<u>Summer Internship (2016)</u>

<u>Report On</u>

<u>Linear Programming and Game Theory</u>

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Preface

This report documents the work done during the summer internship in Indian Statistical Institute,Bangalore, India under the guidance and supervision of Prof. Dr. B.Sury.The report shall give overview of the tasks completed during the period of internship with technical details.

I have tried my best to keep eport simple yet technically correct.I hope I succeed in my Attempt.

Banashree Sarma

Supervisor Student

Acknowledgments

Simply put, I could not have done this work without the lots of help I received from ISI.The work culture and the environment in ISI really motivates.Everybody is such a friendly and cheerful companion here that work stress is never comes in way.

I would specially like to thank my guide and supervisor Prof.B.Sury for providing the nice ideas to work upon.Not only did he advised about my project but also the brief discussions on the topics have evoked a good interest .I am highly indebted to my supervisor ,who seemed to have solutions to all my problems.

Abstract

The report presents the topics studied during summer internship at ISI,Bangalore which are listed below:

- Linear Programming:
  - The Graphical analysis of linearprogramming
  - The Simplex Method
  - The Big M method
  - Concept of Duality
- The Game Theory

Linear Programming:

Introduction:

- ***Mathematical programming*** is used to find the best or optimal solution to a problem that requires a decision or set of decisions about how best to use a set of limited resources to achieve a state goal of objectives.
- ***Steps involved in mathematical programming***
    - Conversion of stated problem into a mathematical model that abstracts all the essential elements of the problem.
    - Exploration of different solutions of the problem.
    - Finding out the most suitable or optimum solution
- ***A Linear Programming*** model seeks to maximize or minimize a linear function, subject to a set of linear constraints. Linear programming requires that all the mathematical functions in the model be linear functions
- ***The linear model consists of the following components***:
    - A set of decision variables.
    - An objective function.
    - A set of constraints.
- ***The Importance of Linear Programming :***
    - Many real world problems lend themselves to linear programming modeling.
    - Many real world problems can be approximated by linear models.
    - There are well-known successful applications in:
        - Manufacturing
        - Marketing
        - Finance (investment)
        - Advertising
        - Agriculture
- There are efficient solution techniques that solve linear programming models.
- The output generated from linear programming packages provides useful "what if" analysis.
- ***Assumptions of the linear programming model:***
    - The parameter values are known with ***certainty.***
    - The objective function and constraints exhibit ***constant returns to scale.***

- There are ***no interactions*** between the decision variables (the additivity assumption).
- The ***Continuity*** assumption: Variables can take on any value within a given feasible range.

The Linear Programming Model:

Let: $x_1, x_2, x_3 \dots \dots \dots .. x_n$= decision variables

Z = Objective function or linear function

Requirement: Maximization of the linear function Z.

$$Z = c_1x_1 + c_2x_2 + c_3x_3 + \cdots \dots \dots \dots + c_nx_n \dots\text{..Eq (1)}$$

subject to the following constraints:

$$a_{11}x_1 + a_{12}x_2 \dots \dots \dots \dots . + a_{1n}x_n \le b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots \dots \dots \dots \dots a_{2n}x_n \le b_2$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots \dots \dots \dots \dots . a_{nn}x_n \le b_n$$

$$all\ x_j \ge 0\ and\ a_{ij}, b_i, c_j\ are\ all\ constants$$

The linear programming model can be written in more efficient notation as:

$$Maximize: \qquad Z = \sum_{j=1}^{n} c_jx_j$$

$$subject\ to\ cnstraints: \qquad \sum_{j=1}^{n} a_{ij}x_j \le b_i \quad where\ \ i = 1,2,3, \dots \dots \dots \dots \dots ., n$$

$$and\ x_j \ge 0 \quad where\ j = 1,2,3, \dots \dots \dots \dots \dots \dots, n$$

The decision variables, $x_1, x_2, x_3 \dots \dots x_n$, represent levels of n competing activities

Now, every linear program can be converted into ***"standard" form.***

Standard form:

$$\text{Max} \qquad Z = c_1x_1 + c_2x_2 + c_3x_3 + \cdots \dots \dots \dots + c_nx_n$$

subject to the following constraints:

$$a_{11}x_1 + a_{12}x_2 \dots\dots\dots\dots. + a_{1n}x_n + s_1 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots \dots\dots\dots\dots a_{2n}x_n + s_2 = b_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots \dots\dots\dots\dots. a_{nn}x_n + s_n = b_n$$

$$all\ x_j \geq 0\ and\ a_{ij}, b_i, c_j\ are\ all\ constants$$

where the objective is maximized, the constraints are equalities and the variables are all nonnegative.

<u>This is done as follows:</u>

- If the problem is min z, convert it to max - z.
- If a constraint is $a_{i1}x_1 + a_{i2}x_2 \dots\dots\dots\dots. + a_{in}x_n \leq b_i$ convert it into an equality constraint by adding a nonnegative slack variable s_i . The resulting constraint is $a_{i1}x_1 + a_{i2}x_2 \dots\dots\dots\dots. + a_{in}x_n + s_i = b_i$ where $s_i \geq 0$
- If a constraint is $a_{i1}x_1 + a_{i2}x_2 \dots\dots\dots\dots. + a_{in}x_n \geq b_i$ convert it into an equality constraint by subtracting a nonnegative surplus variable s_i . The resulting constraint is $a_{i1}x_1 + a_{i2}x_2 \dots\dots\dots\dots. + a_{in}x_n - s_i = b_i$ where $s_i \geq 0$
- If some variable x_i is unrestricted in sign,replace it everywhere in the formulation by $x_i^{'} - x_i^{''}$ where $x_i^{'}, x_i^{''} \geq 0$.

Examples of LP Problems:

1. <u>A Product Mix Problem</u>

   - A manufacturer has fixed amounts of different resources such as raw material, labor, and equipment.
   - These resources can be combined to produce any one of several different products.
   - The quantity of the i^{th} resource required to produce one unit of the j^{th} product is known.

   The decision maker wishes to produce the combination of products that will maximize total income.
2. <u>A Blending Problem</u>
   - Blending problems refer to situations in which a number of components (or commodities) are mixed together to yield one or more products.

- Typically, different commodities are to be purchased. Each commodity has known characteristics and costs.

The problem is to determine how much of each commodity should be purchased and blended with the rest so that the characteristics of the mixture lie within specified bounds and the total cost is minimized

3. A Production Scheduling Problem
   - A manufacturer knows that he must supply a given number of items of a certain product each month for the next *n* months.
   - They can be produced either in regular time, subject to a maximum each month, or in overtime. The cost of producing an item during overtime is greater than during regular time. A storage cost is associated with each item not sold at the end of the month.

The problem is to determine the production schedule that minimizes the sum of production and storage costs.

Developing an LP model:

The variety of situations to which linear programming has been applied ranges from agriculture to zinc smelting.

Steps Involved:

- Determine the objective of the problem and describe it by a criterion function in terms of the decision variables.
- Find out the constraints.
- Do the analysis which should lead to the selection of values for the decision variables that optimize the criterion function while satisfying all the constraints imposed on the problem.

Example:

The Galaxy Industries Production Problem –

- Galaxy manufactures two toy doll models:
  - Space Ray.
  - Zapper.
- Resources are limited to:
  - 1000 pounds of special plastic.
  - 40 hours of production time per week
- Marketing requirement
  - Total production cannot exceed 700 dozens.

- Number of dozens of Space Rays cannot exceed number of dozens of Zappers by more than 350.
- Technological input
  - Space Rays requires 2 pounds of plastic and 3 minutes of labor per dozen.
  - Zappers requires 1 pound of plastic and 4 minutes of labor per dozen
- The current production plan calls for:
  - Producing as much as possible of the more profitable product, Space Ray ($8 profit per dozen).
  - Use resources left over to produce Zappers ($5 profit per dozen), while remaining within the marketing guidelines.
- The current production plan consists of:
  - Space Rays = 450 dozen
  - Zapper = 100 dozen
  - Profit = $4100 per week ($8 \times (450) + 5 \times (1000)$)

Aim-Management is seeking a production schedule that will increase the company's profit.

