: f is increasing on $\mathbb{R} - \mathbb{Q}$.

The previous result and this together prove the theorem.

We may remark, by virtue of our theorem, that the well known Bernoulli inequality [2, p. 477], can be restated simply as $f(\mu) < f(1)$ and $f(\mu) > f(1)$ according as $\mu < 1$ and $\mu > 1$.

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Chord Lengths, Discriminants of Cyclotomic Fields and Reducibility of Cyclotomic Polynomials Modulo Primes

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Abstract. We show that the cyclotomic polynomial Φ_n is reducible modulo all primes if and only if its discriminant is a perfect square, which further happens if and only if $\mathbf{Z}_n^*$ is not cyclic. This is deduced from an elementary exercise in geometry. The exercise in geometry quickly implies the expression for the discriminant of a general cyclotomic field. The geometric problem is the following one. On a unit circle, take n points dividing the circumference into n equal parts. From one of these n points, if we draw the chords to the k-th point from it for each k relatively prime to n, the exercise is to determine the product of their lengths.

1. Introduction

Let us first start with the following simple question. On the unit circle, take n points dividing the circumference into n equal parts. From one of these n points, draw the $n-1$ chords joining it to the other points. It is easy to see that the product of the lengths of these chords is n. A more difficult problem is to start from one of the points and – going in one direction (say, the anticlockwise direction) – drawing the chords joining it to the k-th point from it for each k relatively prime to n, what is the product of the lengths of these chords in this case? After answering this in the proposition below, we derive an expression for the discriminant of any cyclotomic field. The standard method used in textbooks is to compute the discriminant of a cyclotomic field for a prime power and leave the general case as an exercise using properties of discriminants of a compositum of fields (see exercises 4.5.23.4.5.24.4.5.25 of [3]). A direct computation as done below may be of some use. Further, the standard expression for the discriminant is useful also to show that the cyclotomic polynomial Φ_n (for $n > 2$) is reducible modulo all primes if and only if its discriminant is a perfect square, which further happens if and only if $\mathbf{Z}_n^*$ is not cyclic.

2. An Elementary Geometric Problem

Proposition 1. *Let $n > 1$ and let $P_1, \ldots, P_n$ be points on a circle of radius 1 dividing the circumference into n equal parts. Then, we have:*

The product of lengths $\prod_{(l,n)=1, l<n} |\overline{P_1 P_{l+1}}| = p$ or 1 accordingly as to whether $n = p^k$ for a prime p or n is not a power of a prime.

Proof. We may assume that the origin is the centre and that points are $P_{d+1} = e^{2id\pi/n}$ for $d = 0, 1, \ldots, n-1$. Note that the product of lengths of all the chords $P_1 P_i$ is simply $\prod_{d=1}^{n-1} |1 - e^{2id\pi/n}|$. Since the polynomial $1 + X + \cdots + X^{n-1}$ has as roots all the n-th roots of 1 excepting 1 itself, we have

$$\prod_{d=1}^{n-1}(1 - e^{2id\pi/n}) = n$$

by evaluating at $X = 1$. Notice that we have the equality $\prod_{d=1}^{n-1}(1 - e^{2id\pi/n}) = n$ as complex numbers; that is, even without considering absolute values.

Now, let us consider our problem. Here, the product under consideration is

$$\prod_{(d,n)=1} |1 - e^{2id\pi/n}|.$$

First, let us look at the case when $n = p^k$ for some prime p. Then,

$$\prod_{(d,p^k)=1, d<p^k} |1 - e^{2id\pi/p^k}| = \frac{\prod_{d=1}^{p^k-1} |1 - e^{2id\pi/p^k}|}{\prod_{dp<p^k} |1 - e^{2idp\pi/p^k}|}$$
$$= \frac{p^k}{p^{k-1}} = p.$$

Now, suppose that n has at least two prime factors.

Let us start with the identity $\prod_{d=1}^{n-1}(1 - e^{2id\pi/n}) = n$.

If p is a prime dividing n, suppose p^k is the highest power of p dividing n. Then, the product $\prod_{d=1}^{n-1}(1-e^{2id\pi/n})$ contains the products of terms corresponding to d running through multiples of n/p^k; that is, $\prod_{d=1}^{p^k-1}(1 - e^{2id\pi/p^k})$ (which is p^k). We observe that factors occurring for a different prime q dividing n are disjoint from those occurring corresponding to p. Therefore, the factors corresponding to the various primes dividing n contribute $\prod_{p^k \| n} p^k = n$.

On removing these factors corresponding to each prime divisor of n, we will get $\prod_{d\in D}(1 - e^{2id\pi/n}) = 1$, where D consists of those d for which $e^{2id\pi/n}$ does not have prime power order. Thus, if $d \in D$, then $1 - e^{2id\pi/n}$ is a unit since n is not a prime power. Therefore, $1 - e^{2i\pi/n}$ is a unit in the cyclotomic field $\mathbf{Q}(e^{2i\pi/n})$. From Galois theory, we have that the product $\prod_{(d,n)=1}(1 - e^{2id\pi/n})$ is the norm of $1 - e^{2i\pi/n}$ from $\mathbf{Q}(e^{2i\pi/n})$ to $\mathbf{Q}$. As this element is a unit, this product is ± 1. Hence we get $\prod_{(d,n)=1} |1 - e^{2id\pi/n}| = 1$ which proves our assertion in the case when n is not a prime power. The proof is complete.

Remark 1. In the above proof, the second part can also be deduced from the first part of the proof in a different fashion as follows.

Writing $P(n) = \prod_{l=1}^{n-1}(1 - \zeta^l)$ and $Q(n) = \prod_{(d,n)=1} (1 - \zeta^d)$, where $\zeta = e^{2i\pi/n}$, we can see that

$$P(n) = \prod_{r|n} Q(r).$$

By Möbius inversion, $Q(n) = \prod_{d|n} P(d)^{\mu(n/d)} = \prod_{d|n} d^{\mu(n/d)}$ by the simpler first assertion observed at the beginning of the pr of the proposition. The function

$$\log Q(n) = \sum_{d|n} \mu(n/d)\log(d)$$

can be identified with the so-called von Mangoldt function $\Lambda(n)$ which is defined to have the value $\log(p)$ if n is a power of p and 0 otherwise. Using this identification, exponentiation gives also the value asserted in the proposition; viz., $Q(n) = p$ or 1 according as to whether n is a power of a prime p or not.

To see why $\Lambda(n) = \sum_{d|n} \mu(n/d)\log(d)$, we write $n = \prod_{p|n} p^{v_p(n)}$ and note that

$$\log(n) = \sum_{p|n} v_p(n)\log(p)$$

But, the right hand side is clearly $\sum_{d|n} \Lambda(d)$. Hence, Möbius inversion yields

$$\Lambda(n) = \sum_{d|n} \log(d)\mu(n/d).$$

Remark 2. We shall use the above proposition in the next section to compute the discriminant of the field $K := \mathbf{Q}(\zeta_n)$ where ζ_n denotes a primitive n-th root of unity. It implies by the Dedekind-Kummer criterion, the well-known fact that the primes ramifying in $\mathbf{Q}(\zeta_n)$ are exactly those which divide n.

3. Discriminant of $\mathbf{Q}(\zeta_n)$

Lemma 1. *Let $n > 2$ be a positive integer and ζ_n be a primitive n-th root of unity. Then, the discriminant of the cyclotomic field is* $(-1)^{\phi(n)/2}\frac{n^{\phi(n)}}{\prod_{p|n} p^{\phi(n)/(p-1)}}$.

Remark 3. Recall that the ring O_K of algebraic integers of $K = \mathbf{Q}(\zeta_n)$ is $\mathbf{Z}[\zeta_n]$.

The minimal polynomial of ζ_n is the cyclotomic polynomial

$$\Phi_n(X) = \prod_{(r,n)=1} (X - \zeta_n^r).$$

Thus, the discriminant of O_K is that of the polynomial Φ_n up to sign. The polynomial Φ_n has another expression $\Phi_n(X) = \prod_{d|n}(X^d - 1)^{\mu(n/d)}$ which is obtained by Möbius inversion formula to the decomposition

$$X^n - 1 = \prod_{d|n} \Phi_d(X).$$

Now we prove Lemma 1.

Proof of Lemma 1. Since

$$\Phi_n(X) = \prod_{d|n}(X^d - 1)^{\mu(n/d)}$$
$$= (X^n - 1)\prod_{d|n, d<n}(X^d - 1)^{\mu(n/d)},$$

we may write

$$\Psi(X) := \frac{X^n - 1}{\Phi_n(X)} = \prod_{d|n,d<n}(X^d - 1)^{-\mu(n/d)}$$

Now, differentiating $X^n - 1 = \Phi_n(X)\Psi(X)$ and putting $X = \zeta_n$, we get $n\zeta_n^{-1} = \Phi'_n(\zeta_n)\Psi(\zeta_n)$.

We have the discriminant $d(K) = \pm N_{K/\mathbf{Q}}\Phi'_n(\zeta_n) = \pm n^{\phi(n)}$ $N_{K/\mathbf{Q}}(\Psi(\zeta_n))^{-1}$.

Now $\Psi(\zeta_n)^{-1} = \prod_{d|n,d<n}(\zeta_n^d - 1)^{\mu(n/d)}$ which is convenient to write (using n/d instead of d) as:

$$\Psi(\zeta_n)^{-1} = \prod_{d|n,d>1}(\zeta_n^{n/d} - 1)^{\mu(d)}$$

Separating the terms corresponding to $\mu(d) = 1$ and to $\mu(d) = -1$, we have

$$\Psi(\zeta_n)^{-1} = \frac{\prod_{d|n,d>1,\mu(d)=1}(\zeta_n^{n/d} - 1)}{\prod_{d|n,d>1,\mu(d)=-1}(\zeta_n^{n/d} - 1)}$$

Now, for each divisor d of n, $\zeta_n^{n/d}$ is a primitive d-th root of unity. By proposition 1 above, $1 - \zeta_n^{n/d}$ is a unit unless d is a prime power. In the above expression for $\Psi(\zeta_n)^{-1}$, a nontrivial term in the denominator corresponds to $\mu(d) = -1$ which can happen for a prime power d only if d is prime. In the numerator, the condition $\mu(d) = 1$ cannot happen for any prime power d. In other words,

$$\Psi(\zeta_n)^{-1} = (\text{unit}) \cdot \prod_{p|n}(\zeta_n^{n/p} - 1)^{-1}$$

So, its norm is $\pm\prod_{p|n} N_{K/\mathbf{Q}}(\zeta_n^{n/p} - 1)^{-1}$ as units have norm ± 1.

As $\zeta_n^{n/p}$ is a primitive p-th root of unity, it is in the subfield $\mathbf{Q}(\zeta_p)$ generated by a primitive p-th root of unity, and we have

$$N_{K/\mathbf{Q}}(\zeta_n^{n/p} - 1) = (N_{\mathbf{Q}(\zeta_p)/\mathbf{Q}}(\zeta_p - 1))^{[K:\mathbf{Q}(\zeta_p)]}$$
$$= (\pm p)^{\phi(n)/(p-1)}$$

Thus, we get

$$d(K) = \pm\frac{n^{\phi(n)}}{\prod_{p|n} p^{\phi(n)/(p-1)}}$$

Finally, it is well-known (and easy to deduce from the definition) that for any number field L, the discriminant $d(L)$ has sign $(-1)^s$ where s is the number of complex places of L. Our field $K = \mathbf{Q}(\zeta_n)$ has $s = \phi(n)/2$ because primitive n-th roots of unity are all complex.

4. Reducibility of Cyclotomic Polynomials Modulo Primes

The cyclotomic polynomial Φ_n is the monic, irreducible polynomial of a primitive n-th root of unity but it may happen to be reducible modulo certain primes. In this section, we investigate when this happens.

Lemma 2. *For a positive integer $n > 2$, if* $\operatorname{disc}(\Phi_n)$ *is a perfect square, then Φ_n is reducible modulo every prime.*

Proof. This is a standard application of Galois theory. Indeed, it is well-known that if the discriminant of a Galois extension is a square, its Galois group would be contained in the subgroup of even permutations ([2], Lemma 12.3). So, if Φ_n were irreducible modulo some prime p, then the reduction of Φ_n mod p generates over $\mathbf{F}_p$ a Galois extension of degree $\phi(n)$; the Galois group would contain a $\phi(n)$-cycle which is an odd permutation since $\phi(n)$ is even for $n > 2$.

Proposition 2. *For $n > 2$, the polynomial Φ_n is reducible modulo every prime if, and only if,* $\operatorname{disc}(\Phi_n)$ *is a perfect square. If* $\operatorname{disc}(\Phi_n)$ *is not a perfect square – which happens if, and only if, $n = 4$, p^k or $2p^k$ – then there are infinitely many primes p such that Φ_n is irreducible modulo p.*

Proof. We have already seen that if $\operatorname{disc}(\Phi_n)$ is a perfect square in $\mathbf{Z}$, then Φ_n is reducible modulo every prime. Conversely, suppose $\operatorname{disc}(\Phi_n)$ is not a perfect square. Then, looking at the expression $(-1)^{\phi(n)/2}\frac{n^{\phi(n)}}{\prod_{p|n} p^{\phi(n)/(p-1)}}$ for the discriminant, we shall deduce that $n = 4$, p^k or $2p^k$ for some odd prime. Indeed, write

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$$

Firstly, if n is odd and $r > 1$, clearly,

$$\frac{\phi(n)}{2} = \frac{\prod_{i=1}^r p_i^{\alpha_i - 1}(p_i - 1)}{2}$$

is even and the power of p_i dividing the discriminant is

$$(\alpha_i(p_i - 1) - 1)\left(\prod_{k=1}^r p_k^{\alpha_k - 1}\right)\left(\prod_{j\neq i}(p_j - 1)\right)$$

which is even.

Thus, if $n > 2$ is odd, then the discriminant is a perfect square unless $n = p^k$.

If $n = 2p_1^{\alpha_1} \dots p_r^{\alpha_r}$ for some odd primes, $\Phi_n = \Phi_{n/2}$ and the discriminant is a perfect square excepting the case $r = 1$; i.e., $n = 2p^k$.

Now, if $n = 2^\alpha p_1^{\alpha_1} \dots p_r^{\alpha_r}$ with either $\alpha > 2$ or $\alpha = 2$ and $r \geq 1$, then again the powers of 2 and each p_i dividing the discriminant are all even.

Thus, the exceptional case is $n = 4$.

