Central Extensions of a $p$-adic Division Algebra

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1. Introduction

Let $k$ be a finite extension of the field $\mathbb{Q}_p$ of $p$-adic numbers and suppose $D$ is a finite-dimensional central division algebra over $k$.

Then, the group $G = SL_1(D)$ consisting of elements of reduced norm 1 in $D$ acquires a topology from $k$ and is a compact, totally disconnected (i.e., a profinite) group. We are interested in finding the possible (topological) central extensions

$$1 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1.$$

General nonsense tells us that the set of central extensions is determined by a group, denoted $H^2(G, \mathbb{R}/\mathbb{Z})$, which, in this case, is known to be finite by some deep work of Raghunathan on the congruence subgroup problem ([R]). It is expected (although unknown as yet) that, barring a handful of exceptions, $H^2(G, \mathbb{R}/\mathbb{Z}) \cong \mu(k)_p$, the finite cyclic group of $p$-th power roots of unity in $k$. In 1988, Gopal Prasad & M. S. Raghunathan proved ([PR]) that $H^2(G, \mathbb{R}/\mathbb{Z})$ is a finite cyclic group containing an isomorphic copy of $\mu(k)_p$ and is trivial if $\mu(k)_p$ is trivial.

These results were sufficient for their original motivation to solve the so-called metaplectic problem which comes up in the congruence subgroup problem. However, the general computation of $H^2$ is still open.

Our aim here is to stretch the method of [PR] and study the $p^2$-torsion in $H^2(G, \mathbb{R}/\mathbb{Z})$ with a view to proving that if $H^2$ has an element of order $p^2$, then $k$ contains a primitive $p^2$-th root of unity. The computations are rather cumbersome, and we carry them out fully only in a special case when $p = 3$ and $D$ is the quaternion division algebra although we have partial results in more generality.

One probably needs new ideas along with the work of [PR] if one wants to compute $H^2$ in general. Perhaps, on the other hand, the seminal work of Lazard ([L]) on compact $p$-adic Lie groups has not been exploited sufficiently enough.

2. Basic structure of $D$

The structure of $p$-adic division algebras had been investigated by C. Riehm in [Ri]. Let us briefly recall some details.
D contains R, its maximal compact subring which, in turn, contains a unique (two-sided) maximal ideal P. The group $G \cong SL(1, D)$ of elements of reduced norm 1 in D, is a profinite group which is normal in $D^\ast$.

G admits a filtration $G_i = \{ g \in G : g \equiv 1 \mod P^i \}$ for $i \geq 1$. In fact, $G_i$ are all normal in $D^\ast$. $G_1$ is a pro-$p$ group and $G/G_1$ is a finite, cyclic group of order prime to $p$.

For $i, j \geq 1$, we have $[G_i, G_j] \subseteq G_{i+j}$. In particular, $G_1$ is a pro-$p$ group and $G/G_1$ is a finite, cyclic group of order prime to $p$.

By local class field theory, there exists a uniformising parameter $\pi$ in $\mathcal{P}$ which normalises the maximal unramified extension $K$ of degree $d$ over $k$. Also, the automorphism of $K$ given by the conjugation by $\pi$ generates the Galois group $\text{Gal}(K/k) \cong \text{Gal}(F/f)$ where $F, f$ are the residue fields of $K, k$ respectively.

Moreover, the non-zero elements $F^\ast$ can be identified with $\mu(K)$ tame, the cyclic group of prime-to-$p$ roots of unity in $K$.

Each element $g \in R$ can be uniquely expressed as $g = g_0 + \sum_{n \geq 1} g_n \pi^n$ with $g_n \in \mu(K)$ tame $\cup \{0\}$.

A little computation shows also that the abelian group $G_i/G_{i+1}$ can be identified via the map $\rho_i : 1 + \sum_{n \geq i} g_n \pi^n \mapsto g_i$ with $\rho(i)$ which is either $E := \{ x \in F : TF/ff(x) = 0 \}$ or the whole of $F$ according as $d|i$ or $d \nmid i$.

It is also quite easy to show that $G = G_1(G \cap \mu(K)$ tame$)$.

Note that each $F(i)$ is a module for $G \cap \mu(K)$ tame$)$ under the action $\phi \cdot x = \frac{\phi}{\sigma^i(\phi)} x$ where we have identified $\mu(K)$ tame$)$ with $\{0\}$.

A consequence of Hilbert’s theorem 90 is that there is a nontrivial homomorphism (of modules) from $F(i)$ to $F(j)$ if, and only if, $i \equiv j \mod d$.

3. Conditions for roots of unity

It is easy to write down necessary and sufficient conditions for $k$ to contain a primitive $p^2$-th root of unity.

Recall that $k \subset K \subset D$ and $\pi$ is a uniformising parameter in $D$ normalising $K$.

Let $e$ denote the ramification index of $k$ over $\mathbb{Q}_p$ and $d$ denote the degree of $D$ over $k$. Then $\pi^d$ is a uniformising parameter for $k$ and, one can expand $p$ over $k$ as

$$p = \theta \pi^d + \theta_2 \pi^{d+2} + \cdots$$

for some $\theta$’s in $f$.