(A linear programming model can provide an insight and an intelligent solution to this problem.)

The Galaxy Linear Programming Model:

- Decisions variables:
  - x_1 = Weekly production level of Space Rays (in dozens)
  - x_2 = Weekly production level of Zappers (in dozens).
- Objective Function:
  - Weekly profit, to be maximized

The LP model:

Max $8x_1 + 5x_2$ (Weekly profit)

subject to:

$2x_1 + 1x_2 \leq 1000$ (Plastic)

$3x_1 + 4x_2 \leq 2400$ (Production Time)

$x_1 + x_2 \leq 700$ (Total production)

$x_1 - x_2 \leq 350$ (Mix)

$x_1, x_2 \geq 0$

Solution methods of Linear Programming Model:

Linear programming problems can be solved using graphical techniques, **SIMPLEX** algorithms using matrices, or using software, such as ForeProfit software.

The Graphical Analysis of Linear Programming:

- The set of all points that satisfy all the constraints of the model is called a feasible region.
- If there are no values of the decision variables which satisfies all the constraints then that LP model is said to have infeasible solution.
- Multiple optimal solutions or infinite optimal solutions is found when one of the lines making the boundary of the feasible solution region runs parallel to the objective function line .So,all the points on the line making the boundary of the feasible solution region become the optimal solution.
- Reduntant constraint: Every constraint in a LP model form a unique boundary of the feasible region but the constraint which do not contribute to the boundary of the feasible solution region is known as redundant constraint**.**
- When a LP solutions is permitted to be infinitely large it is known as to be unbounded.

A Graphical Solution Procedure (LPs with 2 decision variables can be solved/viewed this way) –

1. Identify the decision variables .eg x_1, x_2
2. Set up the equation of the objective function.Now the objective function could be either minimization or maximization.
3. Set up the equation of the constraints
4. Graph the constraints by reducing them into equality
5. Find the feasible region.

6. Find the optimal solution.For this ,we have two methods:

   Corner point method

   This method is based on the fact that the optimal solution lies on one of the corner points of the feasible solution region.

   Following are the steps to be followed:

   1. Identify the coordinates of the corner points.
   2. Solve the objective with each of the corner point value.
   3. Coordinates giving the highest value of the objective function in case of maximization and lowest value of the objective function in case of the minimization gives the optimal solution.

   Iso profit/iso cost method:

   1. Find the slope of the objective function line and draw a family of parallel lines.
   2. In this method there are two cases:

      For ***minimization objective*** ,the parallel lines are called ***iso cost lines.***Now every point on each of these iso cost lines will yield the same cost.Each line will yield different cost but every point on the same line will yield the cost.So,the nearest point from the origin on the feasible solution region which the iso cost line touches is the point of optimal solution.

      For ***maximization objective*** ,the parallel lines are called ***iso profit lines.***Now every point on each of these iso profit lines will yield the same cost.Each line will yield different profit but every point on the same line will yield the profit.So,the furthest point from the origin on the feasible solution region which the iso profit line touches is the point of optimal solution.

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A Minimization Problem:

- **LP Formulation:**

$$Min\ \ Z = 5x_1 + 2x_2$$

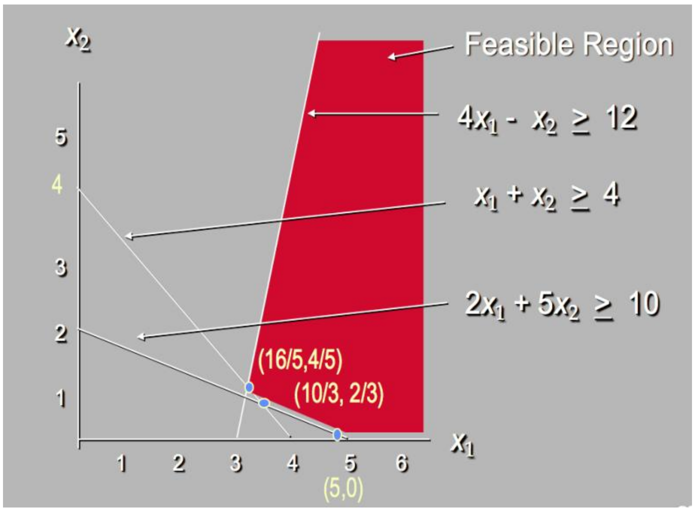
$$subject\ to\ constraints:\quad 2x_1+5x_2 \geq 10$$

$$4x_1 - x_2 \geq 12$$

$$x_1 + x_2 \geq 4$$

$$x_1, x_2 \geq 0$$

- **Graph the Constraints**
  - Constraint 1: When $x_1 = 0$, then $x_2 = 2$; when $x_2 = 0$, then $x_1 = 5$. Connect (5,0) and (0,2). The ">" side is above this line.
  - Constraint 2: When $x_2 = 0$, then $x_1 = 3$. But setting x_1 to 0 will yield $x_2 = -12$, which is not on the graph. Thus, to get a second point on this line, set x_1 to any number larger than 3 and solve for x_2: when $x_1 = 5$, then $x_2 = 8$. Connect (3,0) and (5,8). The ">" side is to the right.
  - Constraint 3: When $x_1 = 0$, then $x_2 = 4$; when $x_2 = 0$, then $x_1 = 4$. Connect (4,0) and (0,4). The ">" side is above this line.
- **Constraints Graphed**



- Solve for the Extreme Point at the Intersection of the second and third Constraints

$$4x_1 - x_2 = 12$$
$$x_1 + x_2 = 4$$

Adding these two equations gives:

$$5x_1 = 16 \; or x_1 = 16/5$$

Substituting this into $x_1 + x_2 = 4$ gives $x_2 = 4/5$

- Solve for the extreme point at the intersection of the first and third constraints

$$2x_1 + 5x_2 = 10$$
$$x_1 + x_2 = 4$$

Multiply the second equation by -2 and add to the first equation, gives

$$3x_2 = 2 \text{ or } x_2 = 2/3$$

Substituting this in the second equation gives $x_1 = 10/3$

point	Z
(16/5, 4/5	88/5
(10/3, 2/3)	18
(5, 0)	25

Maximization problem:

$$Max\ \ Z = 5x_1 + 7x_2$$

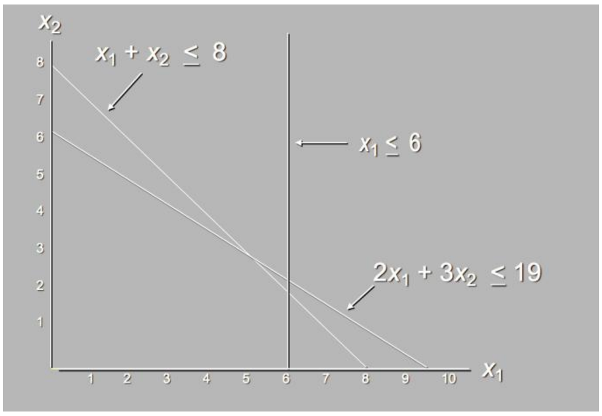
$$subject\ to\ constraints:\ \ x_1 \leq 6 \ldots\ldots\ldots..(1)$$

$$2x_1\ +\ 3x_2 \leq 19 \ldots\ldots.(2)$$

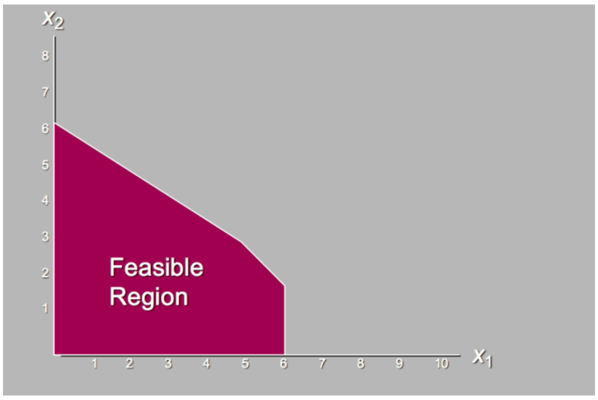
$$x_1 + x_2 \leq 8\ \ldots\ldots\ldots(3)$$