Therefore, we have deduced that the expression for discriminant is a perfect square excepting the cases $n = 4$, p^k and $2p^k$ for an odd prime. These exceptional cases are when the Galois group of the cyclotomic field is cyclic.

The Galois group of Φ_n over $\mathbf{Q}$ is a cyclic group of order $\phi(n)$ and contains a $\phi(n)$-cycle. By the Frobenius density theorem ([1]), there are infinitely many prime numbers l such that the decomposition group at l is cyclic of order $\phi(n)$ which means that Φ_n modulo l is irreducible and generates the extension of degree $\phi(n)$ over $\mathbf{F}_l$. This proves the proposition.

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Complex Hyperbolic Triangle Groups of Type (n, n, ∞)

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Abstract. A complex hyperbolic triangle group is a group generated by three complex reflections fixing complex geodesics in complex hyperbolic space. In this paper we survey our results on complex hyperbolic triangle groups of type (n, n, ∞) in [7,8] and [11].

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1. Introduction

In the real hyperbolic plane $\mathrm{H}^2_{\mathbf{R}}$, a triangle is given by three distinct points in $\overline{\mathrm{H}^2_{\mathbf{R}}}$ joined by geodesics. Let p_1, p_2, p_3 be integers greater than or equal to 2. We allow the possibility that some of the integers are infinite. Let $G(p_1, p_2, p_3)$ be the group generated by three reflections in the sides of a triangle having angles $\pi/p_1, \pi/p_2, \pi/p_3$. This group $G(p_1, p_2, p_3)$ is called a *triangle group* of type (p_1, p_2, p_3) (see [1]).[a]

This paper is concerned with the analogous group in complex hyperbolic 2-space $\mathrm{H}^2_{\mathbf{C}}$. A complex hyperbolic triangle is a triple (C_1, C_2, C_3) of complex geodesics in $\mathrm{H}^2_{\mathbf{C}}$. We assume that C_{k-1} and C_k either meet at the angle π/p_k for some integer $p_k \geq 2$ or else C_{k-1} and C_k are asymptotic, in which case they make an angle 0 and in this case we write $p_k = \infty$, where the indices are taken mod 3. Let Γ be the group of holomorphic isometries of $\mathrm{H}^2_{\mathbf{C}}$ generated by complex reflections i_1, i_2, i_3 fixing complex geodesics C_1, C_2, C_3, respectively. We call Γ a *complex hyperbolic triangle group*. We can index a complex hyperbolic triangle group by a triple (p_1, p_2, p_3). A group Γ with (p_1, p_2, p_3) is said to be a *complex hyperbolic triangle group of type* (p_1, p_2, p_3), which is denoted by $\Gamma(p_1, p_2, p_3)$. In $\mathrm{H}^2_{\mathbf{R}}$, (p_1, p_2, p_3) determines a unique triangle group. On the other hand, in $\mathrm{H}^2_{\mathbf{C}}$ the situation is much different. Actually, for each such triple there is a one real parameter family of complex hyperbolic triangle groups. It is interesting to ask which values of this parameter correspond to discrete groups.

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Complex hyperbolic triangle groups were originally studied by Mostow in [13], where he constructed the first non-arithmetic lattices in PU(2, 1). The deformation theory of complex hyperbolic triangle groups was begun in [4], where complex hyperbolic triangle groups of type (∞, ∞, ∞) were discussed. Since then there have been many developments. In [4] Goldman and Parker investigated for which values of parameter the corresponding representation is a discrete embedding. The Cartan's angular invariant parametrizes this deformation space. Goldman and Parker obtained a necessary condition on this invariant for the corresponding representation to be a discrete embedding. In the same paper [4] they conjectured that this condition above is also sufficient. Schwartz proved this conjecture and sharpened it in [19]. In [20] Schwartz obtained a surprising result showing a certain relation between a complex hyperbolic triangle group and the Whitehead link complement, which is a classic example of finite volume hyperbolic 3-manifold. This also demonstrates the importance of the study on complex hyperbolic triangle groups. Complex hyperbolic triangle groups are the simplest groups, but even in this case we know only a small number of discrete groups. Generally speaking, it is not easy to show a group to be discrete. It seems that this will be done in a case by case fashion. Therefore, we try to restrict the range to search for discrete groups by finding non-discrete groups.

In this paper we restrict our attention to complex hyperbolic triangle groups of type (n, n, ∞) and give a list of non-discrete complex hyperbolic triangle groups of type $(n, n, \infty; k)$. In particular, we show that complex hyperbolic triangle groups of type $(n, n, \infty; k)$ for $n \geq 22$ are not discrete.

This is a survey of our results in [7,8] and [11].

2. Preliminaries

We recall some basic notions of complex hyperbolic geometry. Let $\mathbf{C}^{2,1}$ be the complex vector space of dimension 3, equipped with the Hermitian form

$$\langle Z, W\rangle = Z_0\bar{W}_0 + Z_1\bar{W}_1 - Z_2\bar{W}_2,$$

where $Z = (Z_0, Z_1, Z_2)$, $W = (W_0, W_1, W_2) \in \mathbf{C}^{2,1}$. We call a vector $Z \in \mathbf{C}^{2,1}$ *negative* (respectively *null, positive*) if $\langle Z, Z\rangle < 0$ (respectively $\langle Z, Z\rangle = 0$, $\langle Z, Z\rangle > 0$). Let $\pi : \mathbf{C}^{2,1} - \{0\} \to P^2_{\mathbf{C}}$ (complex projective space) be the projection map defined by $\pi((Z_0, Z_1, Z_2)) = (Z_0/Z_2, Z_1/Z_2)$. The *complex hyperbolic 2-space* $\mathrm{H}^2_{\mathbf{C}}$ is defined as complex projectivization of the set of negative vectors in $\mathbf{C}^{2,1}$. Let PU(2, 1) be the projectivization of SU(2, 1), that is the group of matrices with determinant 1 which are unitary with respect to the Hermitian form. Non-trivial elements in PU(2, 1) fall into three conjugacy classes, depending on the location and the number of fixed points. An element g is *elliptic* if it has a fixed point in $\mathrm{H}^2_{\mathbf{C}}$, *parabolic* if it has a unique fixed point on the boundary $\partial\mathrm{H}^2_{\mathbf{C}}$, *loxodromic* if it fixes a unique pair of points on $\partial\mathrm{H}^2_{\mathbf{C}}$. Furthermore, we say that an elliptic element g is *regular elliptic* if and only if its eigenvalues are distinct. A parabolic element g is *unipotent* if all eigenvalues of g are 1. Using the discriminant function

$$f(z) = |z|^4 - 8\,\mathrm{Re}(z^3) + 18|z|^2 - 27,$$

we can classify elements of PU(2, 1) by traces of the corresponding matrices in SU(2, 1). In [3, Theorem 6.2.4], Goldman states that an element g in SU(2, 1) is regular elliptic if and only if $f(\tau(g)) < 0$, where $\tau(g)$ is the trace of g.

The intrinsic metric on $\mathrm{H}^2_{\mathbf{C}}$ is the Bergman metric. For any pair of points z, w in $\mathrm{H}^2_{\mathbf{C}}$, the *complex hyperbolic distance* $d(z, w)$ is given by:

$$\cosh^2\left(\frac{d(z, w)}{2}\right) = \frac{\langle Z, W\rangle\langle W, Z\rangle}{\langle Z, Z\rangle\langle W, W\rangle}.$$

We see that the group of holomorphic isometries of $\mathrm{H}^2_{\mathbf{C}}$ is exactly PU(2, 1).

The boundary $\partial\mathrm{H}^2_{\mathbf{C}}$ is homeomorphic to S^3 and one of representation we choose for this is $(\mathbf{C} \times \mathbf{R}) \cup \{\infty\}$, with points either ∞ or $(z, r)_H$ with $z \in \mathbf{C}$ and $r \in \mathbf{R}$, where $(0, -1, 1) \in \mathbf{C}^{2,1}$ corresponds to ∞. We call $(z, r)_H$ the $H-$coordinates. Let H denote this representation, that is, $(\mathbf{C} \times \mathbf{R}) \cup \{\infty\}$. We define the *Cygan metric* δ by

$$\delta((z, r)_H, (w, R)_H) = ||z - w|^2 + ir - iR + 2i\mathrm{Im}(z\bar{w})|^{\frac{1}{2}}$$

for $(z, r)_H, (w, R)_H$ in $H - \{\infty\}$. This metric is thought as the counterpart of the Euclidean metric.

In $\mathrm{H}^2_{\mathbf{C}}$ there are two kinds of totally geodesic subspaces, *totally real totally geodesic subspaces* and *totally geodesic complex subspaces*. The former is isometric to $\mathrm{H}^2_{\mathbf{C}} \cap \mathbf{R}^2$. The latter is isometric to $\mathrm{H}^2_{\mathbf{C}} \cap \mathbf{C}$, which is called a *complex geodesic*. A complex geodesic C is uniquely determined by a positive vector $V \in \mathbf{C}^{2,1}$, that is, $C = \pi(\{U \in \mathbf{C}^{2,1} | \langle U, V\rangle = 0\})$. We call V a *polar vector* to C. Two distinct complex geodesics in $\mathrm{H}^2_{\mathbf{C}}$ intersect in either the empty set or a point. Let C_1 and C_2 be distinct complex geodesics corresponding to polar vectors $V_1, V_2 \in \mathbf{C}^{2,1}$, respectively. At a point of intersection, C_1 and C_2 intersect at the *complex angle* ϕ, which is defined as

$$\cos\phi = \frac{|\langle V_1, V_2\rangle|}{\sqrt{\langle V_1, V_1\rangle\langle V_2, V_2\rangle}}.$$

Given a complex geodesic C with polar vector V, there is a unique involution i, that fixes every point in C. We call i the *complex reflection* in C. Explicitly i is given by

$$i(Z) = -Z + \frac{2\langle Z, V\rangle}{\langle V, V\rangle}V.$$

More details on this subject can be found in [3,6] and [9].

3. Complex Hyperbolic Triangle Groups of Type (n, n, ∞)

In this section we show intervals of non-discreteness for different values n. In [19] Schwartz considered ideal triangle groups, that is complex hyperbolic triangle groups of type (∞, ∞, ∞) and proved that if the product $i_1 i_2 i_3$ of generators is regular elliptic, then it is not of finite order, hence the corresponding complex hyperbolic triangle group is not discrete. In [15] Parker explored groups of type (n, n, n) such that $i_1 i_2 i_3$ is regular elliptic. In this case there are some discrete groups. And he classified them. In the same manner as in the proof of Schwartz in [19], Wyss-Gallifent formulated Schwartz's statement for groups of type (n, n, ∞) in [23, Lemma 3.4.0.19]. In [18] Pratoussevitch made a refinement on the proof of Wyss-Gallifent. Here we show the result due to Wyss-Gallifent and Pratoussevitch.

Theorem 1 ([18]). *Let $\Gamma = \langle i_1, i_2, i_3\rangle$ be a complex hyperbolic triangle group of type (n, n, ∞). If the product $i_1 i_2 i_3$ of the three generators is regular elliptic, then Γ is non-discrete.*

By conjugation, we may assume that the forms of i_j as follows:

$$i_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$i_2 = \begin{bmatrix} 1 & -2s & -2s \\ -2s & 2s^2-1 & 2s^2 \\ 2s & -2s^2 & -2s^2-1 \end{bmatrix} \quad \text{and}$$

$$i_3 = \begin{bmatrix} 1 & -2se^{i\theta} & -2se^{i\theta} \\ -2se^{-i\theta} & 2s^2-1 & 2s^2 \\ 2se^{-i\theta} & -2s^2 & -2s^2-1 \end{bmatrix},$$

where $s = \cos(\pi/n)$. Using this theorem, we work out some conditions on $\cos\theta$ for Γ of type (n, n, ∞) to be non-discrete. We see that if $n < 9$, then the product $i_1 i_2 i_3$ is not regular elliptic and that if $n \geq 9$, then it is regular elliptic for $\cos\theta \in (\alpha_n, \beta_n)$. Note that α_n and β_n are increasing functions of n. Denote by $E_{123}(n)$ the interval (α_n, β_n) (see Table 1). By using Theorem 1, we obtain.

Table 1. *Approximations of α_n, β_n and γ_n.*

n	α_n	β_n	γ_n
3	——	——	0.8923
4	——	——	0.9691
5	——	——	0.9819
6	——	——	0.9862
7	——	——	0.9882
8	——	——	0.9893
9	0.9312	0.9319	0.9900
10	0.9367	0.9423	0.9905
11	0.9403	0.9510	0.9908
12	0.9427	0.9580	0.9910
13	0.9445	0.9637	0.9913
14	0.9458	0.9684	0.9914
15	0.9469	0.9722	0.9915
16	0.9477	0.9754	0.9916
17	0.9484	0.9781	0.9917
18	0.9489	0.9804	0.9918
19	0.9494	0.9823	0.9918
20	0.9498	0.9840	0.9919
21	0.9501	0.9854	0.9919
22	0.9504	0.9867	0.9919
23	0.9506	0.9878	0.9920
24	0.9509	0.9887	0.9920
25	0.9510	0.9896	0.9920
26	0.9512	0.9904	0.9920
27	0.9814	0.9911	0.9920
28	0.9515	0.9917	0.9921
29	0.9516	0.9922	0.9921
30	0.9517	0.9927	0.9921
50	0.9527	0.9973	0.9922
200	0.9531	0.9998	0.9922

Theorem 2 ([7,11]). *Let $n \geq 9$. If $\cos\theta \in E_{123}(n)$, then $\Gamma(n, n, \infty)$ is not discrete.*

Next we use a complex hyperbolic version of Jørgensen's inequality to find out some sufficient conditions on $\cos\theta$ for Γ to be non-discrete. Let g be an element of $\mathrm{PU}(2,1)$. We define the translation length $t_g(p)$ of g at $p \in H$ by $t_g(p) = \delta(g(p), p)$. To state Theorem 3, we need the notion of isometric spheres. Let $h = (a_{mn})_{1\leq m,n\leq 3}$ be an element of $\mathrm{PU}(2,1)$ not fixing ∞. The *isometric sphere* of

h is the sphere in the Cygan metric with center $h^{-1}(\infty)$ and radius

$$R_h = \sqrt{\frac{2}{|a_{22} - a_{23} + a_{32} - a_{33}|}}$$

(see [4–6] and [7]).

Here we recall the complex hyperbolic version of Shimizu's lemma due to Parker [14].