Now, well-known properties of the $p$-th power map (see [M], P. 167–168) tells us that $k$ has a primitive $p^2$-th root of unity if, and only if, $p(p - 1)$ divides $e$ and there exists some $Y = 1 + Y_{d/e}/p(p-1) \pi^{d/e/p(p-1)} + Y_{d/e/p(p-1)+d} \pi^{d/e/p(p-1)+d+d} + \cdots$ such that $Y_{d/e/p(p-1)} \neq 0$ and such that $Y^{p^2} \equiv 1 \mod \pi^{d/e/p(p-1)+d+d}$.

This gives (using the expression of $p$) certain polynomial equations in the $Y_i$’s with coefficients as some $\theta$’s. We get finitely many polynomials over $f$.

Our aim is, therefore, to deduce the simultaneous solvability of these equations by somehow getting information over $f$ that can be derived from the assumption that $H^2$ has $p^2$-torsion.

It should be pointed out that the corresponding calculation with $p$ in place of $p^2$ is much easier and was carried out in [PR].
4. Strategy of studying $H^2$ (after [PR])

In this section, we recall the basic method as well as the results of Prasad & Raghunathan from [PR] which we shall be using.

Fact 1. Using the fact that $G$ is profinite, it is easy to deduce that $H^2(G, \mathbb{R}/\mathbb{Z}) \cong H^2(G, J)$ where $J$ is the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of $p$-power order (considered with the discrete topology).

We shall be using the Hochschild-Serre spectral sequence for the situation $G_{i-1}/G_i \leq G/G_i$ for $i > 1$.

A very useful property is that given a central extension $C \subseteq E \rightarrow A$ where $A$ is abelian is that one has a lifted ‘commutator’ map; if $a, b \in A$, then for arbitrary lifts $x, y$ of $a, b \in E$, the commutator $[x, y] = xyx^{-1}y^{-1}$ lands inside $C$ and, is independent of the lifts. One often writes $[a, b]$ for this element of $C$. When $C$ is a divisible group (like our $J$), the extension is ‘trivial’ if, and only if, $E$ itself is abelian.

In our case, we shall use it for $G_{i-1}/G_i$ which is abelian if $2^i \geq j$.

Fact 2. $H^2(G, J) = \lim_{\rightarrow} H^2(G/G_i, J)$ and, the ‘inflation’ maps $\text{inf}(i) : H^2(G/G_i, J) \rightarrow H^2(G, J)$ are injective if, and only if, $d \nmid i$. Moreover, $H^2(G, J)$ is the union of the images $H^2(G_i)$ over all $i$ under the inflation maps; $H^2(G_i)$ is an increasing filtration.

If $d\mid i$, then there is a natural identification of $\text{Ker} \text{inf}(i)$ with the vector space $E$ of elements of trace zero in $F$ over $f$.

look at the inflation maps $H^2(G/Gr_{r-1}, J) \rightarrow H^2(G/Gr_r, J) \rightarrow H^2(G, J)$.

If $d\nmid r$, since the inflation $\text{inf}(r)$ is not injective, it is necessary to know when some $c \in H^2(G/Gr_r, J)$ inflates to $H^2(G, J)$ to an element which comes from $H^2(G/Gr_{r-1}, J)$. This happens if $c$ restricts to the trivial extension over the subgroup $G_{r-1}/G_r$. More precisely, using the Hochschild-Serre sequence corresponding to $G_{r-1}/G_r < G/G_r$, we see that:

Fact 3. If $d\nmid r$, $H^2(G_i) = H^2(G)_{i-1}$.

In fact, one can show that the same equality holds if $dp \nmid j$; this uses some commutator identities due to P. Hall which are valid in any group.

(P. Hall) In any group $G$, for elements $a, b, c$ one has

$$[[a, b], c][[b, c], e][a][[c, a], b]] = 1.$$ 

Look at the inflation maps

$$H^2(G/Gr_{r-1}, J) \rightarrow H^2(G/Gr_r, J) \rightarrow H^2(G, J).$$

If $d\mid r$, since the inflation $\text{inf}(r)$ is not injective, it is necessary to know when some $c \in H^2(G/Gr_r, J)$ inflates in $H^2(G, J)$ to an element which comes from $H^2(G/Gr_{r-1}, J)$. This happens if $c$ restricts to the trivial extension over the subgroup $G_{r-1}/G_r$. More precisely, using the Hochschild-Serre sequence corresponding to $G_{r-1}/G_r < G/G_r$, we see that:

Fact 4. $c$ comes from $H^2(G/Gr_{r-1}, J)$ if, and only if, it is in $\text{Ker}(H^2(G/Gr_r, J) \rightarrow H^2(G/Gr_{r-1}/G_r, J)) \cap E^1_{\infty}$. This is understood better as follows.

For any $c$ in $H^2(G/Gr_r, J)$, let $J \subseteq E \rightarrow G/G_r$ denote the corresponding central extension. The $E^1_{\infty}$-term is

$$H^1(G/Gr_{r-1}, H^1(G/Gr_r, J)) = H^1(G/Gr_{r-1}/G_r, J) = H^1(G/Gr_{r-1}/G_r, J) \cong \text{Hom}(F(1), \text{Hom}(F(r - 1), J)),$$

where $E^1_{\infty} = \text{Hom}(F(1), \text{Hom}(F(r - 1), J))^{G/G_{r-1}}$.
As we noted, this is nontrivial only if \( d | r \). If \( d | r \), this is just the set of all (equivariant) bimultiplicative maps from \( F(1) \times F(r - 1) \) to \( f_0 \) where we shall write \( f_0 \) for the prime field \( \mathbb{Z}/p \).