$$x_1, x_2 \geq 0$$

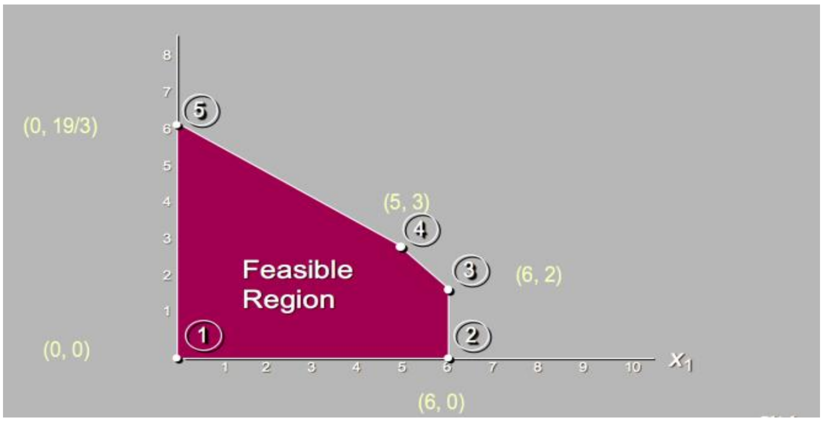
- Combined constraint graph:



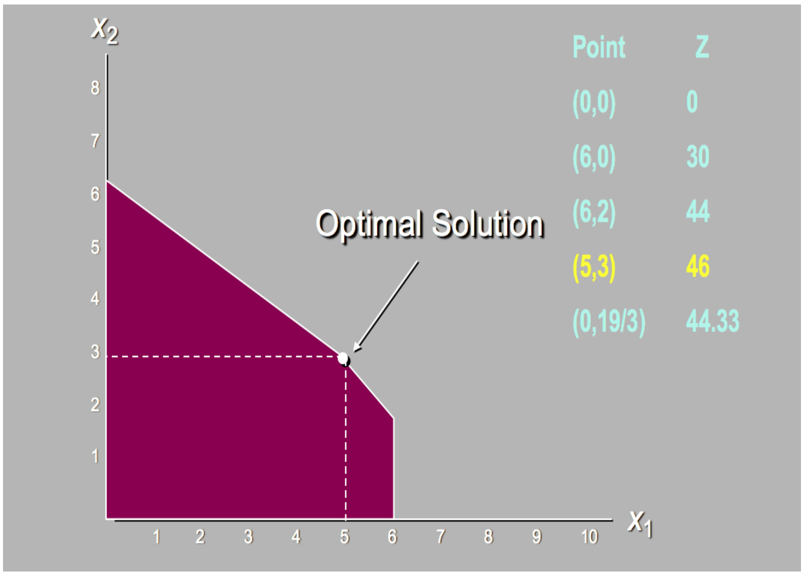
- Feasible Solution Region:



- The five extreme points:



- Having identified the feasible region for the problem, we now search for the optimal solution, which will be the point in the feasible region with the largest (in case of maximization or the smallest (in case of minimization) of the objective function.
- To find this optimal solution, we need to evaluate the objective function at each one of the corner points of the feasible region
- Optimal Solution:



Extreme Points and the Optimal Solution:

- The corners or vertices of the feasible region are referred to as the extreme points.
- An optimal solution to an LP problem can be found at an extreme point of the feasible region.
- When looking for the optimal solution, you do not have to evaluate all feasible solution points.
- We have to consider only the extreme points of the feasible region.

Feasible Region:

- The feasible region for a two-variable linear programming problem can be nonexistent, a single point, a line, a polygon, or an unbounded area.
- Any linear program falls in one of three categories:

  • is infeasible

  • has a unique optimal solution or alternate optimal solutions

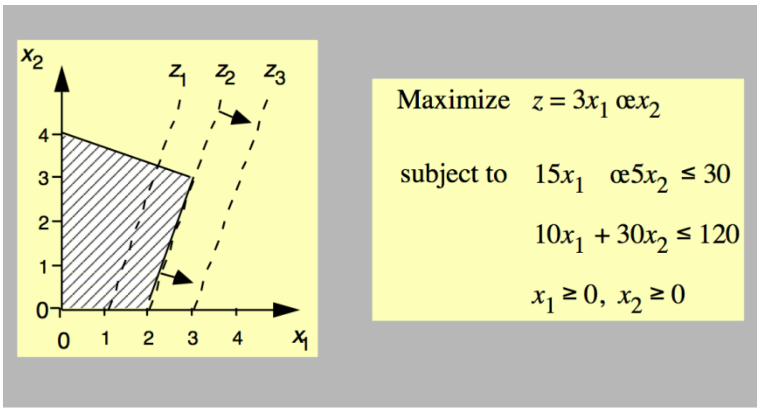
  • has an objective function that can be increased without bound

- A feasible region may be unbounded and yet there may be optimal solutions. This is common in minimization problems and is possible in maximization problems.

Special Cases:

- **Alternative Optimal solutions:**
  In the graphical method, if the objective function line is parallel to a boundary constraint in the direction of optimization, there are
  <u>alternate optimal solutions</u>, with all points on this line segment being optimal.
- **Infeasibility:**
  A linear program which is overconstrained so that no point satisfies all the constraints is said to be <u>infeasible</u>.
- **Unbounded:**
  For a max (min) problem, an unbounded LP occurs if it is possible to find points in the feasible region with arbitrarily large (small) Z.

Example with Multiple Optimal Solutions:



Example of infeasible solution:

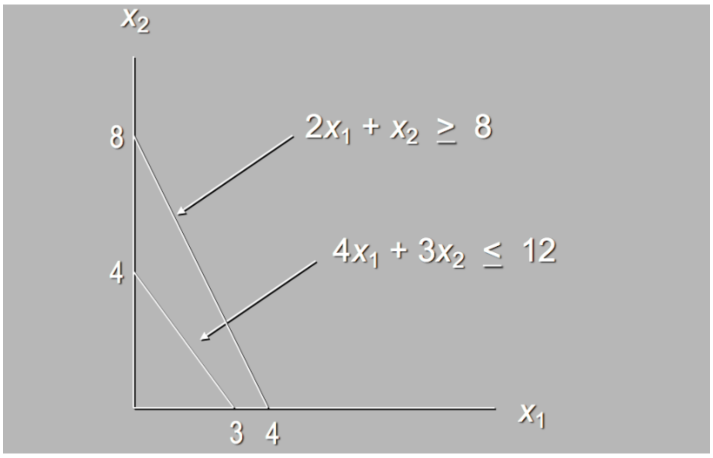
$$\text{Max} \quad Z = 2x_1 + 6x_2$$

$$\text{s.t} \quad 4x_1 + 3x_2 \le 12$$

$$2x_1 + x_2 \ge 8$$

$$x_1, x_2 \ge 0$$

There are no points that satisfy both constraints, hence this problem has no feasible region, and no optimal solution.



Example of Unbounded Solution:

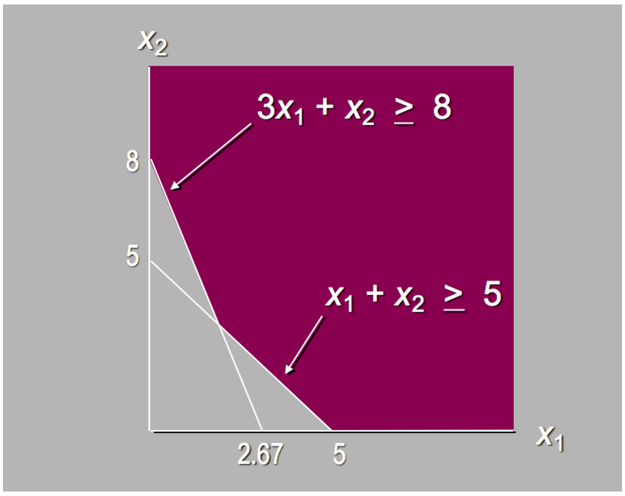
$$\text{Max} \quad Z = 3x_1 + 4x_2$$

$$\text{s.t} \quad x_1 + x_2 \ge 5$$

$$3x_1 + x_2 \ge 8$$

$$x_1, x_2 \ge 0$$

The feasible region is unbounded and the objective function line can be moved parallel to itself without bound so that z can be increased infinitely.