Theorem 3 ([14]). *Let G be a discrete subgroup of* $\mathrm{PU}(2,1)$ *that contains the unipotent parabolic element g with the form*

$$g = \begin{bmatrix} 1 & \tau & \tau \\ -\bar{\tau} & 1-(|\tau|^2 - it)/2 & -(|\tau|^2 - it)/2 \\ \bar{\tau} & (|\tau|^2 - it)/2 & 1+(|\tau|^2 - it)/2 \end{bmatrix}.$$

The element g fixes ∞ and maps the point with H-coordinates $(\zeta, v)_H$ to the point with H-coordinates $(\zeta + \tau, v + t + 2\mathrm{Im}(\tau\bar{\zeta}))_H$. Let h be any element of G not fixing ∞ and with isometric sphere of radius R_h. Then

$$R_h^2 \leq t_g(h^{-1}(\infty))t_g(h(\infty)) + 4|\tau|^2.$$

We apply this theorem to our $\Gamma(n, n, \infty)$. It follows that the above inequality is true only for $\cos\theta$ with $\gamma_n < \cos\theta < 1$, where γ_n is an increasing function of n (see Table 1). Thus we have

Theorem 4 ([11]). $\Gamma(n, n, \infty)$ *is not discrete for* $\cos\theta \in (\gamma_n, 1)$.

We show the beginning of the list of approximations of α_n, β_n and γ_n in Table 1.

4. Complex Hyperbolic Triangle Groups of Type $(n, n, \infty; k)$

Let $\Gamma = \langle i_1, i_2, i_3\rangle$ be a complex hyperbolic triangle group of type (n, n, ∞). If the trace of the element $i_1i_2i_1i_3$ is equal to $1 + 2\cos\frac{2\pi}{k}$, where k is a positive integer ≥ 3, then Γ is said to be of *type* $(n, n, \infty; k)$. This group is denoted by $\Gamma(n, n, ; \infty; k)$.

In this section we discuss non-discrete complex hyperbolic triangle groups of type $(n, n, \infty; k)$.

We have

$$\mathrm{trace}(i_1i_2i_1i_3) = 3 - 16s^2\cos\theta + 16s^4$$
$$= 1 + 2\cos\frac{2\pi}{k}.$$

By Table 1, we can see which values k correspond to non-discrete complex hyperbolic triangle groups of type $(n, n, \infty; k)$.

To find more non-discrete complex hyperbolic triangle groups of type $(n, n, \infty; k)$, we use another complex hyperbolic version of Jørgensen's inequality.

Theorem 5 ([5,10]). *Let $g \in$* $\mathrm{PU}(2,1)$ *be a regular elliptic element of order $n \geq 7$ that preserves a Lagrangian plane (i.e. trace(g) is real). Suppose that g fixes a point $z \in \mathbf{H}^2_{\mathbb{C}}$. Let h be any element of* $\mathrm{PU}(2,1)$ *with $h(z) \neq z$. If*

$$\cosh\left(\frac{d(h(z), z)}{2}\right)\sin\left(\frac{\pi}{n}\right) < \frac{1}{2},$$

then $\langle g, h\rangle$ is not discrete.

Taking i_1i_2 as g in Theorem 5, we obtain

Theorem 6 ([8]). *Let Γ be a complex hyperbolic triangle group of type $(n, n, \infty; k)$ with $n \geq 7$. Let*

$$a_n = -1 + 8\cos^4(\pi/n) - 6\cos^2(\pi/n) - \sin(\pi/n)$$

and

$$b_n = -1 + 8\cos^4(\pi/n) - 6\cos^2(\pi/n) + \sin(\pi/n).$$

If $a_n < \cos(2\pi/k) < b_n$, then Γ is not discrete.

We tabulate some approximations of a_n and b_n.

By Table 2, we have additional 36 non-discrete complex hyperbolic triangle groups of type $(n, n, \infty; k)$.

Table 2. *Approximations of a_n and b_n.*

n	a_n	b_n
7	−1.0329	−0.1652
8	−0.6755	0.0897
9	−0.4023	0.2817
10	−0.1909	0.4270
11	−0.0250	0.5384
12	0.1073	0.6248
13	0.2143	0.6928
14	0.3020	0.7469
15	0.3748	0.7905
16	0.4359	0.8260
17	0.4878	0.8552
18	0.5321	0.8793
19	0.5704	0.8995
20	0.6037	0.9165
21	0.6328	0.9308
22	0.6585	0.9430
23	0.6812	0.9535
24	0.7015	0.9624
25	0.7196	0.9702
26	0.7359	0.9769
27	0.7506	0.9827
28	0.7640	0.9878

Now we show a different way to find non-discrete groups. It is well-known that if a group has an elliptic element of infinite order, then this group is not discrete. To find elliptic element of infinite order in a group, we use the following theorem due to Conway and Jones, which lists all possible trigonometric Diophantine equations with up to four terms.

Theorem 7 ([2]). *Suppose that we are given at most four distinct rational multiples of π lying strictly between 0 and 2π for which some rational linear combination of their cosines is rational, but no proper subsum has this property. Then this linear combination is proportional to one of the following:*

$$\frac{1}{2} = \cos\left(\frac{\pi}{3}\right),$$

$$0 = -\cos(\phi) + \cos\left(\phi - \frac{\pi}{3}\right) + \cos\left(\phi + \frac{\pi}{3}\right), \quad \textit{where} \quad 0 < \phi < \frac{\pi}{6},$$

$$\frac{1}{2} = \cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{2\pi}{5}\right),$$

$$\frac{1}{2} = \cos\left(\frac{\pi}{7}\right) - \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right),$$

$$\frac{1}{2} = \cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{\pi}{15}\right) + \cos\left(\frac{4\pi}{15}\right),$$

$$\frac{1}{2} = -\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{2\pi}{15}\right) - \cos\left(\frac{7\pi}{15}\right),$$

$$\frac{1}{2} = \cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) - \cos\left(\frac{\pi}{21}\right) + \cos\left(\frac{8\pi}{21}\right),$$

$$\frac{1}{2} = \cos\left(\frac{\pi}{7}\right) - \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{2\pi}{21}\right) - \cos\left(\frac{5\pi}{21}\right),$$

$$\frac{1}{2} = -\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{4\pi}{21}\right) - \cos\left(\frac{10\pi}{21}\right),$$

$$\frac{1}{2} = -\cos\left(\frac{\pi}{15}\right) + \cos\left(\frac{2\pi}{15}\right) + \cos\left(\frac{4\pi}{15}\right) - \cos\left(\frac{7\pi}{15}\right).$$

Assume that $i_2i_1i_2i_3$ is a regular elliptic element. Then trace$(i_2i_1i_2i_3)$ is written as

$$\text{trace}(i_2i_1i_2i_3) = 20s^2 - 16s^2\cos\theta - 1 = 1 + 2\cos\phi\pi,$$

which yields that

$$\cos\phi\pi = 10s^2 - 8s^2\cos\theta - 1,$$

where ϕ is a real number. We obtain

$$\begin{aligned}\cos\phi\pi &= -8s^4 + 10s^2 - 2 + \cos\frac{2\pi}{k} \\ &= -\cos\frac{4\pi}{n} + \cos\frac{2\pi}{n} + \cos\frac{2\pi}{k}.\end{aligned}$$

It is seen that in each group of type $(5, 5, \infty; 3)$,$(7, 7, \infty; 4)$, $(9, 9, \infty; 5)$, $(11, 11, \infty; 6)$, $(12, 12, \infty; 7)$ or $(14, 14, \infty; 8)$, $i_2i_1i_2i_3$ is regular elliptic. Theorem 7 tells us that for $(n, k) = (5, 3), (7, 4), (9, 5), (11, 6), (12, 7)$ and $(14, 8)$, there are no rational numbers $\phi's$ satisfying

$$\cos\phi\pi = -\cos\frac{4\pi}{n} + \cos\frac{2\pi}{n} + \cos\frac{2\pi}{k}.$$

It follows that in each group of type $(5, 5, \infty; 3)$, $(7, 7, \infty; 4)$, $(9, 9, \infty; 5)$, $(11, 11, \infty; 6)$, $(12, 12, \infty; 7)$ or $(14, 14, \infty; 8)$, $i_2i_1i_2i_3$ is a regular elliptic element of infinite order. Therefore, the groups $\Gamma(5, 5, \infty; 3)$, $\Gamma(7, 7, \infty; 4)$, $\Gamma(9, 9, \infty; 5)$, $\Gamma(11, 11, \infty; 6)$, $\Gamma(12, 12, \infty; 7)$ and $\Gamma(14, 14, \infty; 8)$ are not discrete.

Next consider elements $i_1i_2i_1i_2i_3i_2$ and $i_3i_1i_3i_1i_2i_1$. In the same manner as the above, we see that in $\Gamma(8, 8, \infty; 5)$, $i_1i_2i_1i_2i_3i_2$ is a regular elliptic element of infinite order. Hence $\Gamma(8, 8, \infty; 5)$ is not discrete. Moreover, $i_3i_1i_3i_1i_2i_1$ is a regular elliptic element of infinite order in $\Gamma(6, 6, \infty; 5)$, $\Gamma(7, 7, \infty; 6)$, $\Gamma(9, 9, \infty; 6)$, $\Gamma(10, 10, \infty; 6)$, and $\Gamma(15, 15, \infty; 10)$. Thus these groups are not discrete.

We summarize the above and show a list of non-discrete groups of type $(n, n, \infty; k)$. In particular, we see that $\Gamma(n, n, \infty; k)$ for $n \geq 22$ is not discrete.

Theorem 8 ([8]). *Let $\Gamma = \langle i_1, i_2, i_3\rangle$ be a complex hyperbolic triangle group of type $(n, n, \infty; k)$. Let $k \geq [n/2] + 1$. The following groups are non-discrete.*

(1) $\Gamma(5, 5, \infty; 3)$.
(2) $\Gamma(6, 6, \infty; 5)$.
(3) $\Gamma(7, 7, \infty; 4)$, $\Gamma(7, 7, \infty; 6)$.
(4) $\Gamma(8, 8, \infty; 5)$.
(5) $\Gamma(9, 9, \infty; 5)$, $\Gamma(9, 9, \infty; 6)$.
(6) $\Gamma(10, 10, \infty; 6)$, $\Gamma(10, 10, \infty; 9)$.
(7) $\Gamma(11, 11, \infty; 6)$, $\Gamma(11, 11, \infty; 10)$, $\Gamma(11, 11, \infty; 11)$.
(8) $\Gamma(12, 12, \infty; 7)$, *and* $\Gamma(12, 12, \infty; k)$ *for* $11 \leq k \leq 16$.

(9) $\Gamma(13, 13, \infty; 7)$, *and* $\Gamma(13, 13, \infty; k)$ *for* $12 \leq k \leq 38$.

(10) $\Gamma(14, 14, \infty; 8)$, *and* $\Gamma(14, 14, \infty; k)$ *for* $k \geq 12$.

(11) $\Gamma(15, 15, \infty; 8)$, $\Gamma(15, 15, \infty; 9)$, $\Gamma(15, 15, \infty; 10)$, *and* $\Gamma(15, 15, \infty; k)$ *for* $k \geq 13$.

(12) $\Gamma(16, 16, \infty; 9)$, $\Gamma(16, 16, \infty; 10)$, *and* $\Gamma(16, 16, \infty; k)$ *for* $k \geq 14$.

(13) $\Gamma(17, 17, \infty; 9)$, $\Gamma(17, 17, \infty; 10)$, $\Gamma(17, 17, \infty; 11)$, *and* $\Gamma(17, 17, \infty; k)$ *for* $k \geq 15$.

(14) $\Gamma(18, 18, \infty; 10)$, $\Gamma(18, 18, \infty; 11)$, $\Gamma(18, 18, \infty; 12)$, $\Gamma(18, 18, \infty; k)$ *for* $k \geq 16$.

(15) $\Gamma(19, 19, \infty; 10)$, $\Gamma(19, 19, \infty; 11)$, $\Gamma(19, 19, \infty; 12)$, $\Gamma(19, 19, \infty; 13)$, *and* $\Gamma(19, 19, \infty; k)$ *for* $k \geq 17$.

(16) $\Gamma(20, 20, \infty; 11)$, $\Gamma(20, 20, \infty; 12)$, $\Gamma(20, 20, \infty; 13)$, $\Gamma(20, 20, \infty; 14)$, $\Gamma(20, 20, \infty; 15)$ *and* $\Gamma(20, 20, \infty; k)$ *for* $k \geq 18$.

(17) $\Gamma(21, 21, \infty; 11)$, $\Gamma(21, 21, \infty; 12)$, $\Gamma(21, 21, \infty; 13)$, $\Gamma(21, 21, \infty; 14)$, $\Gamma(21, 21, \infty; 15)$, $\Gamma(21, 21, \infty; 16)$, *and* $\Gamma(21, 21, \infty; k)$ *for* $k \geq 19$.

(18) $\Gamma(22, 22, \infty; k)$ *for any* $k(\geq 12)$.

(19) $\Gamma(n, n, \infty; k)$ *for any* $n(> 22)$.

Remark 4. *The following 10 groups are discrete:*
$\Gamma(3, 3, \infty; 4)$, $\Gamma(3, 3, \infty; 6)$, $\Gamma(3, 3, \infty; \infty)$;
$\Gamma(4, 4, \infty; 3)$, $\Gamma(4, 4, \infty; 4)$, $\Gamma(4, 4, \infty; 6)$, $\Gamma(4, 4, \infty; \infty)$;
$\Gamma(6, 6, \infty; 4)$, $\Gamma(6, 6, \infty; 6)$, $\Gamma(6, 6, \infty; \infty)$.

Remark 5. *Our list in Theorem 8 is not complete. Except for groups in Theorem 8 and Remark 1, we do not know whether a group of type* $(n, n, \infty; k)$ *is discrete or not.*

5. Problems

Schwartz has given a conjectural overview on complex hyperbolic triangle groups in [21]. In [16] Parker has given an excellent survey on complex hyperbolic lattices. We can find many conjectures and open problems on complex hyperbolic triangle groups in [16,17,21,22] and [23]. As we are particularly interested in complex hyperbolic triangle groups of type (n, n, ∞), we give some questions only on them.

(1) Complete the list of Theorem 8.

(2) When a complex hyperbolic triangle group of type $(n, n, \infty; k)$ is given, we ask if this group is discrete.