In fact, it is easy to write down all its elements. These are the maps \( F(1) \times F(r - 1) \to f_0 \); \((X, Y) \mapsto Tr_{F/F_0} \left( \lambda X \sigma(Y) \right) \) for some \( \lambda \in F \). With this identification, it follows that \( c \) comes from the previous level if, and only if, the corresponding \( \lambda \) has trace zero over \( f \) i.e., we get:

**Fact 5.** \( c \) in \( H^2(G/G_r, J) \) is the inflation of an element of \( H^2(G/G_{r-1}, J) \) if, and only if, there exists \( \lambda \in F \) of trace zero over \( f \) such that \( c \) can be ‘regarded’ as (this is the image in the \( E_2^{1,1} \)-term) the map from \( F(1) \times F(r - 1) \) to \( f_0 = \mathbb{Z}/p \) given by

\[
\wedge^c (X, Y) = Tr_{F/F_0} \left( \lambda X \sigma(Y) \right) \forall X \in F(1), Y \in F(r - 1).
\]

It should be noted that the map above is induced by the ‘commutator’ map from \( E_1 \times E_{r-1} \) to \( J \) where \( E_i \) is the inverse image of \( G_i/G_r \) in \( E \).

More generally, from P. Hall’s identity, one can easily show that \( E_i \) commutes with \( E_{r-i+1} \) for all \( i \leq r \). Thus the central extension splits over \( G_i/G_r \) whenever \( 2i > r \). Thus:

**Fact 6.** If \( r > 2 \) and \( \epsilon < \frac{r}{2} \) then the restriction map \( H^2(G/G_r, J) \to H^2(G_{r-\epsilon}/G_r, J) \) is the zero map.

Once again, if \( \epsilon < \frac{r}{2} \), we can consider the Hochschild-Serre sequence corresponding to \( G_{r-\epsilon}/G_r < G/G_r \). By the above, \( (E_{i+n}^{0,2} \)-term is zero and so) we have a homomorphism

\[
H^2(G/G_r, J) \to E_1^{0,1} \subseteq E_2^{1,1} = H^1(G/G_r-\epsilon, \text{Hom} \left( (G_{r-\epsilon}/G_r, J) \right)).
\]

One can show that the above-mentioned image in the \( E_2^{1,1} \)-term actually comes from \( (H^1(G/G_{\epsilon+1}, \text{Hom} \left( (G_{r-\epsilon}/G_r, J) \right)) \). More precisely:

**Fact 7.** With \( \epsilon < r/2 \), the image in \( E_2^{1,1} \) is contained in the image of

\[
\text{inf} l : H^1(G/G_{\epsilon+1}, \text{Hom} \left( (G_{r-\epsilon}/G_r, J) \right)) \to E_2^{1,1}.
\]

A key point (discovered in [PR]) is that one can describe these 1-cocycles very explicitly. We describe this now.

**Fact 8.** Let \( \epsilon, s, t \) be positive integers s.t. \( \epsilon \leq \text{min}(de, \frac{1}{2}dt) \) and \( s \geq \epsilon + 1 \). Let \( f = \left[ \begin{smallmatrix} \epsilon \\ -d \\ \frac{1}{2} \\ t \end{smallmatrix} \right] \). For \( (\lambda_0, \ldots, \lambda_f) \in F^{f+1} \),

\[
Z_{(\lambda_0, \ldots, \lambda_f)}(a)(b) = Tr_{F/F_0} \left\{ \sum_{0 \leq u \leq f} \sum_{\ell+m \leq \epsilon - du} \lambda_u(\ell) a_{\ell} \sigma^\ell(a_m') \sigma^{\ell+m}(b_{dt-du-\ell-m}) \right\}
\]

(here \( a = \sum a_{\ell} \pi^\ell \in D_1/D_s, b = \sum b_{n} \pi^m \in D_{dt-\epsilon}/D_{dt} \)) is a \( F \)-invariant 1-cocycle on \( D_1/D_s \) with values in \( \text{Hom} \left( (D_{dt-\epsilon}/D_{dt}) \right) \); these restricted to \( G_1/G_s \times G_{dt-\epsilon}/G_{dt} \) and then extended to \( G/G_s \times G_{dt-\epsilon}/G_{dt} \) by defining them to be zero on \( (G \cap F) \times G_{dt-\epsilon}/G_{dt} \), give all the cohomology classes in \( H^1(G/G_s, \text{Hom} \left( (G_{dt-\epsilon}/G_{dt}, J) \right)) \).

Also, here \( \lambda(l) \) stands for \( \lambda + \sigma(\lambda) + \cdots + \sigma^{l-1}(\lambda) \).

We shall be using this only for \( \epsilon = de, s = de + 1 \) and \( t = \frac{ep}{p+1} + \epsilon \).
Finally, let us note:

**Fact 9.** \(H^2(G) = 0\) for \(r < \frac{\text{dep}}{p-1}\) and \(H^2(G)_{\text{dep}/(p-1)}\) constitutes the elements of order at most \(p\) in \(H^2(G, \mathbb{R}/\mathbb{Z})\).

Moreover, if \(H^2(G, \mathbb{R}/\mathbb{Z})\) has an element of order \(p^2\), then \(r = \frac{\text{dep}}{p-1} + \text{de}\) where \(H^2(G)\) is the earliest where an element of order \(p^2\) shows up.