Note: The graphical method of solution may be extended to a case in which there are three variables. In this case, each constraint is represented by a plane in three dimensions, and the feasible region bounded by these planes is a polyhedron

However,if decision variables are more than 2, it is always advisable to use **Simplex Method** to avoid lengthy graphical procedure.The simplex method is not used to examine all the feasible solutions. It deals only with a small and unique set of feasible solutions, the set of vertex points (i.e., extreme points) of the convex feasible space that contains the optimal solution.

The Simplex Method:

Theory:

A linear-programming algorithm that can solve problems having more than two decision variables.

The simplex technique involves generating a series of solutions in tabular form, called **tableaus**. By inspecting the bottom row of each tableau, one can immediately tell if it represents the optimal solution. Each tableau corresponds to a corner point of the feasible solution space. The first tableau corresponds to the origin. Subsequent tableaus are developed by shifting to an adjacent corner point in the direction that yields the highest (smallest) rate of profit (cost). This process continues as long as a positive (negative) rate of profit (cost) exists.

Teminology:

Constraint Boundary Equation:

an equation obtained by replacing its sign (≥,≤ or =) by an equality sign (=).

Boundary: the boundary of the feasible region contains (1) the feasible solutions that satisfy one or more of the constraint boundary equations.

CPF:is a feasible solution that exists at the constraints of more than one constraint boundary equations (i.e., it does not lie on any line segment connecting two other feasible solutions)

For any linear programming problem with n decision variables, each CPF solution lies at the intersection of n constraint boundaries.

Adjacent CPF solutions: A CPF solution lies at the intersection of n constraint boundaries. An edge of the feasible region is feasible line segment that lies at the intersection of n-1 constraint boundaries. Two CPF solutions are adjacent if the line segment connecting them is an edge of the feasible region. Each CPF solution has n edges each one leading to one of the n adjacent CPF solutions.

The key solution concepts

- the simplex method focuses on CPF solutions.
- the simplex method is an iterative algorithm (a systematic solution procedure that keeps repeating a fixed series of steps, called, an iteration, until a desired result has been obtained) .
- whenever possible, the initialization of the simplex method chooses the origin point (all decision variables equal zero) to be the initial CPF solution.
- given a CPF solution, it is much quicker computationally to gather information about its adjacent CPF solutions than about other CPF solutions. Therefore, each time the simplex method performs an iteration to move from the current CPF solution to a better one, it always chooses a CPF solution that is adjacent to the current one.
- After the current CPF solution is identified, the simplex method examines each of the edges of the feasible region that emanate from this CPF solution. Each of these edges leads to an adjacent CPF solution at the other end, but the simplex method doesn't even take the time to solve for the adjacent CPF solution. Instead it simply identifies the rate of improvement in Z that would be obtained by

moving along the edge. And then chooses to move along the one with largest positive rate of improvement.

- A positive rate of improvement in Z implies that the adjacent CPF solution is better than the current one, whereas a negative rate of improvement in Z implies that the adjacent CPF solution is worse. Therefore, the optimality test consists simply of checking whether any of the edges give a positive rate of improvement in Z. if none do, then the current CPF solution is optimal.

The following are steps for simplex method:

1. **Initialization:**

    - transform all the constraints to equality by introducing slack, surplus, and artificial variables as follows:

Constraint type	Variable to be added
≤	+ slack (s)
≥	- Surplus (s) + artificial (A)
=	+ Artificial (A)

    - Construct the initial simplex tableau:

		C_j	Objective function coefficients			
B Basic variables	C_b Coefficients of the basic variables in the objective function	X_b RHS of the constraints	$x_1 x_2 x_3 \dots \dots x_n$	$s_1 s_2 s_3 \dots \dots s_n$	$A_1 A_2 A_3 \dots \dots A_n$	Min ration $\frac{X_b}{x_i}$
			Coeffiecients of the constraints			
Z		Z_j				
		$C_j - Z_j$				

- Optimality:
  - Case 1: Maximization problem
    the current BF solution is optimal if every coefficient in the objective function row is nonnegative
  - Case 2: Minimization problem
    the current BF solution is optimal if every coefficient in the objective function row is nonpositivity
- Iteration
  1. Step 1: determine the entering basic variable by selecting the variable (automatically a nonbasic variable) with the most negative value (in case of maximization) or with the most positive (in case of minimization) in the last row (Z-row). Put a box around the column below this variable, and call it the "pivot column
  2. Step 2: Determine the leaving basic variable by applying the minimum ratio test as following:
     - Pick out each coefficient in the pivot column that is strictly positive (>0)
     - Divide each of these coefficients into the right hand side entry for the same row
     - Identify the row that has the smallest of these ratios
     - The basic variable for that row is the leaving variable, so replace that variable by the entering variable in the basic variable column of the next simplex tableau. Put a box around this row and call it the "pivot row"
  3. Step 3: Solve for the new BF solution by using elementary row operations (multiply or divide a row by a nonzero constant; add or subtract a multiple of one row to another row) to construct a new simplex tableau, and then return to the optimality test. The specific elementary row operations are:
     - Divide the pivot row by the "pivot number" (the number in the intersection of the pivot row and pivot column)
     - For each other row that has a negative coefficient in the pivot column, add to this row the product of the absolute value of this coefficient and the new pivot row.

- For each other row that has a positive coefficient in the pivot column, subtract from this row the product of the absolute value of this coefficient and the new pivot row

Example: **Product Mix Problem**

The N. Dustrious Company produces two products: I and II. The raw material requirements, space needed for storage, production rates, and selling prices for these products are given below:

	product	
	I	II
Storage space($ft^2/unit$)	4	5
Raw material(lb/unit)	5	3
Production rate(units/hr)	60	30
Selling price($/unit)	13	11

The total amount of raw material available per day for both products is 15751b. The total storage space for all products is 1500 ft^2, and a maximum of 7 hours per day can be used for production. The company wants to determine how many units of each product to produce per day to maximize its total income.

Solution

- **Step 1:** Convert all the inequality constraints into equalities by the slack variables. Let:

$$S_1 = unused\ storage\ capacity$$

$$S_2 = unused\ raw\ material$$

$$S_3 = unused\ production\ time$$

As already developed, the LP model is:

$$Maximize\ Z = 13x_1 + 11x_2$$

Subject to: $4x_1 + 5x_2 \leq 1500$

$$5x_1 + 3x_2 \leq 1575$$

$$x_1 + 2x_2 \leq 420$$

$$x_1, x_2 \geq 0$$

Introducing these slack variables into the inequality constraints and rewriting the objective function such that all variables are on the left-hand side of the equation. Equation 4 can be expressed as:

:

$$Z - 13x_1 - 11x_2 = 0 \text{.......................(A1)}$$

$$4x_1 + 5x_2 + S_2 = 150 \ldots\ldots\ldots\ldots\ldots\ldots..(B1)$$

$$5x_1 + 3x_2 + S_2 = 1575 \text{..................(C1)}$$

$$x_1 + x_2 + S_3 = 420 \text{..................(D1)}$$

$$x_1, x_2, S_1, S_2 \geq 0$$

From the equations above ,it is obvious that one feasible solution that satisfies all the constraints is :$x_1 = 0, x_2 = 0, S_1 = 1500, S_2 = 1575, S_3 = 420 \; and \; Z = 0$