(3) Assume that neither $i_1i_2i_3$ nor $i_1i_2i_1i_3$ is elliptic in $\Gamma(n, n, \infty)$. Is this group discrete?
Let $E_{123}(n) = (\alpha_n, \beta_n)$ as in Section 3. Let $E_{1213}(n)$ be the interval $(\kappa_n, 1)$ such that if $\cos\theta > \kappa_n$, then $i_1i_2i_1i_3$ is regular elliptic.
We have the following properties:
   1. For $n < 9$, $E_{123}(n) = \emptyset$;
   2. For $n = 9, 10, 11, 12, 13, \kappa_n < \alpha_n < \beta_n$ (i.e. $E_{123}(n) \subset E_{1213}(n)$);
   3. For $n > 14$, $\alpha_n < \kappa_n < \beta_n$ (i.e. $E_{123}(n) \cap E_{1213}(n) \neq \emptyset$ and $E_{123}(n) - (E_{123}(n) \cap E_{1213}(n)) \neq \emptyset$).

(4) Find new discrete complex hyperbolic triangle groups of type (n, n, ∞).

(5) In [12], Knapp obtained a complete list of Fuchsian groups generated by two elliptic elements. By this result, Parker proved some complex hyperbolic triangle group of type (n, n, n) to be non-discrete in [15]. We would like to know if we can use the same result in our case. Our question is: Does there exist a complex hyperbolic triangle group of type (n, n, ∞) containing such a Fuchsian group as a subgroup?

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Counting Carefree Couples

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Abstract. A pair of natural numbers (a, b) such that a is both squarefree and coprime to b is called a carefree couple. A result conjectured by Manfred Schroeder (in his book 'Number theory in science and communication') on carefree couples and a variant of it are established using standard arguments from elementary analytic number theory. Also a related conjecture of Schroeder on triples of integers that are pairwise coprime is proved.

1. Introduction

It is well known that the probability that an integer is squarefree is $6/\pi^2$. Also the probability that two given integers are coprime is $6/\pi^2$. (More generally the probability that n positive integers chosen arbitrarily and independently are coprime is well-known [17,22,27] to be $1/\zeta(n)$, where ζ is Riemann's zeta function. For some generalizations see e.g. [3,4,12,23,25].) One can wonder how 'statistically independent' squarefreeness and coprimality are. To this end one could for example consider the probability that of two random natural numbers a and b, a is both squarefree and coprime to b.

Let us call such a couple (a, b) *carefree*. If b is also squarefree, we say that (a, b) is a *strongly carefree* couple. Let us denote by $C_1(x)$ the number of carefree couples (a, b) with both $a \le x$ and $b \le x$ and, similarly, let $C_2(x)$ denote the number of strongly carefree couples (a, b) with both $a \le x$ and $b \le x$.

The purpose of this note is to establish the following result, part of which was conjectured, on the basis of heuristic arguments, by Manfred Schroeder [26, p. 54]. (In it and in the rest of the paper the mathematical symbol p is exclusively used to denote primes.)

Theorem 1. *We have*

$$C_1(x) = \frac{x^2}{\zeta(2)} \prod_p \left(1 - \frac{1}{p(p+1)}\right) + O(x \log x), \qquad (1)$$

and

$$C_2(x) = \frac{x^2}{\zeta(2)^2} \prod_p \left(1 - \frac{1}{(p+1)^2}\right) + O(x^{3/2}). \qquad (2)$$

The interpretation of Theorem 1 is that the probability for a couple to be carefree is

$$K_1 := \frac{1}{\zeta(2)} \prod_p \left(1 - \frac{1}{p(p+1)}\right)$$
$$\approx 0.4282495056770944 4022 \qquad (3)$$

and to be strongly carefree is

$$K_2 := \frac{1}{\zeta(2)^2} \prod_p \left(1 - \frac{1}{(p+1)^2}\right)$$
$$\approx 0.28674742843447873411 \qquad (4)$$

Using the identity $\zeta(n) = \prod_p (1 - p^{-n})^{-1}$ valid for $n > 1$ we can alternatively write

$$K_2 = \frac{1}{\zeta(2)} \prod_p \left(1 - \frac{2}{p(p+1)}\right) = \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p}\right). \qquad (5)$$

For $m \ge 3$ and $0 \le k \le m$ we put

$$Z_k(m) = \prod_p \left(1 + \frac{k-1}{p^m} - \frac{k}{p^{m-1}}\right). \qquad (6)$$

Note that $Z_2(3) = K_1$ and $Z_3(3) = K_2$.

The constants K_1 and K_2 we could call the *carefree*, respectively *strongly carefree constant*, cf. [10, Section 2.5].

Assuming independence of squarefreeness and coprimality we would expect that $K_1 = \zeta(2)^{-2}$ and $K_2 = \zeta(2)^{-3}$. Now note that

$$K_1 = \frac{1}{\zeta(2)^2} \prod_p \left(1 + \frac{1}{(p+1)(p^2-1)}\right),$$

$$K_2 = \frac{1}{\zeta(2)^3} \prod_p \left(1 + \frac{2p+1}{(p+1)^2(p^2-1)}\right).$$

We have $\zeta(2)^2 K_1 \approx 1.15876$ and $\zeta(2)^3 K_2 \approx 1.27627$. Thus, there is a positive correlation between squarefreeness and coprimality.

Let $I_3(x)$ denote the number of triples (a, b, c) with $a \le x,\ b \le x, c \le x$ such that $(a, b) = (a, c) = (b, c) = 1$. Schroeder [26, Section 4.4] claims that $I_3(x) \sim K_2 x^3$. Indeed, in Section 2.2 we will prove the following result.

Theorem 2. *We have* $I_3(x) = K_2 x^3 + O(x^2 \log^2 x)$.

The work described in this note was carried out in 2000 and with some improvement in the error terms was posted on the arXiv in September of 2005 [21], with the remark that it was not intended for publication in a research journal as the methods used involve only rather elementary and standard analytic number theory. Over the years various authors referred to [21], and this induced me to try to publish it in a mathematical newsletter. (For publications in this area after 2005 see, e.g, [1,6–9,14–16,30,31].) In [21] there was a mistake in the proof of (2) leading to an error term of $O(x \log^3 x)$, rather than $O(x^{3/2})$. Except for this, the present version has essentially the same mathematical content as the earlier one, but is written in a less carefree way and with the mathematical details more spelled out.

2. Proofs

As usual we let μ denote the Möbius function and φ Euler's totient function. Note that n is squarefree if and only if $\mu(n)^2 = 1$. We will repeatedly make use of the basic identities

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1; \\ 0 & \text{otherwise,} \end{cases} \qquad (7)$$

and

$$\frac{\varphi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d} = \prod_{p|n} \left(1 - \frac{1}{p}\right). \qquad (8)$$

We will also use several times that if s is a complex number and f a multiplicative function such that $\sum_p \sum_{\nu\geq 1} |f(p^\nu)p^{-\nu s}| < \infty$, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_p \sum_{\nu\geq 1} \frac{f(p^\nu)}{p^{\nu s}}. \tag{9}$$

(For a proof see, e.g., Tenenbaum [28, p. 107].)

In the proof of Theorem 1 we will make use of the following lemma.

Lemma 3. *Let $d \geq 1$ be arbitrary. Put*

$$S_d(x) = \sum_{\substack{n\leq x\\(d,n)=1}} \mu(n)^2.$$

We have

$$S_d(x) = \frac{x}{\zeta(2)\prod_{p|d}\left(1+\frac{1}{p}\right)} + O(2^{\omega(d)}\sqrt{x}), \tag{10}$$

where $\omega(d)$ denotes the number of distinct prime divisors of d.

Proof. Let $T_d(x)$ denote the number of natural numbers $n \leq x$ that are coprime to d. Using (7) and (8) and $[x] = x + O(1)$ we deduce that

$$T_d(x) = \sum_{\substack{n\leq x\\(n,d)=1}} 1 = \sum_{n\leq x} \sum_{\substack{\alpha|n\\ \alpha|d}} \mu(\alpha)$$

$$= \sum_{\alpha|d} \mu(\alpha)\left[\frac{x}{\alpha}\right] = \frac{\varphi(d)}{d}x + O(2^{\omega(d)}). \tag{11}$$

By the principle of inclusion and exclusion we find that

$$S_d(x) = \sum_{\substack{m\leq\sqrt{x}\\(d,m)=1}} \mu(m)T_d\left(\frac{x}{m^2}\right).$$

Hence, on invoking (11), we find

$$S_d(x) = x\frac{\varphi(d)}{d} \sum_{\substack{m\leq\sqrt{x}\\(d,m)=1}} \frac{\mu(m)}{m^2} + O(2^{\omega(d)}\sqrt{x}).$$

and hence, on completing the sum,

$$S_d(x) = x\frac{\varphi(d)}{d} \sum_{\substack{m=1\\(d,m)=1}}^{\infty} \frac{\mu(m)}{m^2} + O(2^{\omega(d)}\sqrt{x})$$

Note that

$$\sum_{\substack{m=1\\(d,m)=1^{\infty}}} \frac{\mu(m)}{m^2} = \prod_{p\nmid d}\left(1-\frac{1}{p^2}\right) = \frac{1}{\zeta(2)\prod_{p|d}(1-1/p^2)}.$$

Using this and (8) the proof is completed. □

Let $d(n)$ denote the number of divisors of n. We have $2^{\omega(n)} \leq d(n)$ with equality iff n is squarefree. The estimates below also hold with $2^{\omega(n)}$ replaced by $d(n)$.

Lemma 4. *We have*

$$\sum_{d\leq x}^{\infty} \frac{2^{\omega(d)}}{d^{3/2}} = O(1), \quad \sum_{d\leq x} \frac{2^{\omega(d)}}{\sqrt{d}} = O(\sqrt{x}\log x),$$

$$\sum_{d\leq x} \frac{4^{\omega(d)}}{d} = O(\log^3 x).$$

Proof. Using the convergence of $\sum_p p^{-3/2}$ we find by (9) that $\sum_{d=1}^{\infty} 2^{\omega(d)}d^{-3/2} = O(1)$. The remaining estimates follow on invoking Theorem 1 at p. 201 of Tenenbaum's book [28] together with partial integration. □

2.1 *Proof of Theorem 1*

Note that

$$C_1(x) = \sum_{a\leq x}\sum_{b\leq x} \mu(a)^2 \sum_{d|a,\ d|b} \mu(d)$$

$$= \sum_{d\leq x} \mu(d) \sum_{\substack{a\leq x\\ d|a}} \mu(a)^2 \sum_{b_1\leq x/d} 1,$$

after swapping the summation order. Using $[x/d] = x/d + O(1)$, we then obtain

$$C_1(x) = x\sum_{d\leq x} \frac{\mu(d)}{d} \sum_{\substack{a\leq x\\ d|a}} \mu(a)^2 + O(x\log x).$$

On noting that

$$\sum_{\substack{a\leq x\\ d|a}} \mu(a)^2 = \mu(d)^2 \sum_{\substack{n\leq x/d\\(d,n)=1}} \mu(n)^2 = \mu(d)^2 S_d\left(\frac{x}{d}\right) \tag{12}$$

and $\mu(d) = \mu(d)^3$, we find

$$C_1(x) = x\sum_{d\leq x} \frac{\mu(d)}{d} S_d\left(\frac{x}{d}\right) + O(x\log x).$$

On using Lemma 1 we obtain the estimate

$$C_1(x) = \frac{x^2}{\zeta(2)} \sum_{d\leq x} \frac{\mu(d)}{d^2\prod_{p|d}(1+1/p)}$$
$$+ O\left(\sqrt{x}\sum_{d\leq x}\frac{2^{\omega(d)}}{\sqrt{d}}\right) + O(x\log x).$$

On completing the latter sum and noting that

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2 \prod_{p|d}(1+1/p)} = \prod_p \left(1 - \frac{1}{p(p+1)}\right),$$

we obtain

$$C_1(x) = \frac{x^2}{\zeta(2)} \prod_p \left(1 - \frac{1}{p(p+1)}\right) + O\left(\sqrt{x} \sum_{d \le x} \frac{2^{\omega(d)}}{\sqrt{d}}\right) + O(x \log x).$$

Estimate (1) now follows on invoking Lemma 2.

The proof of (2) is very similar to the proof of (1). We start by noting that

$$C_2(x) = \sum_{a \le x} \sum_{b \le x} \mu(a)^2 \mu(b)^2 \sum_{d|a,\ d|b} \mu(d).$$

On swapping the summation order, we obtain

$$C_2(x) = \sum_{d \le x} \mu(d) \sum_{\substack{a \le x \\ d|a}} \mu(a)^2 \sum_{\substack{b \le x \\ d|b}} \mu(b)^2. \tag{13}$$

On noting that $\mu(d) = \mu(d)^5$ and invoking (12) we obtain

$$C_2(x) = \sum_{d \le x} \mu(d) S_d \left(\frac{x}{d}\right)^2. \tag{14}$$

On using Lemma 1 we obtain the estimate

$$C_2(x) = \frac{x^2}{\zeta(2)^2} \sum_{d \le x} \frac{\mu(d)}{d^2 \prod_{p|d}(1+1/p)^2} + O\left(x^{3/2} \sum_{d \le x} \frac{2^{\omega(d)}}{d^{3/2}}\right) + O\left(x \sum_{d \le x} \frac{4^{\omega(d)}}{d}\right).$$

On completing the first sum and noting that

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2 \prod_{p|d}(1+1/p)^2} = \prod_p \left(1 - \frac{1}{(p+1)^2}\right),$$

we find

$$C_2(x) = \frac{x^2}{\zeta(2)^2} \prod_p \left(1 - \frac{1}{(p+1)^2}\right) + O\left(x^{3/2} \sum_{d \le x} \frac{2^{\omega(d)}}{d^{3/2}}\right) + O\left(x \sum_{d \le x} \frac{4^{\omega(d)}}{d}\right).$$

On invoking Lemma 2 estimate (2) is then established. □

2.2 *Proof of Theorem 2*

We write $[n, m]$ for the least common multiple of n and m, and (n, m) for the greatest common divisor. Recall that $(n, m)[n, m] = nm$.