This is because the \(p\)-th power gives \(G_{\frac{\text{dep}}{p-1} + \text{de}} \cong G_{\frac{\text{dep}}{p-1} + \text{de}}\) and an element \(c\) of \(H^2(G)_{\frac{\text{dep}}{p-1} + i - \text{de}}\) is of order \(p^j\) ⇔ \(cp\) in \(H^2(G)_{\frac{\text{dep}}{p-1} + i - \text{de}}\) is of order \(p^{j-1}\). Also, we have \(p|\text{e}\) because cohomology ‘pops up’ at stages which are multiples of \(pd\). Thus, \(pd/\text{de}\) i.e. \(p|e\).

### 5. Outline of proof

Here is how we obtain conditions over \(f\) using the assumption that \(H^2\) has \(p^2\)-torsion.

We look at the corresponding element in

\[H^1(G/G_{1+\text{de}}, H^1(G_{\text{dep}/(p-1)}/G_{\text{dep}/(p-1) + \text{de}}, J)).\]

We consider the abelian subgroup \(A = K \cap G\) of \(G\) and, as elements of the above cohomology group, we may write down the equations \([X, Y^{p^2}] = 0\) \(\forall X, Y \in A\), where \(A\) is the image of \(A\) in \(G/G_{\frac{\text{dep}}{p-1} + \text{de}}\). These commutators are computed with the help of the above explicit expressions for \(\epsilon = \text{de}, dt = \frac{\text{dep}}{p-1} + \text{de}, s = \text{de} + 1\) as written down in fact 8.

We note that the triviality of these commutators is due to their bilinearity. Thus we get equations over \(f\) from which we try to deduce the required equations (for \(p^2\)-th root to exist in \(k\)) in \(f\). Note that in the computation of commutators in the central extension corresponding to \(c\), the \(\lambda_0\) which figures is such that \(T_{r/f}(\lambda_0) \neq 0\) since \(\frac{\text{dep}}{p-1} + \text{de}\) is the smallest level where \(p^2\)-torsion occurs.

### 6. The prime \(p = 3\)

We carry out the computations only in the following special case.

We assume \(p = 3, d = 2, e = p(p - 1) = 6\).

As \(\pi^2\) is a uniformising parameter for \(k\), we can write \(\frac{3}{\pi^2} = \theta + \theta_2 \pi^2 + \theta_4 \pi^4 + \ldots\) where \(\theta, \theta_2, \ldots \in f\).

For a \(p^2\)-th root of 1 to exist in \(k\), it is necessary and sufficient that there exists \(Y \in U_{\frac{\text{dep}}{p-1} + \text{de} + 1}\) such that \(Y^{p^2} \in U_{\frac{\text{dep}}{p-1} + \text{de} + 1}\) and such that \(Y \notin U_{\frac{\text{dep}}{p-1}}\) i.e. \(Y\) is not a \(p\)-th root.

In our case, we want \(Y \in U_2 \setminus U_6\) such that \(Y^9 \equiv 1\) mod \(\pi^3\). Then,

\[\theta Y^3 + Y^9 = 0\]
\[\theta_2 = 0\]
\[\theta_4 = 0\]
\[ \theta(Y_4 + Y_2^3) + \theta_6 Y_2^3 = 0 \]
\[ \theta^2 Y_2 + \theta_8 Y_2^3 = 0 \]
\[ \theta^2(Y_4 + Y_2^2) + \theta_{10} Y_2^3 = 0 \]
\[ \theta^2 Y_6 + \theta Y_6^3 + 2(\theta^2 Y_2 Y_4 + \theta Y_2^3 Y_4^3) + \theta_6(Y_4 + Y_2^2) + \theta_{12} Y_2^3 = 0. \]

We can find such a \( Y \) if, and only if, the following ‘compatibility’ conditions hold

\[ \theta_2 = 0 \]
\[ \theta_4 = 0 \]
\[ \theta + X^2 = 0 \] has a solution \( \mu \) in \( f \)
\[ \theta^3 = \theta^5 \]
\[ \theta_1^3 + \theta^2 \theta_6 = 0 \]

\( X^9 + \theta^3 X^3 = \theta_1^3 + \theta_6^6 \mu^{-9} - \theta_6 \mu^6 + \theta_1^3 \mu^{-3} \) has a solution over \( f \).

If these hold, then the solutions for \( Y \) are:

\[ Y_2 = \text{A solution of } Y_2^6 + \theta = 0 \]
\[ Y_4 = \text{A solution of } Y_4^3 = \theta_6 Y_2^{-3} - Y_2^6 \]
\[ Y_6 = \text{A solution of } Y_6^9 + \theta^3 Y_6^3 = \theta_6^3 + \frac{\theta_6^6}{Y_2^3} + \theta^3 \theta_6 + \frac{\theta_{12}}{Y_2^3}. \]

Let us expand the relevant powers of \( Y \) now using the expression of \( p \) in \( k \).