- ❖ Since the coefficients of x_1 and x_2 in Eq. (A1) are both negative, the value of Z can be increased by giving either x_1 or x_2 some positive value in the solution.
- ❖ In Eq. (B1), if $x_2 = S_1 = 0$, then $x_1 = 1500/4 = 375$. That is, there is only sufficient storage space to produce 375 units at product I.
- ❖ From Eq. (C1), there is only sufficient raw materials to produce 1575/5 = 315 units of product I.
- ❖ From Eq. (D1), there is only sufficient time to produce 420/1 = 420 units of product I.
- ❖ Therefore, considering all three constraints, there is sufficient resource to produce only 315 units of x_1. Thus the maximum value of x_1 is limited by Eq. (C1).
- ❖ **Step 2:** From Equation CI, which limits the maximum value of x_1.

$$x_1 = -\frac{3}{5}x_2 - \frac{1}{5}S_2 + 315$$

Substituting this equation into Eq. (5) yields the following new formulation of the model.

$$Z - \frac{16}{5}x_2 + \frac{13}{5}S_2 = 4095 \text{......................}(A2)$$

$$+\frac{13}{5}x_2 + S_1 - \frac{4}{5}S_2 = 240 \text{...........................}(B2)$$

$$x_1 + \frac{3}{5}x_2 + \frac{1}{5}S_2 = 315 \text{............................}(C2)$$

$$\frac{7}{5}x_2 - \frac{1}{5}S_2 + S_3 = 105 \text{..............}(D2)$$

- ❖ It is now obvious from these equations that the new feasible solution is:

$$x_1 = 315,\ x_2 = 0,\ S_1 = 240,\ S_2 = 0,\ S_3 = 105, \text{ and } Z = 4095$$

- ❖ It is also obvious from Eq.(A2) that it is also not the optimum solution. The coefficient of x_1 in the objective function represented by A2 is negative (-16/5), which means that the value of Z can be further increased by giving x_2 some positive value.
- ❖ Following the same analysis procedure used in step 1, it is clear that:
- ❖ In Eq. (B2), if $S_1 = S_1 = 0$, then $x_2 = (5/13)(240) = 92.3$.
- ❖ From Eq. (C2), x_2 can take on the value (5/3)(315) = 525 if $x_1 = S_2 = 0$
- ❖ From Eq. (D2), x_2 can take on the value (5/7)(105) = 75 if $S_2 = S_3 = 0$
- ❖ Therefore, constraint D_2 limits the maximum value of x_2 to 75. Thus a new feasible solution includes $x_2 = 75$, $S_2 = S_3 = 0$.
- ❖ **Step 3:** From Equation D2:

$$x_2 = \frac{1}{7}S_2 - \frac{5}{7}S_3 + 75$$

Substituting this equation into Eq. (7) yield:

$$Z + \frac{15}{7}S_2 + \frac{16}{7}S_3 = 4335 \text{..............}(A3)$$

$$S_1 - \frac{3}{7}S_2 - \frac{13}{7}S_3 = 45 \text{..............}(B3)$$

$$x_1 + \frac{2}{7}S_2 - \frac{3}{7}S_3 = 270 \text{..............}(C3)$$

$$x_2 - \frac{1}{7}S_2 + \frac{5}{7}S_3 = 75 \ldots\ldots\ldots\ldots\ldots (D3)$$

From these equations, the new feasible solution is readily found to be: x_1 = 270, x_2 = 75, S_1 = 45, S_2 = 0, S_3 = 0, Z = 4335.

- Because the coefficients in the objective function represented by Eq. (A3) are all positive, this new solution is also the optimum solution.

Now using simplex tableau:

- **Step I:** Set up the initial tableau using Eq. (5).

:

$$Z - 13x_1 - 11x_2 = 0$$

$$4x_1 + 5x_2 + S_2 = 1500$$

$$5x_1 + 3x_2 + S_2 = 1575$$

$$x_1 + x_2 + S_3 = 420$$

$$x_1, x_2, S_1, S_2 \geq 0$$

		C_j	13	11	0	0	0	
B	C_b	X_b	x_1	x_2	S_1	S_2	S_3	Min ratio
S_1	0	1500	4	5	1	0	0	375
S_2	0	1575	(5)	3	0	1	0	315
S_3	0	420	1	2	0	0	1	420
Z=0		Z_j	0	0				
		$C_j - Z_j$	13	11	0	0	0	

Entering=x_1,departing =S_1,key element=5

		C_j	13	11	0	0	0	
B	C_b	X_b	x_1	x_2	S_1	S_2	S_3	Min ratio
S_1	0	240	0	13/5	1	-4/5	0	1200/13
x_1	0	315	1	3/5	0	1/5	0	525
S_3	0	105	0	(7/5)	0	-1/5	1	95
Z=4095		Z_j	13	39/5	0	-13/5	0	
		$C_j - Z_j$	0	16/5	0	-13/5	0	

Entering=x_2,departing =S_3,key element=7/5

		C_j	13	11	0	0	0	
B	C_b	X_b	x_1	x_2	S_1	S_2	S_3	Min ratio
S_1	0	45	0		1	-3/7	-13/7	
x_1	0	270	1	0	0	2/7	-3/7	
x_2	0	75	0	1	0	-1/7	5/7	
Z=4335		Z_j	13	11	0	15/7	16/7	
		$C_j - Z_j$	0	0	0	-15/7	-16/7	

Since all $C_j - Z_j \leq 0$

Optimum solution is arrived with value of variables as:

$$x_1 = 270$$

$$x_2 = 75$$

Maximize Z=4335

Special Cases

Degeneracy in Linear programming:

- occurs whenever there is a tie for departing variable
- at next iteration, entering variable will be constrained to enter at value zero
- simplex algorithm will move to a new basic feasible solution, but it's geometrically the same point, and the objective doesn't change.

Example: max $Z = x_1 + x_2 + x_3$

Subject to constraints:

$$x_1 + x_2 \leq 1$$
$$-x_2 + x_3 \leq 0$$
$$x_1, x_2, x_3 \geq 0$$

		C_j	13	11		0	0	
B	C_b	X_b	x_1	x_2	x_3	S_1	S_2	Min ratio
S_1	0	1	1	1	0	1	0	1/1=1
S_2	0	0	0	-1	1	0	1	
Z=0		Z_j	0	0	0	0	0	
		$C_j - Z_j$	1	1	1	0	0	

For the first iteration of simplex method, there are 3 choices of entering variables: x_1, x_2, x_3. We choose x_1 . (In practice, the choice of entering variable is determined by the pivot rule used.) The leaving variable should be S_1 .

		C_j	1	1	1	0	0	
B	C_b	X_b	x_1	x_2	x_3	S_1	S_2	Min ratio
x_1	1	1	1	1	0	1	0	
S_2	0	0	0	-1	1	0	1	0/1=0
Z=1		Z_j	1	1		0	0	
		$C_j - Z_j$	0	0	1	-1	0	

Entering=x_3,departing =S_2,Key element=1

		C_j	1	1	1	0	0	
B	C_b	X_b	x_1	x_2	x_3	S_1	S_2	Min ratio
x_1	1	1	1	1	0	1	0	1/1=1
x_3	1	0	0	-1	1	0	1	
Z=1		Z_j	1	0	1	1	1	
		$C_j - Z_j$	0	1	0	-1	0	

Entering=x_2,departing =x_1,Key element=1

		C_j	1	1	1	0	0	
B	C_b	X_b	x_1	x_2	x_3	S_1	S_2	Min ratio
x_2	1	1	1	1	0	1	0	
x_3	1	1	1	0	1	1	1	
Z=2		Z_j	2	1	1	2	1	
		$C_j - Z_j$	-1	0	0	-2	-1	

Since all $C_j - Z_j \leq 0$

Optimum solution is arrived with value of variables as:

$$x_1 = 0$$
$$x_2 = 1$$
$$x_3 = 1$$

Maximize Z=2

The Finiteness of the Simplex Algorithm when there is no degeneracy:

the simplex algorithm tries to increase a non-basic variable x_j . If there is no degeneracy, then x_j will be positive after the pivot, and the objective value will improve. Each solution produced by the simplex algorithm is a basic feasible solution with m basic variables, where m is the number of constraints. There are a finite number of ways of choosing the basic variables. So, the simplex algorithm moves from bfs to bfs. And it never

repeats a bfs because the objective is constantly improving. This shows that the simplex method is finite, so long as there is no degeneracy.
Cycling:
If a sequence of pivots starting from some basic feasible solution ends up at the exact same basic feasible solution, then we refer to this as "cycling." If the simplex method cycles, it can cycle forever. Klee and Minty [1972] gave an example in which the simplex algorithm really does cycle.
Example:

$$\max Z = 100x_1 + 10x_2 + x_3$$

Subject to constraints:

$$x_1 \leq 1$$
$$20x_1 + x_2 \leq 100$$
$$200x_1 + 20x_2 + x_3 \leq 10000$$
$$x_1, x_2, x_3 \geq 0$$