Note that

$$I_3(x) = \sum_{a,b,c \le x} \sum_{\substack{d_1|a \\ d_1|b}} \mu(d_1) \sum_{\substack{d_2|a \\ d_2|c}} \mu(d_2) \sum_{\substack{d_3|b \\ d_3|c}} \mu(d_3),$$

which can be rewritten as

$$I_3(x) = \sum_{\substack{[d_1,d_2] \le x \\ [d_1,d_3] \le x \\ [d_2,d_3] \le x}} \mu(d_1)\mu(d_2)\mu(d_3) \left[\frac{x}{[d_1, d_2]}\right] \times \left[\frac{x}{[d_1, d_3]}\right]\left[\frac{x}{[d_2, d_3]}\right].$$

Now put

$$J_1(x) = \sum_{\substack{[d_1,d_2] \le x \\ [d_1,d_3] \le x \\ [d_2,d_3] \le x}} \frac{\mu(d_1)\mu(d_2)\mu(d_3)}{[d_1, d_2][d_1, d_3][d_2, d_3]},$$

$$J_2(x) = \sum_{\substack{[d_1,d_2] \le x \\ [d_1,d_3] \le x \\ [d_2,d_3] \le x}} \frac{1}{[d_1, d_2][d_1, d_3]},$$

$$J_3(x) = \sum_{\substack{[d_1,d_2] \le x \\ [d_1,d_3] \le x \\ [d_2,d_3] \le x}} \frac{1}{[d_1, d_2]} \text{ and } J_4(x) = \sum_{\substack{[d_1,d_2] \le x \\ [d_1,d_3] \le x \\ [d_2,d_3] \le x}} 1.$$

Using that $[x] = x + O(1)$ we find that

$$I_3(x) = x^3 J_1(x) + O(x^2 J_2(x)) + O(x J_3(x)) + O(J_4(x)). \tag{15}$$

We will show first that

$$J_1(x) = \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{\infty} \sum_{d_3=1}^{\infty} \frac{\mu(d_1)\mu(d_2)\mu(d_3)}{[d_1, d_2][d_1, d_3][d_2, d_3]} + O\left(\frac{\log x}{x}\right).$$

To this end it is enough, by symmetry of the argument of the sum, to show that

$$\sum_{[d_1,d_2] > x} \sum_{d_3 \ge 1} \frac{1}{[d_1, d_2][d_1, d_3][d_2, d_3]} = O\left(\frac{\log x}{x}\right). \tag{16}$$

Put $(d_1, d_2) = \alpha$, $(d_1, d_3) = \beta$ and $(d_2, d_3) = \gamma$. Since $\alpha | d_1$ and $\beta | d_1$, we can write $d_1 = [\alpha, \beta]\delta_1$ for some integer $\delta_1 \ge 1$, and similarly $d_2 = [\alpha, \gamma]\delta_2, d_3 = [\beta, \gamma]\delta_3$. Note that any triple (d_1, d_2, d_3) corresponds to a uniqe 6-tuple $(\alpha, \beta, \gamma, \delta_1, \delta_2, \delta_3)$. Since $\alpha(\delta_1, \delta_2)$ divides $([\alpha, \beta]\delta_1, [\alpha, \gamma]\delta_2)$ on the one hand and $([\alpha, \beta]\delta_1, [\alpha, \gamma]\delta_2) = (d_1, d_2) = \alpha$ on the other, it follows that $(\delta_1, \delta_2) = 1$ and likewise $(\delta_1, \delta_3) = (\delta_2, \delta_3) = 1$. Write $u = \alpha\beta\gamma/(\alpha, \beta, \gamma)^2$. On noting that $((d_1, d_2), (d_2, d_3)) = (d_1, d_2, d_3) = ((d_1, d_2), (d_1, d_3), (d_2, d_3))$ we infer that $(\alpha, \beta) = (\alpha, \gamma) = (\beta, \gamma) = (\alpha, \beta, \gamma)$ and hence we find that $[d_1, d_2] = u\delta_1\delta_2$, $[d_1, d_3] = u\delta_1\delta_3$ and $[d_2, d_3] = u\delta_2\delta_3$. Now

$$\sum_{[d_1,d_2]>x} \sum_{d_3 \geq 1} \frac{1}{[d_1, d_2][d_1, d_3][d_2, d_3]} \leq \sum_{\alpha,\beta,\gamma} \frac{1}{u^3} \sum_{\delta_1\delta_2>x/u} \sum_{\delta_3 \geq 1} \frac{1}{(\delta_1\delta_2\delta_3)^2},$$

where the triple sum is over all 6-tuples $(\alpha, \beta, \gamma, \delta_1, \delta_2, \delta_3)$ and is of order

$$O\left(\sum_{\alpha,\beta,\gamma} \frac{1}{u^3} \sum_{\delta_1\delta_2>x/u} \frac{1}{(\delta_1\delta_2)^2}\right) = O\left(\sum_{\alpha,\beta,\gamma} \frac{1}{u^3} \sum_{n>x/u} \frac{d(n)}{n^2}\right) = O\left(\frac{\log x}{x} \sum_{\alpha,\beta,\gamma} \frac{1}{u^2}\right),$$

where we used the well-known estimate $\sum_{n>x} d(n)n^{-2} = O(\log x/x)$. Now

$$\sum_{\alpha,\beta,\gamma} \frac{1}{u^2} = \sum_{\alpha,\beta,\gamma} \frac{(\alpha, \beta, \gamma)^4}{(\alpha\beta\gamma)^2} = O\left(\sum_{d=1}^{\infty} \frac{1}{d^2} \sum_{\alpha',\beta',\gamma'} \frac{1}{(\alpha'\beta'\gamma')^2}\right) = O(1), \qquad (17)$$

where we have written $(\alpha, \beta, \gamma) = d$, $\alpha = d\alpha'$, $\beta = d\beta'$ and $\gamma = d\gamma'$. Thus we have established equation (16).

In the same vein $J_2(x)$ can be estimated to be

$$J_2(x) = O\left(\sum_{\alpha,\beta,\gamma} \sum_{\substack{\delta_1\delta_2 \leq x/u \\ \delta_1\delta_3 \leq x/u \\ \delta_2\delta_3 \leq x/u}} \frac{1}{[d_1, d_2][d_1, d_3]}\right) = O\left(\sum_{\alpha,\beta,\gamma} \frac{1}{u^2} \sum_{\delta_1\delta_2\delta_3 \leq (x/u)^{3/2}} \frac{1}{\delta_1^2\delta_2\delta_3}\right)$$

$$= O\left(\sum_{\alpha,\beta,\gamma} \frac{1}{u^2} \sum_{\delta_2\delta_3 \leq (x/u)^{3/2}} \frac{1}{\delta_2\delta_3}\right) = O\left(\sum_{\alpha,\beta,\gamma} \frac{1}{u^2} \sum_{n \leq (x/u)^{3/2}} \frac{d(n)}{n}\right)$$

Using the classical estimate $\sum_{n \leq x} d(n)/n = O(\log^2 x)$ and (17), one obtains $J_2(x) = O(\log^2 x)$.

Note that $0 \leq J_4(x) \leq xJ_3(x) \leq x^2 J_2(x)$. Using (15) we see that it remains to evaluate the triple infinite sum, which we rewrite as

$$\sum_{d_1=1}^{\infty} \sum_{d_2=1}^{\infty} \sum_{d_3=1}^{\infty} \frac{\mu(d_1)\mu(d_2)\mu(d_3)(d_1, d_2)(d_1, d_3)(d_2, d_3)}{(d_1 d_2 d_3)^2},$$

which can be rewritten as

$$\sum_{d_1=1}^{\infty} \frac{\mu(d_1)}{d_1^2} \sum_{d_2=1}^{\infty} \frac{\mu(d_2)(d_1, d_2)}{d_2^2} \sum_{d_3=1}^{\infty} \frac{\mu(d_3)(d_1, d_3)(d_2, d_3)}{d_3^2}.$$

Note that the argument of the inner sum is multiplicative in d_3. By Euler's product identity (9) it is zero if $(d_1, d_2) > 1$ and $\zeta(2)^{-1} \prod_{p|d_1d_2}(1 + 1/p)^{-1}$ otherwise. Thus the latter triple sum is seen to yield

$$\frac{1}{\zeta(2)} \sum_{d_1=1}^{\infty} \frac{\mu(d_1)}{d_1^2 \prod_{p|d_1}(1 + 1/p)} \sum_{\substack{d_2=1 \\ (d_1,d_2)=1}}^{\infty} \frac{\mu(d_2)}{d_2^2 \prod_{p|d_2}(1 + 1/p)},$$

the argument of the inner sum is multiplicative in d_2 and proceeding as before we obtain that it equals

$$\frac{1}{\zeta(2)} \prod_p \left(1 - \frac{1}{p(p+1)}\right) \times \sum_{d_1=1}^{\infty} \frac{\mu(d_1)}{d_1^2 \prod_{p|d_1}\left(1 + \frac{1}{p}\right) \prod_{p|d_1}\left(1 - \frac{1}{p(p+1)}\right)},$$

which is seen to equal

$$\frac{1}{\zeta(2)} \prod_p \left(1 - \frac{2}{p(p+1)}\right),$$

which by equation (5) equals K_2. □

3. Numerical Aspects

Direct evaluation of the constants K_1 and K_2 through (3), respectively (4) yields only about five decimal digits of precision. By expressing K_1 and K_2 as infinite products involving $\zeta(k)$ for $k \geq 2$, they can be computed with high precision. To this end Theorem 1 of [20] can be used. The error analysis can be dealt with using Theorem 2 of [20]. Using [20, Theorem 1] it is inferred that

$$K_1 = \prod_{k\geq 2} \zeta(k)^{-e_k}, \quad \text{where} \quad e_k = \frac{\sum_{d|k} b_d \mu\left(\frac{k}{d}\right)}{k} \in \mathbb{Z},$$

with the sequence $\{b_k\}_{k=0}^{\infty}$ defined by $b_0 = 2$ and $b_1 = -1$ and $b_{k+2} = -b_{k+1} + b_k$. Using the same theorem, it is seen that

$$K_2 = \frac{1}{2}\prod_{k\geq 2}\{\zeta(k)(1-2^{-k})\}^{-f_k}, \quad \text{where}$$

$$f_k = \frac{\sum_{d|k}(-2)^d \mu\left(\frac{k}{d}\right)}{k} \in \mathbb{Z}.$$

Typically in analytic number theory constants of the form $\prod_p f(1/p)$ with f rational arise as densities. Their numerical evaluation was considered by the author in [20]. By similar methods any constant of the form $\prod_p f(1/p)$ with f an analytic function on the unit disc satisfying $f(0) = 1$ and $f'(0) = 0$ can be evaluated [19].

4. Related Problems

Let us call a couple (a, b) with $a,\ b \leq x$, a and b coprime and either a or b squarefree, *weakly carefree*. A little thought reveals that $C_3(x) = 2C_1(x) - C_2(x)$. By Theorem 1 it then follows that the probability K_3 that a couple is weakly carefree equals $K_3 = 2K_1 - K_2 \approx 0.5697515829$.

The problem of estimating $I_3(x)$ has the following natural generalisation. Let $k \geq 2$ be an integer and let $I_k(x)$ be the number of k-tuples $(a_1, \ldots, a_k)$ with $1 \leq a_i \leq x$ for $1 \leq i \leq k$ such that $(a_i, a_j) = 1$ for every $1 \leq i \neq j \leq k$. The number of k-tuples such that none of the gcd's is divisible by some fixed prime p is easily seen to be

$$\sim x^k\left(\left(1-\frac{1}{p}\right)^k + \frac{k}{p}\left(1-\frac{1}{p}\right)^{k-1}\right)$$
$$= x^k\left(1-\frac{1}{p}\right)^{k-1}\left(1+\frac{k-1}{p}\right).$$

Thus, it seems plausible that

$$I_k(x) \sim x^k \prod_p \left(1-\frac{1}{p}\right)^{k-1}\left(1+\frac{k-1}{p}\right), \quad (x\to\infty). \tag{18}$$

For $k = 2$ and $k = 3$ (by Theorem 2 and equation (5) this is true. In 2000 I did not see how to prove this for arbitrary k, however the conjecture (18) was established soon afterwards (in 2002) by L. Tóth [29], who proved that for $k \geq 2$ we have

$$I_k(x) = x^k \prod_p \left(1-\frac{1}{p}\right)^{k-1} \times \left(1+\frac{k-1}{p}\right) + O(x^{k-1}\log^{k-1} x). \tag{19}$$

Let $I_k^{(u)}(x)$ denote the number of k-tuples $(a_1, \ldots, a_k)$ with $1 \leq a_i \leq x$ that are pairwise coprime and moreover satisfy $(a_i, u) = 1$ for $1 \leq i \leq k$. It is easy to see that

$$I_{k+1}^{(u)}(n) = \sum_{\substack{j=1 \\ (j,u)=1}}^{n} I_k^{(ju)}(n).$$

Note that $I_1^{(u)}(n) = T_u(n)$ can be estimated by (11). Then by recursion with respect to k an estimate for $I_k^{(u)}(n)$ can be established that implies (19).

In [13] Havas and Majewski considered the problem of counting the number of n-tuples of natural numbers that are pairwise not coprime. They suggested that the density δ_n of these tuples should be

$$\delta_n = \left(1-\frac{1}{\zeta(2)}\right)^{\binom{n}{2}}. \tag{20}$$

The probability that a pair of integers is not coprime is $1 - 1/\zeta(2)$. Since there are $\binom{n}{2}$ pairs of integers in an n-tuple, one might naively espect the probability for this problem to be as given by (20).

T. Freiberg [11] studied this problem for $n = 3$ using my approach to estimate $I_3(x)$ (it seems that the recursion method of Tóth cannot be applied here). Freiberg showed that the density of triples (a, b, c) with $(a, b) > 1$, $(a, c) > 1$ and $(b, c) > 1$ equals

$$F_3 = 1 - \frac{3}{\zeta(2)} + 3K_1 - K_2 \approx 0.1742197830347247005,$$

whereas $(1 - 1/\zeta(2))^3 \approx 0.06$. Thus the guess of Havas and Majewski for $n = 3$ is false. Indeed, it is easy to see (as Peter Pleasants pointed out to the author [24]) that for every $n \geq 3$ their guess is false. Since all n-tuples of even numbers are pairwise not coprime, δ_n, if it exists, satisfies $\delta_n \geq 2^{-n}$. Since $\binom{n}{2} \geq n$ and $1 - 1/\zeta(2) < 0.4$ the predicted density by Havas and Majewski [13] satisfies $\delta_n < 2^{-n}$ for $n \geq 3$ and so must be false.