\[ Y = 1 + Y_2 \pi^2 + Y_4 \pi^4 + Y_6 \pi^6 + \ldots \in U_2 \]
\[ Y^3 = 1 + (\theta \pi^{12} + \theta_2 \pi^{14} + \theta_4 \pi^{16} + \ldots)(Y_2 \pi^2 + Y_4 \pi^4 + Y_6 \pi^6 + \ldots) \]
\[ + (\theta \pi^{12} + \theta_2 \pi^{14} + \ldots)(Y_2^2 \pi^4 + 2Y_2 Y_4 \pi^6 + (Y_4^2 + 2Y_2 Y_6)\pi^8 + \ldots) \]
\[ + (Y_2^3 \pi^6 + Y_4^3 \pi^{12} + Y_6^3 \pi^{18} + \ldots) \]
\[ = 1 + Y_2^3 \pi^6 + Y_4^3 \pi^{12} + \theta Y_2 \pi^{14} + (\theta Y_2^2 + \theta Y_4 + \theta_2 Y_2)\pi^{16} \]
\[ + (Y_6^3 + 2\theta Y_2 Y_4 + \theta Y_6 + \theta_2 Y_2^2 + \theta_2 Y_4 + \theta_4 Y_2)\pi^{18} + \ldots \]
\[ Y^9 = 1 + (\theta \pi^{12} + \theta_2 \pi^{14} + \ldots)(Y_2^3 \pi^6 + Y_4^3 \pi^{12} + \theta Y_2 \pi^{14} + (\theta Y_2^2 + \theta Y_4 + \theta_2 Y_2)\pi^{16} \]
\[ + (Y_6^3 + 2\theta Y_2 Y_4 + \theta Y_6 + \theta_2 Y_2^2 + \theta_2 Y_4 + \theta_4 Y_2)\pi^{18} + \ldots) \]
\[ + (\theta \pi^{12} + \theta_2 \pi^{14} + \ldots)(Y_2^6 \pi^{12} + 2Y_2^3 Y_4 \pi^{18} + \ldots) + (Y_2^9 \pi^{18} \text{ mod } \pi^{36}) \]
\[ = 1 + (\theta Y_2^2 + \theta Y_2^2)\pi^{18} + \theta Y_2^3 \pi^{20} + \theta_4 Y_2^3 \pi^{22} + (\theta Y_6^3 + \theta Y_4 + \theta_6 Y_2^3)\pi^{24} \]
\[ + (\theta Y_2^2 + \theta Y_2^2 + \theta Y_2^2)\pi^{26} \]
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\[ + (\theta^2 Y_2^2 + \theta^2 Y_4 + 2\theta \theta_2 Y_2 + \theta_4 Y_2^6 + \theta_4 Y_3 + \theta_{10} Y_2^3) \pi^{28} \]
\[ + (\theta^2 Y_6 + \theta Y_6^3 + 2(\theta^2 Y_2 Y_4 + \theta Y_2^3 Y_4^3) + \theta_6(Y_4 + Y_2 Y_2^3) \]
\[ + \theta_{12} Y_2^3 + 2\theta \theta_2 Y_2^2 + 2\theta \theta_4 Y_4 + 2\theta_2 Y_2 + \theta_2 Y_2) \pi^{30} + \ldots . \]

Now \( H^1(G/G_{1+}\text{de}, \text{Hom}(G_{1+}/G_{1^{+}}, J)) = H^1(G/G_{13}, \text{Hom}(G_{18}/G_{30}, J)) \).

Consider \( X, Y \in K^1 \) such that \( Y^9 \in G_{18}/G_{30} \).

One knows then that \( [X, Y^9] = [X, Y]^9 = 1 \) and \( [X, Y^9] \) can be calculated from our knowledge of \( H^1(G/G_{13}, \text{Hom}(G_{18}/G_{30}, J)) \).

In fact, if \( X = 1 + \sum b_i X_i \) and \( Y = 1 + \sum Y_i \), if \( Y^9 = 1 + \sum b_i X_i \), then

\[ [X, Y^9] = Tr_{F/f} \left\{ \sum_{0 \leq u \leq 5} \sum_{\ell + m \leq 12 - 2u} \lambda_u(\ell) X_\ell \bar{X}_m b_{30 - 2u - \ell - m} \right\}. \tag{1} \]

Here \( X^{-1} = 1 + \sum \bar{X}_i \). Since \( Y^9 \in G_{18}/G_{30} \), we have \( b_i = 0 \) for \( i \neq 18, 20 \).

Contributions to the right side of (1) are as follows:

For \( u = 5 \), it is

\[ Tr_{F/f} \{ \lambda_5(2) X_2 b_{18} \} = Tr_{F/f} \{ Tr_{F/f} (\lambda_5) X_2 (\theta Y_2^3 + Y_2) \} . \]

For \( u = 4 \), it is

\[ Tr_{F/f} \{ \lambda_4(2)(X_2 b_{20} + X_2 \bar{X}_2 b_{18}) + \lambda_4(4) X_4 b_{18} \} \]
\[ = Tr_{F/f} \{ Tr_{F/f} (\lambda_4) [X_2 \theta_2 Y_2^3 + (2X_4 + X_2 \bar{X}_2)(\theta Y_2^3 + Y_2)] \} . \]

For \( u = 3 \), it is

\[ Tr_{F/f} \{ \lambda_3(2)(X_2 b_{22} + X_2 \bar{X}_2 b_{20} + X_2 \bar{X}_4 b_{18}) \]
\[ + \lambda_3(4)(X_4 b_{20} + X_4 \bar{X}_2 b_{18}) + \lambda_3(6) X_6 b_{18} \} \]
\[ = Tr_{F/f} \{ Tr_{F/f} (\lambda_3) [X_2 \theta_2 Y_2^3 + (2X_4 + X_2 \bar{X}_2) \theta_2 Y_2^3 \]
\[ + (2X_4 \bar{X}_2 + X_2 \bar{X}_4)(\theta Y_2^3 + Y_2)] \} . \]

Note that \( \lambda_3(6) = 3 Tr_{F/f} (\lambda_3) = 0 \).