		C_j	100	10	1	0	0	0	
B	C_b	X_b	x_1	x_2	x_3	S_1	S_2	S_3	Min ratio
S_1	0	1	1	0	0	1	0	0	1
S_2	0	100	20	1	0	0	1	0	5
S_3	0	10000	200	20	1	0	0	1	50
Z=0		Z_j	0	0	0	0	0	0	
		$C_j - Z_j$	100	10	1	0	0	0	

At first iteration the basic columns are x_1, S_2, S_3
At 2nd iteration the basic columns arex_1, x_2, S_3
At 3rd iteration the basic columns are x_1, S_1, S_3
At 4th iteration the basic columns are x_2, x_3, S_1
At 5th iteration the basic columns are x_1, x_2, x_3
At 6th iteration the basic columns are x_1, x_3, S_2
At 7th iteration the basic columns are x_3, S_1, S_2

$$(x_1, x_2, x_3, S_1, S_2, S_3) = (0,0,10^4,1,10^2,0)$$

is the optimal and that the objective function value is 10,000
Along the way there are 7 pivot steps.The objective function made a strict increase with each change of basis.

Is the simplex method finite?

There is a technique that prevents bases from repeating in the simplex method, even if they are degenerate bases. This will guarantee the finiteness of the simplex algorithm, provided that the technique is used. There are several approaches to guaranteeing that the simplex method will be finite, including one developed by Professors Magnanti and Orlin. And there is the perturbation technique that entirely avoids degeneracy. Bland's rule, developed by Bob Bland. It's the simplest rule to guarantee finiteness of the simplex method. Bland's rule ensures there is **no cycling**.

Bland’s Rule.

1.The entering variable should be the lowest index variable with positive reduced cost.

2.The leaving variable (in case of a tie in the min ratio test) should be the lowest index row. (It is the row closest to the top, regardless of the leaving variable.

Degeneracy is important because we want the simplex method to be finite, and the generic simplex method is not finite if bases are permitted to be degenerate. In principle, cycling can occur if there is degeneracy. In practice, cycling does not arise, but no one really knows why not. Perhaps it does occur, but people assume that the simplex algorithm is just taking too long for some other reason, and they never discover the cycling. Researchers have developed several different approaches to ensure the finiteness of the simplex method, even if the bases can be degenerate. Bob Bland developed a very simple rule that prevents cycling.

Alternative optimum solutions:

- when optimality is reached, one (or more) of the non-basic variables has coefficient zero in objective
- each one can enter into the set of basic variables, without changing the objective value.

Unbounded Solution:

- when ratio test is being used to determine constraints on entering variable,all ratios are either negative or infinity.
- the current entering variable is the one that can be made as large as desired.

Artificial Variable Technique
(The Big-M Method)
Introduction to the Big M Method:

In this section, we will present a generalized version of the simplex method that will solve both maximization and minimization problems with any combination of ≤, ≥, = constraints.

The Big-M method of handling instances with artificial variables is the“commonsense approach”. Essentially, the notion is to make the artificial variables, through their coefficients in the objective function, so costly or unprofitable that any feasible solution to the real problem would be preferred....unless the original instance possessed no feasible solutions at all. But this means that we need to assign, in theobjective function, coefficients to the artificial variables that are either very small (maximization problem) or very large(minimization problem); whatever this value,let us call it ***Big M.*** In fact, this notion is an old trick in optimization in general; we simply associate a penalty value with variables that we do not want to be part of an ultimate solution(unless such an outcome is unavoidable). Indeed, the penalty is so costly that unless any of the respective variables' inclusion is warranted algorithmically,such variables will never be part of any feasible

solution.
This method removes artificial variables from the basis. Here, we assign a large undesirable (unacceptable penalty) coefficients to artificial variables from the objective function point of view. If the objective function (Z) is to be minimized, then a very large positive price (penalty, M) is assigned to each artificial variable and if Z is to be minimized, then a very large negative price is to be assigned. The penalty will be designated by +M for minimization problem and by –M for a maximization problem and also M>0.

<u>Example</u>: Minimize $Z = 600X_1 + 500X_2$

subject to constraints,

$$2X_1 + X_2 \geq 80$$

$$X_1 + 2X_2 \geq 60$$

$$X_1, X_2 \geq 0$$

<u>Step1</u>: Convert the LP problem into a system of linear equations.
We do this by rewriting the constraint inequalities as equations by subtracting new "surplus & artificial variables" and assigning them ***zero & +M*** coefficientsrespectively in the objective function as shown below.
So the Objective Function would be:

$Z = 600X_1 + 500X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

subject to constraints,

$$2X_1 + X_2 - S_1 + A_1 = 80$$

$$X_1 + 2X_2 - S_2 + A_2 = 60$$

$$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$$

<u>Step 2</u>: Obtain a Basic Solution to the problem.
We do this by putting the decision variables

$$X_1 = X_2 = S_1 = S_2 = 0$$

so that.

$$A_1 = 80, A_2 = 60$$

These are the initial values of ***artificial variables.***

<u>Step 3</u>: Form the Initial Tableau as shown

Key element is coloured yellow

		C_j	600	500	0	0	M	M	
B	C_b	X_b	X_1	X_2	S_1	S_2	A_1	A_2	Min ratio
A_1	M	80	2	1	-1	0	1	0	80
A_2	M	60	1	2	0	-1	0	1	60
Z=0		Z_j	3M	3M	-M	-M	M	M	
		$C_j - Z_j$	600-3M	500-3M	M	M	0	0	

		C_j	600	500	0	0	M	
B	C_b	X_b	X_1	X_2	S_1	S_2	A_1	Min ratio
A_1	M	50	3/2	0	-1	1/2	1	100/3
X_2	500	30	1/2	1	0	-1/2	0	60
		Z_j	3M/2+250	500	-M	M/2-250	M	
		$C_j - Z_j$	350-3M/2	0	M	M	0	

		C_j	600	500	0	0	
B	C_b	X_b	X_1	X_2	S_1	S_2	Min ratio
X_1	600	100/3	1	0	-2/3	1/3	
X_2	500	40/3	0	1	1/3	-2/3	
Z=0		Z_j	600	500	-700/3	-400/3	
		$C_j - Z_j$	0	0	700/3	400/3	

Since all the values of **(Cj-Zj)** are either zero or positive and also both the artificial variables have been removed, an optimum solution has been arrived at with X_1=100/3 , X_2=40/3 and Z=80,000/3.

Concept of Duality:

It is the elegant and important concept within the field of operations research. This theory was first developed in relation to linear programming, but it has many applications, and perhaps even a more natural and intuitive interpretation, in several related areas such as nonlinear programming, networks and game theory.

One part of a Linear Programming Problem (LPP) is called the ***Primal*** and the other part is called the ***Dual.*** In other words, each maximization problem in LP has its corresponding problem, called the dual, which is a minimization problem. Similarly, each minimization problem has its corresponding dual, a maximization problem. For example, if the primal is concerned with maximizing the contribution from the three products A, B, and C and from the three departments X, Y, and Z, then the dual will be concerned with minimizing the costs associated with the time used in the three departments to produce those three products. An optimal solution from the primal and the dual problem would be same as they both originate from the same set of data.