In 2006 the author learned [18] that the result of Freiberg is implicit in the PhD thesis of R. N. Buttsworth [2] and indeed can be found there in more general form. Buttsworth showed that the density of relatively prime m-tuples for which k prescribed $(m-1)$-tuples have gcd 1 equals $Z_k(m)$ given in (6). Consequently by inclusion and exclusion the set of relatively prime m-tuples such that every $(m-1)$-tuple fails to be relatively prime has density

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}Z_k(m).$$

For $m=3$ this yields $1/\zeta(3)-3/\zeta(2)+3K_1-K_2$. So the density of relatively prime 3-tuples such that at least one 2-tuple is relatively prime, is equal to $3/\zeta(2)-3K_1+K_2$. However this is also equal to the density of 3-tuples such that at least one 2-tuple is relatively prime. Hence the density of 3-tuples such that all 2-tuples are not relatively prime is $1-3/\zeta(2)+3K_1-K_2$, which is Freiberg's formula.

To close this discussion, we like to remark that Freiberg established his result with error term $O(x^2\log^2 x)$ and that Buttsworth's result gives only a density.

Some related open problems are as follows:

Problem 1.

a) To compute the density of n-tuples such that at least k pairs are coprime.
b) To compute the density of n-tuples such that exactly k pairs are coprime.

Problem 2. To compute the density of n-tuples such that all pairs are not coprime.

Remark. Recently Jerry Hu [16] announced that he solved Problem 1.

5. Conclusion

In stark constrast to what experience from daily life suggests, (strongly) carefree couples are quite common.

Acknowledgement

The author likes to thank Steven Finch for bringing Schroeder's conjecture to his attention and his instignation to write down these results. Also Finch and de Weger pointed out that one has $\sum_{n\le x}k(n)=\zeta(2)K_1x^2/2+O(x^{3/2})$, where $k(n)=\prod_{p|n}p$, and that in [21] the K_1 was inadvertently dropped. For a proof of this formula see Eckford Cohen [5, Theorem 5.2].

The author likes to thank Tristan Freiberg and Jerry Hu for pointing out some references and helpful comments. Keith Matthews provided me kindly with very helpful information concerning the relevant results of his former PhD student Buttsworth. In particular he pointed out how Freiberg's result follows from that of Buttsworth.

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Problems and Solutions

Edited by Amritanshu Prasad

E-mail: `problems@imsc.res.in`

This section of the Newsletter contains problems contributed by the mathematical community. These problems range from mathematical brain-teasers through instructive exercises to challenging mathematical problems. Contributions are welcome from everyone, students, teachers, scientists and other maths enthusiasts. We are open to problems of all types. We only ask that they be reasonably original, and not more advanced than the MSc level. A cash prize of Rs. 500 will be given to those whose contributions are selected for publication (for administrative reasons, payments will be made within India only). Please send solutions along with the problems. The solutions will be published in the next issue. Please send your contribution to `problems@imsc.res.in` with the word "**problems**" somewhere in the subject line. Please also send solutions to these problems (with the word "**solutions**" somewhere in the subject line). Selected solutions will be featured in the next issue of this Newsletter.

1. **Hitesh Jain, T. I. M. E. Aurangabad.** For each integer $N > 1$, find N positive integers (not necessarily distinct) whose sum is equal to their product.
2. **Jon Stammers, AMRC with Boeing.** Consider a machine tool with a circular table that rotates about an unknown central point. An operator wishes to determine the centre of rotation of the table according a fixed coordinate system. The operator has marked the position of a point mounted on the table at various rotations of the table to obtain a number of coordinates (x_i, y_i) for this point relative to the fixed coordinate system (see Figure 1). Determine the coordinates of the centre of rotation of the table in terms of the coordinates (x_i, y_i).

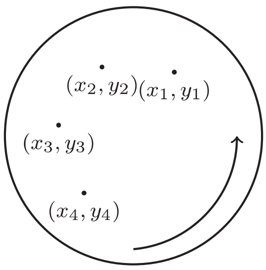

Figure 1.

3. **K. N. Raghavan, IMSc Chennai.** Let $K(x)$ and $L(x)$ be relatively prime polynomials of degrees k and ℓ respectively. Show that, given a polynomial $f(x)$ of degree less than $k + \ell$, there exist unique polynomials $S(x)$ and $T(x)$ of degrees less than k and ℓ respectively, such that
$$\frac{f(x)}{K(x)L(x)} = \frac{S(x)}{K(x)} + \frac{T(x)}{L(x)} \tag{1}$$
How would you construct $S(x)$ and $T(x)$?
4. **K. N. Raghavan, IMSc Chennai.** Let $d_1, \dots, d_m$ be a sequence of positive integers and $a_1, \dots, a_m$ a sequence of distinct real numbers. Show that there exists a unique polynomial of degree less than $d_1 + \cdots + d_m$ whose derivatives of all orders less than d_i have arbitrarily specified values at a_i, for every i, $1 \leq i \leq m$. The case when all the d_i are 1 is Lagrange interpolation and the case when $m = 1$ is Taylor expansion. How will you explicitly construct the polynomial?
5. **K. N. Raghavan, IMSc Chennai.** Can a real $m \times n$ matrix A be recovered from AA^tA (where A^t is of course the transpose of A)?
6. **Amritanshu Prasad, IMSc Chennai** A bijection $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ is said to be a cyclic permutation of n if there exists $i \in \{1, \dots, n\}$ such that the set
$$\{i, \sigma(i), \sigma^2(i), \dots\}$$
is all of $\{1, \dots, n\}$. Find the total number of cyclic permutations of n.

Solutions to Problems from the December 2013 Issue

1. **Parameswaran Sankaran, IMSc Chennai.** If $a_1, a_2, a_3, \dots$ is a sequence of non-zero numbers such that each member of this sequence (starting from the second one) is one less than the product of its neighbors, show that $a_{n+5} = a_n$ for all $n \geq 1$. Find all such sequences which are constant.

Solution. Define f to be the function:
$$(u, v) \mapsto \left(\frac{u+1}{v}, u\right).$$
One easily calculates
$$f^2(u, v) = f \circ f(u, v) = \left(\frac{u+v+1}{uv}, \frac{u+1}{v}\right),$$
and then
$$f^4(u, v) = f^2 \circ f^2(u, v) = \left(v, \frac{v+1}{u}\right),$$
whence
$$f^5(u, v) = f \circ f^4(u, v) = (u, v). \tag{$*$}$$
Now our sequence $\{a_n\}$ satisfies
$$(a_{n+1}, a_n) = f(a_n, a_{n-1}),$$
whence
$$(a_{n+5}, a_{n+4}) = f^5(a_n, a_{n-1}) = (a_n, a_{n-1}).$$
Therefore the identity $(*)$ implies that $a_{n+1} = a_n$ for all $n \geq 1$.

Solution Received. A correct solution to this problem was received from Aditi Phadke of Nowrosjee Wadia College, Pune. Aditi points out that if the numbers are not all required to be non-zero, then one possible sequence is $2, -1, 0, -1, 3, -4, -1, 0, -1, \dots$ where $a_1 \neq a_{1+5}$.

2. **Amritanshu Prasad, IMSc Chennai** If a stick is broken into three pieces randomly, what is the probability that these three pieces can be used to form the sides of a triangle?

Solution. (Due to Aditi S. Phadke and Pramod N. Shinde of Nowrosjee Wadia College, Pune.)

If the stick is broken at two points to form three pieces of length x, y and z respectively, then the lengths of the pieces satisfy the constraints

$$x+y+z=1, 0 \le x \le 1, 0 \le y \le 1 \quad \text{and} \quad 0 \le z \le 1. \tag{1}$$

Moreover, if these pieces are to form a triangle then x, y and z must satisfy the additional inequalities

$$x+y \ge z, x+z \ge y \quad \text{and} \quad y+z \ge x. \tag{2}$$

Under (1), the constraints (2) are equivalent to

$$0 \le x \le 1/2, 0 \le y \le 1/2 \quad \text{and} \quad 0 \le z \le 1/2. \tag{3}$$

In Figure 2, the ΔABC is the set of all points satisfying (1), and ΔPQR is the set of all points satisfying both (1) and (2). Since ΔPQR is the triangle whose vertices are the bisectors of the sides of ΔABC, $\frac{A(\Delta PQR)}{A(\Delta ABC)} = \frac{1}{4}$, which is the probability that the sides of the broken stick form a triangle.

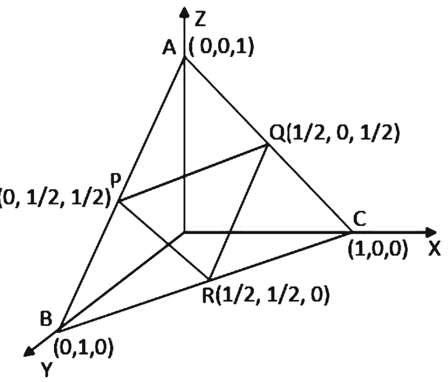

Figure 2.

3. **Amritanshu Prasad, IMSc Chennai** Let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$. For each positive integer i and each permutation w, let $x_i(w)$ denote the number of cycles of length i in w. For example, if w is the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 4 & 6 & 7 & 8 & 9 & 10 \\ 9 & 6 & 4 & 3 & 1 & 5 & 7 & 10 & 2 & 8 \end{pmatrix}$$

then w has cycle decomposition $(1, 9, 2, 6, 5)(3, 4)(8, 10)(7)$, so $x_1(w) = 1$, $x_2(w) = 2$, $x_5(w) = 1$ and $x_i(w) = 0$ for all other i. Show that every class function on S_n can be represented by a polynomial in $x_1, x_2, \ldots, x_n$ with rational coefficients.

Solution. The possible values of $(x_1(w), \ldots, x_n(w))$ form the finite set

$$P = \left\{ (m_1, \ldots, m_n) \in \mathbb{Z}^n \mid m_i \ge 0, \sum_i i m_i = n \right\}.$$

It suffices to find, for each $u \in P$, a polynomial $f_u \in \mathbb{Q}[x_1, \ldots, x_n]$ such that $f_u(u) \neq 0$ and $f_u(v) = 0$ for all $v \in P$ different from u (linear combinations of such polynomials will give all the class functions on S_n). But such polynomials are not too hard to find. For example take:

$$f_u(x_1, \ldots, x_n) = \prod_{(x_1,\ldots,x_n) \in P - \{u\}} \sum_{i=1}^{n} (x_i - u_i)^2.$$

4. **B. Sury, ISI Bangalore.** Write $\{1, 2, \cdots, 2^{2014}\} = A \cup B$ into subsets A, B of equal size such that

$$\sum_{a \in A} a = \sum_{b \in B} b$$
$$\sum_{a \in A} a^2 = \sum_{b \in B} b^2$$
$$\sum_{a \in A} a^3 = \sum_{b \in B} b^3$$
$$\vdots \ \vdots \ \vdots$$
$$\sum_{a \in A} a^{2013} = \sum_{b \in B} b^{2013}.$$

Solution. (Due to Aditi S. Phadke and Pramod N. Shinde of Nowrosjee Wadia College, Pune.) More generally, we prove for any n that the set $\{1, 2, \cdots, 2^{n+1}\}$ can be partitioned into two subsets A_n, B_n each of size 2^n such that

$$\sum_{a \in A_n} a^k = \sum_{b \in B_n} b^k \ \ \forall \ k = 1, 2, \cdots n.$$

For instance, if $n = 1$, take $A = \{1, 4\}$, $B = \{2, 3\}$.

We may prove the general case by induction on n.

If one has chosen A_n, B_n of size 2^n with $A_n \sqcup B_n = \{1, 2, \cdots, 2^{n+1}\}$ and such that

$$\sum_{a \in A_n} a^k = \sum_{b \in B_n} b^k \ \ \forall \ k = 1, 2, \cdots n,$$

simply take

$$A_{n+1} = A_n \cup (2^{n+1} + B_n), \quad B_{n+1} = B_n \cup (2^{n+1} + A_n).$$

Here, of course, $d + S$ denotes the set $\{d + s : s \in S\}$. The proof of

$$\sum_{a \in A_n} a^k + \sum_{b \in B_n} (2^{n+1} + b)^k$$
$$= \sum_{b \in B_n} b^k + \sum_{a \in A_n} (2^{n+1} + a)^k \ \forall \ k = 1, 2, \cdots n + 1$$

is evident by the binomial expansion.

5. **B. Sury, ISI Bangalore.** Recall:
A Fermat prime is a prime number of the form $2^{2^n} + 1$.
A Mersenne prime is a prime number of the form $2^n - 1$.
A Wieferich prime is a prime number p satisfying $2^{p-1} \equiv 1 \pmod{p^2}$.
Prove that a Wieferich prime cannot be a Fermat prime or a Mersenne prime.

Solution. Note that any prime which is Fermat or Mersenne is of the form

$$1 + 2^k + 2^{2k} + \cdots + 2^{nk}$$

for some $n, k \geq 1$. We will show more generally that any prime with such an expansion cannot be a Wieferich prime. The argument given below works with base 2 replaced by any base b and will show that a prime of the form $1 + b^k + b^{2k} + \cdots + b^{nk}$ cannot be Wieferich.

We will prove that for a prime

$$p = 1 + 2^k + 2^{2k} + \cdots + 2^{nk}$$

for some $n, k \geq 1$, the number $(n + 1)k$ divides $p - 1$ and

$$2^{p-1} \equiv 1 + \frac{p-1}{(n+1)k}(2^k - 1)p \not\equiv 1 \bmod p^2.$$

Now $p = 1 + 2^k + \cdots + 2^{nk} = \frac{2^{(n+1)k}-1}{2^k-1}$.

Now, p and $2^k - 1$ are relatively prime because p is a prime and $p \geq 2^k + 1 > 2^k - 1$.

Since p divides $2^{(n+1)k} - 1$, the order of 2 mod p is a divisor of $(n+1)k$. If it were smaller, say mr, with $m|(n+1)$ and $r|k$, then either $m < n + 1$ or $r < k$.

If $r < k$, then the assertion $2^{(n+1)r} \equiv 1 \bmod p$ means p divides

$$(1 + 2^r + \cdots + 2^{nr})(2^r - 1).$$

Now, p and $2^r - 1$ are relatively prime because p is a prime and $p \geq 2^k + 1 > 2^r - 1$.

Hence $p = 1 + 2^k + \cdots + 2^{nk}$ divides $1 + 2^r + \cdots + 2^{nr}$, which is impossible as p is the bigger number.