For \( u = 2 \), it is

\[ Tr_{F/f} \{ Tr_{F/f} (\lambda_2) \left[ X_2 (\theta Y_4^6 + \theta Y_4^3 + \theta_6 Y_2^3) + (2X_4 + X_2 \bar{X}_2) \theta_2 Y_2^3 \right. \]
\[ \left. + (2X_4 \bar{X}_2 + X_2 \bar{X}_4) \theta_2 Y_2^3 + (X_8 + 2X_4 \bar{X}_4 + X_2 \bar{X}_6)(\theta Y_2^3 + Y_2^3) \right] \} . \]
For $u = 1$, it is

$$Tr_{F/\mathfrak{h}} \begin{bmatrix} X_2(\theta^2 Y_2 + \theta_2 Y_4^3 + \theta Y_2^6 + \theta_8 Y_2^3) \\
+(2X_4 + X_2 \bar{X}_2)(\theta Y_2^6 + \theta Y_4^3 + \theta_6 Y_2^3) \\
+(2X_4 \bar{X}_2 + X_2 \bar{X}_4)\theta_4 Y_2^3 + (X_8 + 2X_4 \bar{X}_4 + X_2 \bar{X}_6)\theta_2 Y_2^3 \\
+(2X_{10} + X_8 \bar{X}_2 + 2X_4 \bar{X}_6 + X_2 \bar{X}_8)(\theta Y_2^3 + Y_2^7) \end{bmatrix}.$$  

For $u = 0$, it is

$$Tr_{F/\mathfrak{h}} \begin{bmatrix} X_2(\theta^2 Y_2^2 + \theta^2 Y_4 + 2\theta_2 Y_2 + \theta_4 Y_2^6 + \theta_4 Y_4^3 + \theta_10 Y_2^3) \\
+(2X_4 + X_2 \bar{X}_2)(\theta^2 Y_2 + \theta_2 Y_2^3 + \theta_2 Y_2^6 + \theta_8 Y_2^3) \\
+(2X_4 \bar{X}_2 + X_2 \bar{X}_4)(\theta Y_2^6 + \theta Y_4^3 + \theta_6 Y_2^3) \\
+(X_8 + 2X_4 \bar{X}_4 + X_2 \bar{X}_6)\theta_4 Y_2^3 \\
+(2X_{10} + X_8 \bar{X}_2 + 2X_4 \bar{X}_6 + X_2 \bar{X}_8)\theta_2 Y_2^3 \\
+(2X_{10} \bar{X}_2 + X_8 \bar{X}_4 + 2X_4 \bar{X}_8 + X_2 \bar{X}_{10})(\theta Y_2^3 + Y_2^7) \end{bmatrix}.$$  

We also see (since $X$ is of norm 1 and $p = 3$) that

$$2X_4 + X_2 \bar{X}_2 = -(X_4 + X_2^2)$$  
$$2X_4 \bar{X}_2 + X_2 \bar{X}_4 = X_2^3$$  
$$2X_{10} \bar{X}_2 + X_8 \bar{X}_4 + 2X_4 \bar{X}_8 + X_2 \bar{X}_{10} = -(X_4^3 + X_6^3).$$

We have $0 = [X, Y^9] = $ Sum of these 6 terms corresponding to the values $u = 0, 1, 2, 3, 4, 5$. Now, we start proving the compatibility conditions hold good.

Define $\bar{X}$ by changing $X_{10}$ to $\bar{X}_{10} = X_{10} + \mu$ for some $\mu$ of trace 0. Then, we can have $\bar{X} \in K^1$ with $\bar{X}_i = X_i$ for $i < 10$. Now,

$$0 = [\bar{X}, Y^9] - [X, Y^9] = Tr_{F/\mathfrak{h}} \left\{ Tr_{F/\mathfrak{h}}(\lambda_1)2\mu(\theta Y_2^3 + Y_2^9) + Tr_{F/\mathfrak{h}}(\lambda_0)2\mu\theta_2 Y_2^3 \right\} = 4Tr_{F/\mathfrak{h}} \left\{ Tr_{F/\mathfrak{h}}(\lambda_1)\mu(\theta Y_2^3 + Y_2^9) + Tr_{F/\mathfrak{h}}(\lambda_0)\mu\theta_2 Y_2^3 \right\}.$$  

We have used the fact that when $d = 2, E \cdot E = f$.

For $Y_2 \neq 0$, since $\mu Y_2^3$ could be any arbitrary element of $f$, not depending on $Y_2^3$, we must have

$$Tr_{F/\mathfrak{h}}(\lambda_1)(\theta + Y_2^6) + Tr_{F/\mathfrak{h}}(\lambda_0)\theta_2 = 0 \ \forall \ Y_2 \in E.$$  

If we take $\tilde{Y}_2$ whose square is not $Y_2^3$, then we get, on subtraction,

$$Tr_{F/\mathfrak{h}}(\lambda_1)(\tilde{Y}_2^6 - Y_2^6) = 0 \ i.e. \ Tr_{F/\mathfrak{h}}(\lambda_1) = 0. \quad (2)$$
Therefore (as the existence of $p^2$-torsion implies that $Tr_{F/\mathcal{F}}(\lambda_0) \neq 0$), we obtain

$$\theta_2 = 0.$$  

Then, the first compatibility condition for the existence of a primitive $p^2$-th root of unity in $k$ is proved.