Given a primal problem:

P: min c^Tx subject to $Ax \geq b$, $x \geq 0$

The dual is:

D: max b^Ty subject to $A^Ty \leq c$, $y \geq 0$

Example:

$$
\begin{aligned}
&\text{P: min } 7x_1 + x_2 + 5x_3 \\
&\text{subject to } x_1 - x_2 + 3x_3 \geq 10 \\
&\qquad 5x_1 + 2x_2 - x_3 \geq 6 \\
&\qquad x_1, x_2, x_3 \geq 0 \\
&\text{D: max } 10y_1 + 6y_2 \\
&\text{subject to } y_1 + 5y_2 \leq 7 \\
&\qquad -y_1 + 2y_2 \leq 1 \\
&\qquad 3y_1 - y_2 \leq 5 \\
&\qquad y_1, y_2 \geq 0
\end{aligned}
$$

Rules for Constructing the Dual from Primal:

1. A dual variable is defined for each constraint in the primal problem,i.e, the no. of variables in the dual problem is equal to no. of constraints in the primal problem and vice-versa. If there are m constraints and n variables in the primal problem then there would be m variables and n constraints in the dual problem.

2. The RHS of primal,i.e, $b_1, b_2, b_3 \dots \dots b_m$ become the coefficients of dual variables($Y_1, Y_2, \dots \dots Y_m$) in the dual objective function(ZY). Also the coefficients of primal variables($X_1, X_2, X_3 \dots \dots X_n$),i.e, $c_1, c_2, c_3 \dots \dots c_n$, become RHS of the dual constraints.

3. For a maximization primal problem(with all < or = constraints), there exists a minimization dual problem(with all > or = constraints) and vice-versa.

4.The matrix of coefficients of variables in dual problem the transpose of matrix of coefficients in the primal problem and vice-versa.

5. If any of the primal constraint (say ith) is an equality then the corresponding dual variable is unrestricted in sign and vice-versa.

The Primal-Dual Relationship

The Primal-Dual Relationship

		Primal Variables				Relation	Constraints
		X_1	X_2	…….	X_n		
Dual Variables	Y_1	a_{11}	a_{12}	…….	a_{1n}	< or =	b_1
	Y_2	a_{21}	a_{22}	…….	a_{2n}	< or =	b_2
	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	Y_m	a_{m1}	a_{m2}	…….	a_{mn}	< or =	b_m
Relation		> or =	> or =	…….	> or =		Min Z_Y
Constraint		c_1	c_2	…….	c_n	Max Z_X	

<u>**Example**</u>**:**

$$Z = 6000X_1 + 4000X_2$$

s.t $\quad 4X_1 + X_2 \leq 12$

$$9X_1 + X_2 \leq 20$$

$$7X_1 + 3X_2 \leq 18$$

$$10X_1 + 40X_2 \leq 40$$

$$X_1, X_2 \geq 0$$

Primal Dual Relationship					
		Primal variables		relation	constraints
		X_1	X_2		
Dual Variable	Y_1	4	1	$\leq$	12
	Y_2	9	1	$\leq$	20
	Y_3	7	3	$\leq$	18
	Y_4	10	40	$\leq$	40
	Relation	$\geq$	$\geq$		MinZ_Y
	constraint	6000	4000	MaxZ_X	

Hence the Dual Problem looks like,

Hence the Dual Problem looks like,

Minmize $\quad Z = 12Y_1 + 20Y_2 + 18Y_3 + 40Y_4$

s.t $\quad 4Y_1 + 9Y_2 + 7Y_3 + 10Y_4 \geq 6000$

$Y_1 + Y_2 + 3Y_3 + 40Y_4 \geq 4000$

$Y_1, Y_2 \geq 0$

Duality theorems:

- It is easy to show that we can move from one pair of primal-dual problems to the other.
- It is also easy to show that the dual of the dual problem is the primal problem.
- Thus we are showing the duality theorems using the pair where the primal primal is in the standard form:

Weak duality theorem :

Let x and y be the feasible solutions for P and D respectively, then

$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

Proof: Follows immediately from the constraints

$$c^T x \geq (A^T y)^T x = y^T Ax \geq y^T b = b^T y$$

- This theorem is very useful
- Suppose there is a feasible solution y to D. Then any feasible solution of P has value lower bounded by $b^T y$. This means that if P has a feasible solution, then it has an optimal solution
- Reversing argument is also true
- Therefore, if both P and D have feasible solutions, then both must have an optimal solution.

Furthermore,suppose we happen to have feasible solutions x^* and y^* to the primal and dual problems respectively ,such that $c^T x^* = b^T y^*$,then both these solutions are optimal for their respective problems,since for any x that is feasible for P $c^T x \geq b^T y^* = c^T x^*$

Strong duality theorem If one of the two primal or dual problem has a finite value optimal solution, then the other problem has the same property, and the optimal values of the two problems are equal. If one of the two problems is unbounded, then the feasible domain of the other problem is empty.

Proof The second part of the theorem follows directly from the weak duality theorem. Indeed, suppose that the primal problem is unbounded below, and thus $c^T x \to -\infty$. For contradiction, suppose that the dual problem is feasible. Then there would exist a solution $y \in \{y : A^T y \leq c\}$, and from the weak duality theoren, it would follow that $b^T y \leq c^T x$; i.e., $b^T y$ would be a lower bound for the value of the primal objective function $c^T x$, a contradiction.

Complementary Slackness :

Theorem: Let x and y be primal and dual feasible solutions respectively. Then x and y are both optimal iff two of the following conditions are satisfied:

$(A^T y - c)_j \, x_j = 0$ for all j = 1…n

$(Ax - b)_i \, y_i = 0$ for all i = 1…m

Proof:

As in the proof of the weak duality theorem, we have:

$c^T x \geq (A^T y)^T x = y^T Ax \geq y^T b$ (1)

From the strong duality theorem, we have:

x and y are optimal $\Leftrightarrow c^T x = b^T y$

$\Leftrightarrow c^T x = y^T Ay = y^T b$ ....(2)

$\Leftrightarrow (y^T A - c^T)x = 0 and\ y^T(b - Ax) = 0$ ....(3)

Note that $(y^T A - c^T)x = \sum_{j=1}^{n} (y^T A - c^T)_j x_j = \sum_{j=1}^{n} (A^T y - c)_j x_j$ (4)

And $y^T(b - Ax) = \sum_{i=1}^{m} (b - Ax)_i y_i$ (5)

We have:

x and y optimal $\Leftrightarrow$ (2) and (3) hold

$\Leftrightarrow$ both sums (4) and (5) are zero

$\Leftrightarrow$ all terms in both sums are zero

$\Leftrightarrow$ Complementary slackness holds

- It's an easy way to check whether a pair of primal/dual feasible solutions are optimal
- Given one optimal solution, complementary slackness makes it easy to find the optimal solution of the dual problem
- May provide a simpler way to solve the primal

Comparing
(primal) simplexe alg. and dual simplexe alg.

Simplex alg

- . Search in the feasible domain
- Search for an entering variable to reduce the value of the objective function
- Search for a leaving variable preserving the feasibility of the new solution
- Stop when an optimal solution is found or when the problem is not bounded below .

Dual simplex alg.

- Search out of the feasible domain
- Search for a leaving variable to eliminate a negative basic variable
- Search for an entering variable preserving the non negativity of the relative costs
- Stop when the solution becomes feasible or when the problem is not feasible .

The Game Theory:

Game theory is the study of how people interact and make decisions.This broad definition applies to most of the social sciences, but game theory applies mathematical models to this interaction under the assumption that each person's behavior impacts the well-being of all other participants in the game. These models are often quite simplified abstractions of real-world interactions**.**

Game theoretic notions go back thousands of years

– Talmud and Sun Tzu's writings.

Modern theory credited to John von Neumann and Oskar Morgenstern 1944.

– *Theory of Games and Economic Behavior*. In the early 1950s,

John Nash ("A Beautiful Mind" fame) generalized these results and provided the basis of the modern field.

Game Theory in the Real World:

- **Economists**
  innovated antitrust policy
  auctions of radio spectrum licenses for cell phone
  program that matches medical residents to hospitals.
- **Computer scientists**
  new software algorithms and routing protocols
  Game AI
- **Military strategists**
  nuclear policy and notions of strategic deterrence.
- **Sports coaching staff**
  run versus pass or pitch fast balls versus sliders.
- **Biologists**
  what species have the greatest likelihood of extinction

For Game Theory, our focus is on games where:

- There are 2 or more *players*.
- There is some choice of action where *strategy* matters.
- The game has one or more *outcomes*, e.g. someone wins, someone loses.
- The outcome depends on the strategies chosen by all players; there is *strategic interaction*.