Now, if $m < n + 1$, then the condition $2^{mk} \equiv 1$ means p divides $1 + 2^k + \cdots + 2^{(m-1)k} = \frac{2^{mk}-1}{2^k-1}$ as p and $2^k - 1$ are relatively prime because p is a prime and $p \geq 2^k + 1 > 2^k - 1$.

This is impossible, as $p = 1 + 2^k + \cdots + 2^{nk}$ is larger than $1 + 2^k + \cdots + 2^{(m-1)k}$.

We have shown that the order of 2 mod p is $(n + 1)k$; hence, this order $(n + 1)k$ divides $p - 1$.

Now, raise $2^{(n+1)k} = 1 + p(2^k - 1)$ to the $\frac{p-1}{(n+1)k}$-th power. We have

$$2^{p-1} \equiv 1 + p(2^k - 1)\frac{p-1}{(n+1)k} \bmod p^2$$

Now, again the observation that p is relatively prime to $2^k - 1$ implies that p does not divide $(2^k - 1)\frac{p-1}{(n+1)k}$.

Announcement

A special volume of the Journal of the Indian Mathematical Society to commemorate the 125th. Birth Anniversary of Srinivasa Ramanujan and the National Mathematics Year-2012 was released on 28th December, 2013 during the inaugural function of the 79th Annual Conference of the Indian Mathematical. Society held at Rajagiri School of Engineering & Technology, Cochin, 28–31 December, 2013.

Editor: A. K. Agarwal

Publisher: The Indian Mathematical Society

Contributors: S. D. Adhikari, S. Bhargava, M. D. Hirschhorn, J. Lovejoy & R. Osburn, A. M. Mathai, M. Ram Murty, N. Robbins, J. A. Sellers.

Preface

Srinivasa Ramanujan at One Hundred Twenty Five

On December 26, 2011, the Hon'ble Prime Minister of India, Dr. Manmohan Singh declared the Year 2012 as the 'National Mathematics Year' to commemorate the 125th birth anniversary of the legendary Indian mathematician Srinivasa Ramanujan and the date December 22, being his birthday has been declared as the 'National Mathematics Day'. Consequently, several academic programs in his memory were organized during the Year 2012. The Indian Mathematical Society has always been in forefront for the celebration of such events, specially, if they are related to Ramanujan – rightly so, after all, Ramanujan was the discovery of its founder Ramaswamy Aiyar and that Ramanujan published his first paper entitled "Some properties of Bernoulli's numbers" in its journal [J. Indian Math. Soc. 3(1911), 219–234]. This time also the Society rose to the occasion and decided to bring out a special issue of its journal dedicated to Ramanujan. I was given the responsibility of its editorship. I contacted several mathematicians working in areas influenced by Ramanujan such as hypergeometric functions, partition theory, modular equations, continued fractions and mock theta functions through E-mails and invited them for their contributions. I also met many of them personally in two conferences: (1) International Conference on the Works of Srinivasa Ramanujan and Related Topics, University of Mysore, December 12–14, 2012, and (2) International Conference on the Legacy of Srinivasa Ramanujan, University of Delhi, December 17–22, 2012 and reminded them of my invitation. In all eight mathematicians responded positively to our invitation. The present volume is the collection of their articles. The Indian Mathematical Society is presenting this special volume as its tribute to the great mathematician who was described by G. H. Hardy as "a natural genius". Hardy also compared him with the eminent mathematicians Jacobi, Gauss and Euler. There are two sides of Ramanujan's personality. We all know that he was a great mathematician, as my personal tribute to him on this occasion, I recall the other side of his personality, which is: he was a great human being, most humble and completely free from egoism. He lived and worked in the true spirit of the Bhagavad Geeta wherein Lord Krishna says (Chapter 3, verse 30):

mayi sarvani karmani
amnyasa dhyatmacetasa
nirasir nirmano bhutva
yudhyasva vigatajvarah

(Renouncing all actions in Me, with the mind centred on Me free from lethargy and egoism, discharge your duties without claims of proprietorship).

Ramanujan did not take the credit for his research but renounced it in the Goddess Namagiri of Namakhal.

I thank all the invited mathematicians for their valuable contributions and those experts who have helped us in refereeing the articles but preferred to remain anonymous.

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Lecture Notes Series in Mathematics

Department of Science and Technology, Government of India sponsored Lecture Notes Series in Mathematics (LNSM) is published by the Ramanujan Mathematical Society (RMS). The type of material considered for publication in LNSM includes (1) High-level research monographs covering a broad spectrum of topics (2) Proceedings of Conferences (3) "Collected Works" and "Selected Works" of eminent mathematicians (4) Current research oriented summer schools and intensive courses.

Editorial Board

Editorial Policy

While the monograph manuscripts should be primarily based on the author's own research contributions to the subject, they should also present the related work by other people making them self-contained. They should include:

(a) a table of contents;
(b) an introduction with historical background and motivation;
(c) author's own contributions with sufficient examples and applications;
(d) some directions for future work;

(e) a subject index;
(f) an extensive up-to-date list of references.

Initially, the authors must submit a soft copy (in pdf format) by E-mail as well as a hard copy by post to the Editor-in-Chief (aka@pu.ac.in) Centre for Advanced Study in Mathematics, Panjab University, Chandigarh-160014, India. All the manuscripts will be peer-reviewed, if accepted, the authors will be asked to submit a source (latex) file.

For the Proceedings of conferences, the organizers must submit a complete list of all the authors along with the title and abstract of their talk.

The decision of the Editorial Board regarding the suitability of a monograph/Proceedings of a conference for publication in this series will be final.

Abel Prize 2014

Yakov G. Sinai (Princeton University, USA, and the Landau Institute for Theoretical Physics, Russian Academy of Sciences) received the Abel Prize, awarded by the Norwegian Academy of Science and Letters, from His Royal Highness The Crown Prince at an award ceremony in Oslo on 20 May. The Prize was awarded "for his fundamental contributions to dynamical systems, ergodic theory, and mathematical physics".

Yakov Sinai is one of the most influential mathematicians of the twentieth century. He has achieved numerous groundbreaking results in the theory of dynamical systems, in mathematical physics and in probability theory. Many mathematical results are named after him, including Kolmogorov-Sinai entropy, Sinai's billiards, Sinai's random walk, Sinai-Ruelle-Bowen measures, and Pirogov-Sinai theory. Sinai is highly respected in both physics and mathematics communities as the major architect of the most bridges connecting the world of deterministic (dynamical) systems with the world of probabilistic (stochastic) systems.

Yakov G. Sinai has spoken four times at the International Congress of Mathematicians, in 1962, 1970, 1978 and 1990 (plenary talk). In 2001, he was appointed Chairman of the Fields Medal Committee of International Mathematical Union, which decided on the awards of the Fields Medals at ICM2002 in Beijing.

For more information: http://www.abelprize.no/

This is from IMU-Net 65. Courtesy from IMU

International Conference on Vibration Problems (ICOVP-2015)

18–20 February, 2015

Organizer: Department of Mathematics and Department of Mechanical Engineering, Kakatiya University, Warangal, Telengana, India, in collaboration with Von-Karman Society, West Bengal, India.

Scope: Problems on Elastic, Poroelastic, Thermoelastic, Piezoelectric, Viscoelastic materials, Fluid Mechanics, Computational Fluid Dynamics (CFD), Mechanics of Solids, Structural Mechanics, Seismic Analysis and Earthquake Engineering, Neural Waves, Thermal Stresses, Thermal Buckling and Post-Buckling Analysis, Mechanical Behaviour of Structures at Elevated Temperature, Bio-medical Engineering, Bone Mechanics, Applications of Electromagnetic Waves, Dynamics of Rotating Systems and Vehicle Dynamics, Vibration Control and Isolation, Vibration of Beams, Plates and Shells including Random Vibrations, Engineering, Physical, Structural, Architectural Acoustics, Computational Methods, and Recent Developments in Applied Mathematics.

Important Dates:

Deadline for submission of abstract (200 words)	:	August 31, 2014
Notification of acceptance	:	September, 2014
Deadline for camera ready copy	:	October 31, 2014
Deadline for registration	:	December 31, 2014

For Further Details Visit URL:

www.kakatiya.ac.in/icovp-2015

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(Conference Co Director),

lp_raj8@yahoo.com

(Organizing Secretary)

Geometry, Arithmetic and Analysis on Hyperbolic Spaces

10–23 December, 2014

Brief Description: There will be a series of two workshops. First workshop will be on geometric and analytic aspects of the hyperbolic spaces at Delhi University during December 10–15 and, the second workshop Lattices: Geometry and Dynamics will be held at IISER Mohali during December 17–23, 2014.

The second workshop on lattices is a ATM Workshop of the NCM India. Both workshops will include a combination of mini courses, with tutorial sessions and research expository talks by experts. More details of the workshops will be available in due course in the following webpage:

https://sites.google.com/site/indiahyperbolic/

Contact: math.iiserm@gmail.com

For participation, interested people should contact before September 30, 2014.

Details of Workshop/Conferences in India

For details regarding Advanced Training in Mathematics Schools
Visit: http://www.atmschools.org/

Name: International Conference on Data Mining and Intelligent Computing
Date: September 05–06, 2014
Location: Indira Gandhi Delhi Technical University for Women, Delhi
Visit: http://www.igit.ac.in/ICDMIC-2014/

Name: Second International Workshop on Mathematical Modelling and Scientific Computing (MMSC-2014)
Date: September 24–27, 2014
Location: Galgotias College of Engineering & Technology, Greater Noida
Visit: http://icacci-conference.org/website/MMSC2014

Name: National Conference on Advances in Mathematics
Date: November 21–23, 2014
Location: Amity University, Jaipur
Visit: http://bgpaur2014.webs.com/

Name: Recent Advances in Operator Theory and Operator Algebras-2014
Date: December 9–19, 2014
Location: Bangalore, India
Visit: http://www.isibang.ac.in/~jay/OTOA2014/OTOA14.html

Name: International Conference on Linear Algebra and its Applications
Date: December 18–20, 2014
Location: Manipal University, Manipal, Karnataka, India
Visit: http://conference.manipal.edu/ICLAA2014/

Name: International Conference on Current Developments in Mathematics and Mathematical Sciences (ICCDMMS-2014)
Date: December 19–21, 2014
Location: Calcutta Mathematical Society, AE-374, Sector-1, Salt Lake City, Kolkata-700064 West Bengal, India
Visit: http://www.calmathsoc.org/

Name: 2014 Fourth International Conference on Emerging Applications of Information Technology (EAIT 2014)
Date: December 19–21, 2014

Location: Indian Statistical Institute, Kolkata, India.
Visit: https://sites.google.com/site/csieait/

Name: 8th International Conference of IMBIC on "Mathematical Sciences for Advancement of Science and Technology (MSAST 2014)"
Date: December 21–23, 2014
Location: IMBIC, Salt Lake City, Kolkata, India
Visit: http://imbic.org/forthcoming.html

Details of Workshop/Conferences in Abroad

For details regarding ICTP (International Centre for Theoretical Physics)
Visit: http://www.ictp.it/

Name: International School on Mathematical Epidemiology-ISME 2014
Date: September 1–5, 2014
Location: Strathmore University, Nairobi, Kenya
Visit: http://www.strathmore.edu/carms

Name: Trimester program on Non-commutative Geometry and its Applications
Date: September 1–December 19, 2014
Location: Hausdorff Research Institute for Mathematics, Bonn, Germany
Visit: http://www.him.uni-bonn.de/programs/future-programs/future-trimester-programs/non-commutative-geometry-2014/description/

Name: Black-Box Global Optimization: Fast Algorithms and Engineering Applications (part of the CST2014 Conference)
Date: September 2–5, 2014
Location: Hotel Royal Continental, Naples, Italy
Visit: http://www.civil-comp.com/conf/cstect2014/cst2014-s23.htm

Name: Introductory Workshop: Geometric Representation Theory
Date: September 2–5, 2014
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: http://www.msri.org/web/msri/scientific/workshops/all-workshops/show/-/event/Wm9804

Name: NUMAN2014 Recent Approaches to Numerical Analysis: Theory, Methods and Applications
Date: September 2–5, 2014
Location: Chania, Crete, Greece
Visit: http://numan2014.amcl.tuc.gr

Name: 12th AHA Conference-Algebraic Hyperstructures and its Applications
Date: September 2–7, 2014
Location: Democritus University of Thrace, School of Engineering, Department of Production and Management Engineering 67100, Xanthi, Greece International Algebraic Hyperstructures Association (IAHA)
Visit: http://aha2014.pme.duth.gr

Name: 4th IMA Numerical Linear Algebra and Optimisation
Date: September 3–5, 2014

Location: University of Birmingham, Birmingham, United Kingdom
Visit: http://www.ima.org.uk/conferences/conferences_calendar/4th_ima_conference_on_numerical_linear_algebra_and_optimisation.cfm

Name: International Workshop on Operator Theory 2014 (iWOP2014)
Date: September 3–5, 2014
Location: Queen's University Belfast, Belfast, Northern Ireland
Visit:http://iwop2014.martinmathieu.net/

Name: Symposium on Trustworthy Global Computing
Date: September 5–6, 2014
Location: Rome, Italy
Visit: http://www.cs.le.ac.uk/events/tgc2014/

Name: Workshop on "Exceptional Orthogonal Polynomials and Exact Solutions in Mathematical Physics"
Date: September 7–12, 2014
Location: Segovia, Spain
Visit: http://www.icmat.es/congresos/2014/xopconf/

Name: CICAM 7, Seventh China-Italy Colloquium on Applied Mathematics
Date: September 8–11, 2014
Location: Palermo, Italy
Visit: http://www.math.unipa.it/∼cicam7

Name: Workshop on Special Geometric Structures in Mathematics and Physics
Date: September 8–12, 2014
Location: University of Hamburg, Hamburg, Germany
Visit: http://www.math.uni-hamburg.de/sgstructures/

Name: ICERM Semester Program: High-Dimensional Approximation
Date: September 8–December 5, 2014
Location: Brown University, Providence, Rhode Island
Visit: http://icerm.brown.edu/sp-f14/

Name: Mathematics of Turbulence
Date: September 8–December 12, 2014
Location: Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California
Visit: http://www.ipam.ucla.edu/programs/mt2014/

Name: Summer School on Spectral Geometry
Date: September 9–12, 2014
Location: University of Göttingen, Göttingen, Germany

Name: IMA Conference on Mathematical Modelling of Fluid Systems
Date: September 10–12, 2014
Location: University of Bath, United Kingdom
Visit: http:// www.ima.org.uk/conferences/conferences_calendar/ima_conference_on_mathematical_modelling_of_fluid_systems.html