Let us do the same with $X_8$ now i.e. call $\tilde{X} = 1 + X_2\pi^2 + X_4\pi^4 + X_6\pi^6 + (X_8 + \mu)\pi^8 + (X_{10} + \mu X_2)\pi^{10} + \cdots \in K^1$.

$$0 = [\tilde{X}, Y^9] - [X, Y^9]$$

$$\Rightarrow 0 = Tr_{f/f_0} \left( Tr_{F/\mathcal{F}}(\lambda_2) \mu (\theta Y^3 + Y^9) + Tr_{F/\mathcal{F}}(\lambda_4) \mu \theta_4 Y^3 \right).$$

Changing $\mu$ to $\alpha \mu$ for any $\alpha \in f$, we get

$$0 = Tr_{F/\mathcal{F}}(\lambda_2)(Y^9 + \theta Y^3) + Tr_{F/\mathcal{F}}(\lambda_4) \theta_4 Y^3 \forall Y_2 \in E.$$  

Again, as before, we will get

$$Tr_{F/\mathcal{F}}(\lambda_2) = 0$$  

and therefore,

$$\theta_4 = 0.$$  

This is the 2nd compatibility condition.

Now, putting $X_2 = 0$ in the original formula (which now consists of four terms since $Tr(\lambda_1) = 0 = Tr(\lambda_3)$), we get for all $X, Y \in E$

$$Tr_{f/f_0} \left[ Tr_{F/\mathcal{F}}(\lambda_4) X (Y^9 + \theta Y^3) + Tr_{F/\mathcal{F}}(\lambda_0) [X (\theta^2 Y + \theta_8 Y^3) + X^3(\theta Y^3 + Y^9)] \right] = 0.$$  

(5)

Again putting $X_4 = -X_2^2$ in the original formula, we get

$$Tr_{f/f_0} \left\{ Tr_{F/\mathcal{F}}(\lambda_3) X_2^2(Y_2^9 + \theta Y_2^3) + Tr_{F/\mathcal{F}}(\lambda_5) X_4^2(Y_2^9 + \theta Y_2^3) + Tr_{F/\mathcal{F}}(\lambda_0) [X_2^2(\theta^2 Y_2^3 + \theta_4 Y_2^3) + X_4^2(\theta^2 Y_2^3 + \theta_4 Y_2^3 + \theta_6 Y_2^3)] \right\} = 0.$$  

Putting $Y_2 = 0$ will give

$$Tr_{f/f_0}(Tr_{F/\mathcal{F}}(\lambda_0)(\theta^2 XY + \theta X^3 Y^3)) = 0 \forall X, Y \in E.$$  

(6)

Since $Tr_{F/\mathcal{F}}(\lambda_0) \neq 0$ and since $Z \mapsto \theta^2 Z + \theta Z^3$ is an endomorphism of $f$, therefore, it follows from (6) that $\exists Z \neq 0$ in $f$ such that $\theta^2 Z + \theta Z^3 = 0$. Thus, the 3rd compatibility condition has been proved.

In the above, instead of putting $Y_2 = 0$, let us, instead, put $Y_4 = -Y_2^2$ which will give $\forall X, y \in E$

$$Tr_{f/f_0} \left\{ Tr_{F/\mathcal{F}}(\lambda_5)(\theta Y^3 + Y^9) + Tr_{F/\mathcal{F}}(\lambda_3) X^3(\theta Y^3 + Y^9) + Tr_{F/\mathcal{F}}(\lambda_0)(\theta_10 XY^3 + \theta_6 X^3 Y^3) \right\} = 0.$$  

(7)
We shall use this later in the proof of the fifth compatibility condition.

Next our aim is to prove \( \Theta_1 = \Theta_5 \) which is the 4th condition. Now since \( F \) is a quadratic extension of \( f \) and \( E \) is the trace zero elements in \( F \), we have \( E = f \cdot \alpha \) for some \( \alpha \in F \) such that \( \alpha^2 \in f^* \setminus (f^*)^2 \). Let us expand (5) by putting \( X = x\alpha \), \( Y = y\alpha \) where \( x, y \in f \). We get

\[
0 = Tr_{f/f_0} \left[ \alpha^4 Tr(\lambda_4)Z\psi^2(\theta + y^6\alpha^6) + \alpha^2 Tr(\lambda_0)[Z(\psi^2 + \theta\psi y^2\alpha^2) + \alpha^4 Z^3(\theta + y^6\alpha^6)] \right] \tag{8}
\]

where we have put \( Z = xy \) and the traces inside are \( Tr_{f/f} \).

Replacing \( y \) by \( y + 1 \) and subtracting, we get

\[
0 = Tr_{f/f_0} \left[ \alpha^4 Tr(\lambda_4)Z[(1 - y)\theta + \alpha^6((y + 1)^8 - y^8)] + \alpha^2 Tr(\lambda_0)(Z\theta\alpha^2(1 - y) + \alpha^{10} Z^3(1 - y^3)) \right].
\]

Again, putting \( y + 1 \) for \( y \) and subtracting,

\[
0 = Tr_{f/f_0} \left[ \alpha^4 Tr(\lambda_4)Z(-\theta - \alpha^6(1 + y^2 + y^4 + y^6)) + \alpha^2 Tr(\lambda_0)(-Z\theta\alpha^2 - Z^3\alpha^{10}) \right].
\]

Assuming that \( \# f \) is large enough so that \( \exists y_1, y_2 \) s.t. \( 1 + y_1^2 + y_1^4 + y_1^6 \neq 1 + y_2^2 + y_2^4 + y_2^6 \), we will have

\[
0 = Tr_{f/f_0} \left[ \alpha^{10} Tr(\lambda_4)Z(y_1^2 + y_1^4 + y_1^6 - y_2^2 - y_2^4 - y_2^6) \right] \quad \forall \ Z \in f
\]

and so

\[
Tr_{f/F}(\lambda_4) = 0. \tag{9}
\]

The equation (8) becomes

\[
0 = Tr_{f/f_0} \left[ \alpha^2 Tr(\lambda_0)(Z(\psi^2 + \theta\psi y^2\alpha^2) + \alpha^4 Z^3(\theta + y^6\alpha^6)) \right].
\]

Putting \( y + 1 \) for \( y \) and subtracting,

\[
0 = Tr_{f/f_0} \left[ \alpha^2 Tr(\lambda_0)(Z\theta\alpha^2(1 - y) + Z^3\alpha^{10}(1 - y^3)) \right].
\]

Putting \(-y \) for \( y \) and adding

\[
0 = Tr_{f/f_0} \left[ \alpha^2 Tr(\lambda_0)(Z\theta\alpha^2 + Z^3\alpha^{10}) \right].
\]

This makes the previous equation

\[
0 = Tr_{f/f_0} \left[ \alpha^2 Tr(\lambda_0)(Z\theta\alpha^2 y + Z^3\alpha^{10}y^3) \right].
\]

Replace \( y \) by \( y^2 \) and substitute in (8) to get

\[
0 = Tr_{f/f_0} \left[ \alpha^2 Tr(\lambda_0)(Z\psi^2 + Z^3\alpha^4\psi) \right] \quad \forall \ Z \in f.
\]

Adding the last 2 equations

\[
0 = Tr_{f/f_0} \left[ \alpha^2 Tr(\lambda_0)(Z\psi^2 + Z\theta\alpha^2 y + Z^3\alpha^4\psi + Z^3\alpha^{10}y^3) \right].
\]
Central Extensions of a $p$-adic Division Algebra

Write $xZ$ in place of $Z$ and then write $y = Z^2\alpha^{-2}$ to get

$$0 = Tr_{f/k}[\alpha^2 Tr(\lambda_0)[x(\theta^2 Z + \theta_8 Z^3) + x^3(Z^3\alpha^4\theta + Z^9\alpha^4)]] \forall x, Z.$$

For each $Z \in f$, $x \mapsto x(\theta^2 Z + \theta_8 Z^3) + x^3(Z^3\alpha^4\theta + Z^9\alpha^4)$ is an endomorphism of $f$. Therefore, $\forall Z \in f$, $\exists$ a corresponding $x \neq 0$ for which it vanishes. Take $Z$ such that $\theta Z^3 + Z^9 = 0$ which is possible by the validity of the 3rd compatibility condition. The corresponding $x$ satisfies

$$x(\theta^2 Z + \theta_8 Z^3) = 0 \text{ i.e. } \theta^2 Z + \theta_8 Z^3 = 0.$$

The 2 equations $\theta Z^3 + Z^9 = 0$ and $\theta^2 Z + \theta_8 Z^3 = 0$ give $\theta_3^2 = \theta^5$ which is the 4th compatibility condition.

Before proving the 5th condition, note that any other uniformising parameter of $K$ is $\pi^2$ times some unit $u$ of $K$ and so, a uniformising parameter of $k$ is $\pi^2$ times the norm of $u$. But, Norm($u$) runs over all units of $k$ as $K$ is unramified. An easy computation shows that, by changing $\pi^2$, we may assume that $\theta_6 = \theta_{12} = 0$. The 6th compatibility condition then just reduces to the 3rd one. Also, the fifth condition becomes then $\theta_{10} = 0$.

To prove this holds, we start with (7) and proceed exactly as we did with (5). We get quite easily that $Tr_{F/f}(\lambda_5) = 0$ and $Tr_{F/f}(\lambda_3) = -\frac{\theta_6}{\theta} Tr_{F/f}(\lambda_0) = 0$. Therefore, (7) reduces to

$$0 = Tr_{f/k}(Tr(\lambda_0)(\theta_{10}XY^3)) \forall X, Y \in E$$

which easily gives $\theta_{10} = 0$, the 5th condition.

Hence, we have proved:

**Theorem.** Let $k$ be an extension of $\mathbb{Q}_3$ whose ramification index is 6. Also, assume that the residue field $f$ of $k$ is so large that there exist $a, b \in f$ such that $(1 + a^2)(1 + a^3) \neq (1 + b^2)(1 + b^4)$. Let $D$ be the quaternion division algebra over $k$. Suppose $H^2(SL(1, D), \mathbb{R}/\mathbb{Z})$ has an element of order 9. Then $k$ contains a primitive 9-th root of 1.

**References**