Name: Second International Conference on Analysis and Applied Mathematics (ICAAM 2014)
Date: September 11–13, 2014

Location: M. Auezov South Kazakhstan State University, Shymkent, Kazakhstan
Visit: `http://www.icaam-online.org/index/`

Name: Getting Started with PDE – Summer Workshop for Undergraduate and Graduate Students
Date: September 14–18, 2014
Location: Department of Mathematics, Technion – I.I.T., 32000 Haifa, Israel
Visit: `http://www.math.technion.ac.il/cms/decade_2011-2020/year_2013-2014/PDE-workshop/`

Name: AIM Workshop: Generalized persistence and applications
Date: September 15–19, 2014
Location: American Institute of Mathematics, Palo Alto, California
Visit: `http://aimath.org/workshops/upcoming/persistence`

Name: ICERM Semester Program Workshop: Information-Based Complexity and Stochastic Computation
Date: September 15–19, 2014
Location: Brown University, Providence, Rhode Island
Visit: `http://icerm.brown.edu/sp-f14-w1/`

Name: Workshop 1: Ecology and Evolution of Cancer
Date: September 15–19, 2014
Location: Mathematical Biosciences, Institute The Ohio State University, Jennings Hall 3rd Floor, 1735 Neil Ave., Columbus, Ohio
Visit: `http://mbi.osu.edu/event/?id=495`

Name: Joint Meeting of the German Mathematical Society (DMV) and the Polish Mathematical Society (PTM)
Date: September 17–20, 2014
Location: The Faculty of Mathematics and Computer Science of the Adam Mickiewicz University, Campus UAM, Morasko, 61-616 Poznan, Poland

Name: Third International Conference of Numerical Analysis and Approximation Theory (NAAT2014)
Date: September 17–20, 2014
Location: Babes – Bolyai University, Faculty of Mathematics and Computer Science, Department of Mathematics, Cluj-Napoca, Romania
Visit: `http://naat.math.ubbcluj.ro/`

Name: Riemann, Einstein and geometry
Date: September 18–20, 2014
Location: Institut de RechercheMathématiqueAvancée, University of Strasbourg, France
Visit: `http://www-irma.u-strasbg.fr/article1377.html`

Name: 12th International Conference of The Mathematics Education into the 21st Century Project: The Future of Mathematics Education in a Connected World
Date: September 21–26, 2014
Location: Hunguest Hotel Sun Resort, Herceg Novi, Montenegro
Visit: `http://www.hunguesthotels.hu/en/hotel/herceg_novi/hunguest_hotel_sun_resort/`

Name: 5th International Workshop on Computational Topology in Image Context
Date: September 22–25, 2014
Location: Timisoara, Romania
Visit: `http://ctic2014.synasc.ro/`

Name: Logic and Applications – LAP 2014
Date: September 22–26, 2014

Location: Inter-University Center, Dubrovnik, Croatia
Visit: `http://imft.ftn.uns.ac.rs/math/cms/LAP2014`

Name: 3rd International Conference on Mathematical Applications in Engineering 2014
Date: September 23–25, 2014
Location: Kuala Lumpur, Malaysia
Visit: `http://www.iium.edu.my/icmae/14/`

Name: MBI Boot Camp: How to Simulate and Analyze Your Cancer Models with COPASI
Date: September 29–October 1, 2014
Location: Mathematical Biosciences Institute, The Ohio State University, Jennings Hall 3rd Floor, 1735 Neil Ave., Columbus, Ohio
Visit: `http://mbi.osu.edu/event/?id=757`

Name: AIM Workshop: Quantum curves, Hitchin systems, and the Eynard-Orantin theory
Date: September 29–October 3, 2014
Location: American Institute of Mathematics, Palo Alto, California
Visit: `http://aimath.org/workshops/upcoming/quantumcurves`

Name: ICERM Semester Program Workshop: Approximation, Integration, and Optimization
Date: September 29–October 3, 2014
Location: Brown University, Providence, Rhode Island
Visit: `http://icerm.brown.edu/sp-f14-w2/`

Name: International Conference on Numerical and Mathematical Modeling of Flow and Transport in Porous Media
Date: September 29–October 3, 2014
Location: Centre for Advanced Academic Studies (CAAS), 20000 Dubrovnik, Croatia
Visit: `http://nm2porousmedia.math.pmf.unizg.hr/index.html`

Name: International Conference on Algebraic Methods in Dynamical Systems (Conference in honour of the 60th birthday of Juan J. Morales-Ruiz)
Date: October 5–11, 2014
Location: Universidad del Norte, Barranquilla, Colombia
Visit: `http://www.scm.org.co/eventos/AMDS2014/`

Name: Methods of Noncommutative Geometry in Analysis and Topology
Date: October 6–9, 2014
Location: Leibniz University Hannover, Hannover, Germany
Visit: `http://www.math-conf.uni-hannover.de/methodsncg14/de/`

Name: AIM Workshop: Positivity, graphical models, and modeling of complex multivariate dependencies
Date: October 13–17, 2014
Location: American Institute of Mathematics, Palo Alto, California
Visit: `http://aimath.org/workshops/upcoming/modelmultivar`

Name: MBI Workshop 2: Metastasis and Angiogenesis
Date: October 13–17, 2014
Location: Mathematical Biosciences Institute, The Ohio State University, Jennings Hall 3rd Floor, 1735 Neil Ave., Columbus, Ohio
Visit: `http://mbi.osu.edu/event/?id=496`

Name: Georgia Algebraic Geometry Symposium 2014
Date: October 17–19, 2014

Location: University of Georgia, Athens, Georgia
Visit: http://gags.torsor.org/conf2014/

Name: Yamabe Memorial Symposium 2014: Current Topics in the Geometry of 3-Manifolds
Date: October 17–19, 2014
Location: School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455
Visit: http://www.math.umn.edu/yamabe/

Name: Autumn school on nonlinear geometry of Banach spaces and applications
Date: October 20–24, 2014
Location: Métabief, France
Visit: http://trimestres-lmb.univ-fcomte.fr/fa

Name: International Conference in Modeling Health Advances 2014
Date: October 22–24, 2014
Location: UC Berkeley, San Francisco Bay Area, California
Visit: http://www.iaeng.org/WCECS2014/ICMHA2014.html

Name: 28th Midwest Conference on Combinatorics and Combinatorial Computing
Date: October 22–24, 2014
Location: University of Nevada, Las Vegas (UNLV), Las Vegas, Nevada
Visit: http://www.mcccc.info

Name: Ahlfors-Bers Colloquium VI
Date: October 23–26, 2014
Location: Yale University, New Haven, Connecticut

Name: AIM Workshop: Configuration spaces of linkages
Date: October 27–31, 2014
Location: American Institute of Mathematics, Palo Alto, California
Visit: http://aimath.org/workshops/upcoming/linkages

Name: Conference on Geometric Functional Analysis and its Applications
Date: October 27–31, 2014
Location: Université de Franche-Comté, Besann, France
Visit: http://trimestres-lmb.univ-fcomte.fr/fa

Name: ICERM Semester Program Workshop: Discrepancy Theory
Date: October 27–31, 2014
Location: Brown University, Providence, Rhode Island
Visit: http://icerm.brown.edu/sp-f14-w3/

Name: Scalar Curvature in Manifold Topology and Conformal Geometry
Date: November 1–December 31, 2014
Location: Institute for Mathematical Sciences, National University of Singapore, Singapore
Visit: http://www2.ims.nus.edu.sg/Programs/014scalar/index.php

Name: AIM Workshop: Combinatorics and complexity of Kronecker coefficients
Date: November 3–7, 2014
Location: American Institute of Mathematics, Palo Alto, California
Visit: http://aimath.org/workshops/upcoming/kroncoeff

Name: MBI Current Topic Workshop on Axonal Transport and Neuronal Mechanics
Date: November 3–7, 2014
Location: Mathematical Biosciences Institute, The Ohio State University, Jennings Hall 3rd Floor, 1735 Neil Ave., Columbus, Ohio
Visit: `http://mbi.osu.edu/event/?id=817`

Name: Fifth Ya.B. Lopatinskii International Conference of Young Scientists on Differential Equations and Its Applications
Date: November 5–8, 2014
Location: Donetsk National University, Donetsk, Ukraine
Visit: `http://math.donnu.edu.ua/en-us/science/conferences/ICL2014`

Name: Inverse Moment Problems: The Crossroads of Analysis, Algebra, Discrete Geometry and Combinatorics
Date: November 11–January 25, 2014
Location: Institute for Mathematical Sciences, National University of Singapore, Singapore
Visit: `http://www2.ims.nus.edu.sg/Programs/014inverse/index.php`

Name: SIAM Conference on Financial Mathematics and Engineering (FM14)
Date: November 13–15, 2014
Location: The Palmer House, A Hilton Hotel, Chicago, Illinois
Visit: `http://www.siam.org/meetings/fm14/`

Name: Conference on Mathematics and its Applications–2014
Date: November 14–17, 2014
Location: Kuwait University, Kuwait City, Kuwait
Visit: `http://cma2014.science.ku.edu.kw`

Name: AIM Workshop: Bounded gaps between primes
Date: November 17–21, 2014
Location: American Institute of Mathematics, Palo Alto, California
Visit: `http://aimath.org/workshops/upcoming/primegaps2`

Name: Categorical Structures in Harmonic Analysis
Date: November 17–21, 2014
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: `http://www.msri.org/workshops/708`

Name: MBI Workshop on Cancer and the Immune System
Date: November 17–21, 2014
Location: Mathematical Biosciences Institute, The Ohio State University, Jennings Hall 3rd Floor, 1735 Neil Ave., Columbus, Ohio
Visit: `http://mbi.osu.edu/event/?id=498`

Name: International Conference on Pure and Applied Mathematics, Goroka 2014: ICPAM-GOROKA (2014)
Date: November 24–28, 2014
Location: University of Goroka, Goroka, Eastern Highlands Province, Papua, New Guinea
Visit: `http://icpam-goroka2014.blogspot.com`

Name: International Congress on Music and Mathematics
Date: November 26–29, 2014
Location: University of Guadalajara, Puerto Vallarta, Mexico
Visit: `http://icmm.cucei.udg.mx/`

Name: The 19th Asian Technology Conference in Mathematics (ATCM 2014)
Date: November 26–30, 2014
Location: State University of Yogyakarta, Yogyakarta, Indonesia
Visit: `http://atcm.mathandtech.org`

Name: Annual meeting of the French research network (GdR) in Noncommutative Geometry
Date: November 27–29, 2014
Location: Besancon, France
Visit: `http://trimestres-lmb.univ-fcomte.fr/Noncommutative-Geometry-meeting.html`

Date: December 1–5, 2014
Name: AIM Workshop: Beyond Kadison-Singer – paving and consequences
Location: American Institute of Mathematics, Palo Alto, California
Visit: `http://aimath.org/workshops/upcoming/beyondks/`

Name: 38th Australasian Conference on Combinatorial Mathematics and Combinatorial Computing (ACCMCC)
Date: December 1–5, 2014
Location: Victoria University of Wellington, Wellington, New Zealand
Visit: `http://msor.victoria.ac.nz/Events/ACCMCC/WebHome?redirectedfrom=Events.38ACCMCC`

Name: Automorphic forms, Shimura varieties, Galois representations and L-functions
Date: December 1–5, 2014
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: `http://www.msri.org/workshops/719`

Name: International Conference on Applied Mathematics – in honour of Professor Roderick S. C. Wong's 70th Birthday
Date: December 1–5, 2014
Location: City University of Hong Kong, Tat Chee Avenue, Kowloon Tong, Hong Kong
Visit: `http://www6.cityu.edu.hk/rcms/icam2014/index.htm`

Name: International Conference on Pure and Applied Mathematics, Goroka 2014: ICPAM-GOROKA (2014)
Date: December 1–5 (NEW DATE), 2014
Location: University of Goroka, Goroka, Eastern Highlands Province, Papua, New Guinea
Visit: `http://icpam-goroka2014.blogspot.in/`

Name: Winter School on Operator Spaces, Non-commutative Probability and Quantum Groups
Date: December 1–12, 2014
Location: Métabief, France
Visit: `http://trimestres-lmb.univ-fcomte.fr/Winter-School?lang=fr`

Name: The Info-Metrics Annual Prize in Memory of Halbert L. White Jr
Date: December 6–31, 2014
Location: Washington, DC
Visit: `http://www.american.edu/cas/economics/info-metrics/prize.cfm`

Name: IMA Conference on Game Theory and its Applications
Date: December 8–10, 2014
Location: St. Anne's College, Oxford, United Kingdom
Visit: `http://ima.org.uk/conferences/conferences_calendar/game_theory_and_its_applications.html`

Name: AIM Workshop: Transversality in contact homology
Date: December 8–12, 2014
Location: American Institute of Mathematics, Palo Alto, California
Visit: http://aimath.org/workshops/upcoming/transcontacthom/

Name: 8th Australia - New Zealand Mathematics Convention
Date: December 8–12, 2014
Location: University of Melbourne, Melbourne, Australia
Visit: http://www.austms2014.ms.unimelb.edu.au/

Name: First call for the training programme "Collaborative Mathematical Research"
Date: December 9–10, 2014
Location: Centre de Recerca Matematica, Bellaterra, Barcelona, Spain

Name: Vertex algebras, W-algebras, and applications
Date: December 9–20, 2014
Location: Centro di Ricerca Matematica Ennio De Giorigi, Pisa, Italy
Visit: http://www.crm.sns.it/event/293/activities.html#title

Name: I Brazilian Congress of Young Researchers in Pure and Applied Mathematics
Date: December 10–12, 2014
Location: Mathematics and Statistics Institute, University of São Paulo, São Paulo, Brazil
Visit: http://jovens.ime.usp.br/jovens/en

Name: Foundations of Computational Mathematics Conference
Date: December 11–20, 2014
Location: Universidad de la República, Montevideo, Uruguay
Visit: http://www.fing.edu.uy/∼jana/www2/focm_2014.html

Name: 10th IMA International Conference on Mathematics in Signal Processing
Date: December 15–17, 2014
Location: Austin Court, Birmingham, United Kingdom
Visit: http://www.ima.org.uk/

Name: 1st International Conference on Security Standardisation Research
Date: December 16–17, 2014
Location: Royal Holloway, University of London (RHUL), Egham Hill, Egham, United Kingdom
Visit: http://www.ssr2014.com/

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