What does this rule out?

- Games of pure chance, e.g. lotteries, slot machines. (Strategies don't matter).
- Games without strategic interaction between players, e.g. Solitaire.

Five Elements of a Game:

- The ***players***
  how many players are there?
  does nature/chance play a role?
- A complete description of what the players can do – ***the set of all possible actions.***
- The ***information that players have available*** when choosing their actions
- A description of ***the payoff consequences*** for each player for every possible combination of actions chosen by all players playing the game.
- A description of all ***players' preferences over payoffs.***

Two-Player Games:

- A game with just two players is a two-player game.
- We will study only games in which there are two players, each of whom can choose between only two strategies.

An Example of a Two-Player Game:

- The players are called A and B.
- Player A has two strategies, called "Up" and "Down".
- Player B has two strategies, called "Left" and "Right".
- the table showing the payoffs to both players for each of the four possible strategy combinations is the game's payoff matrix.

		Player B	
		L	**R**
playerA	**U**	**(3,9)**	**(1,8)**
	D	**(0,0)**	**(2,1)**

this is the game's payoff matrix

Player A's payoff is shown first.
Player B's payoff is shown second.

E.g. if A plays **U**p and B plays **R**ight then A's payoff is 1 and B's payoff is 8.

And if A plays **D**own and B plays **R**ight then A's payoff is 2 and B's payoff is 1.

A play of the game is a pair such as (**U,R**) where the 1st element is the strategy chosen by **Player A** and the 2nd is the strategy chosen by **Player B**

What plays are we likely to see for this game?

Is (**U,R**) a likely play?

If **B** plays **R**ight then **A's** best reply is **D**own since this improves **A**'s payoff from 1 to 2.So (**U,R**) is not a likely play.

Is (**D,R)** a likely play?

If **B** plays **R**ight then **A**'s best reply is **D**own.

If **B** plays **R**ight then **A**'s best reply is **D**own.If **A** plays **D**own then **B'**s best reply is **R**ight.So (**D,R**) is a likely play.

Is (**D,L**) a likely play?

If **A** plays **D**own then **B'**s best reply is **R**ight,so (**D,L**) is not a likely play.

Is **(U,L)** a likely play?

If **A** plays **U**p then **B'**s best reply is **L**eft. If **A** plays **U**p then **B'**s best reply is **L**eft. If **B** plays **L**eft then **A**'s best reply is **U**p. So (**U,L**) is a likely play.

Nash Equilibrium:

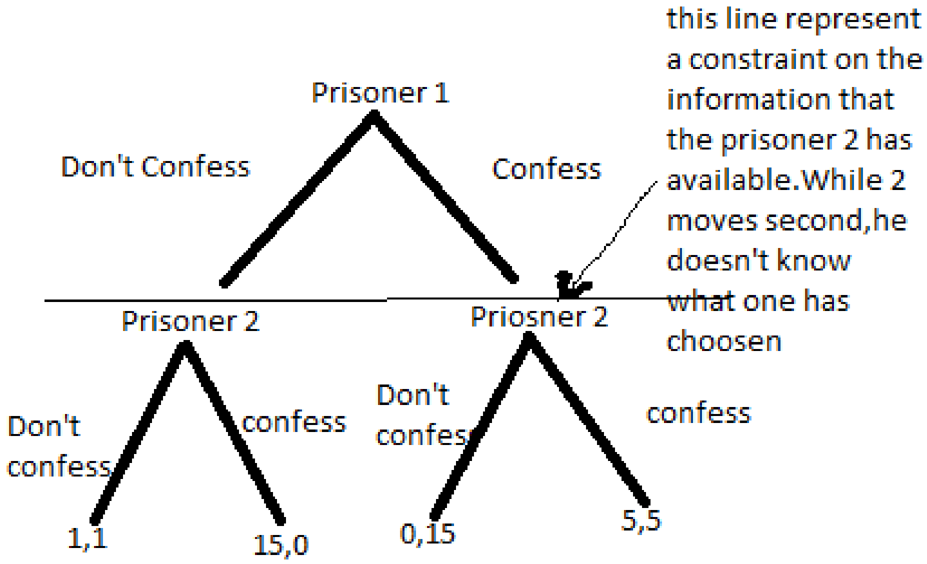
- A play of the game where each strategy is a best reply to the other is a **Nash equilibrium.**
- Our example has two Nash equilibria**; (U,L)** and **(D,R). (U,L)** and **(D,R)** are both Nash equilibria for the game. But which will we see? Notice that **(U,L)** is preferred to **(D,R)** by both players. Must we then see (**U,L)** onl

The Prisoners' Dilemma Game:

- Two players, Clyde and Bonnie
- Each prisoner has two possible actions.
  Clyde: Silence, Confess
  Bonnie: Silence, Confess
- Players choose actions simultaneously without knowing the action chosen by the other.
- S=Silence ,C=Confess
- Payoff consequences quantified in prison years.
  If both silence, each gets 1 year
  If both confess, each gets 5 years
  If 1 confesses, he goes free and other gets 15 years
- Fewer years=greater satisfaction=>higher payoff.
  Prisoner 1 payoff first, followed by prisoner 2 payoff

Clyde(prisoner1)			
		S	C
Bonnie(prisoner2)	S	(1,1)	(15,0)
	C	(0,15)	(5,5)

Prisoner's Dilemma in Extensive Form :



Payoffs are :Priosner 1 payoff,Prisoner 2 payoff

What plays are we likely to see for this game?

If Bonnie plays **S**ilence then Clyde's best reply is **C**onfess.

If Bonnie plays **C**onfess then Clyde's best reply is **C**onfess.

So no matter what Bonnie plays, Clyde's best reply is always Confess. Similarly, no matter what Clyde plays, Bonnie's best reply is always Confess.

Confess is a dominant strategy for Bonnie also Confess is a dominant strategy for Clyde.

So the only ***Nash equilibrium*** for this game is **(C,C**), even though (**S,S**) gives both Bonnie and Clyde better payoffs.The only Nash equilibrium is inefficient.

Prisoners' Dilemma :
Example of Non-Zero Sum Game

- A zero-sum game is one in which the players' interests are in direct conflict, e.g. in football, one team wins and the other loses; payoffs sum to zero.
- A game is non-zero-sum, if players interests are not always in direct conflict, so that there are opportunities for both to gain.
- For example, when both players choose Don't Confess in the Prisoners' Dilemma

Prisoners' Dilemma :
Application to other areas

- Nuclear arms races.
- Dispute Resolution and the decision to hire a lawyer.
- Corruption/political contributions between contractors and politicians.

Simultaneous versus Sequential Move Games

- Games where players choose actions simultaneously are ***simultaneous move games***.
  - Examples: Prisoners' Dilemma,.
  - Must anticipate what your opponent will do right now, recognizing that your opponent is doing the same. **In both examples the players chose their strategies simultaneously.**
    Such games are *simultaneous play games*
- Games where players choose actions in a particular sequence are ***sequential move games.*** The player who plays first is the ***leader***. The player who plays second is the ***follower***
  - Examples: Chess, Bargaining/Negotiations.
- Must look ahead in order to know what action to choose now.
- Many strategic situations involve both sequential and simultaneous moves.

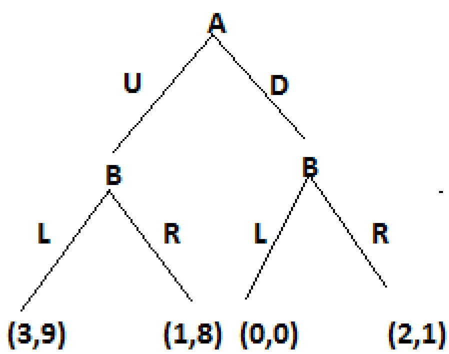
A Sequential Game Example:

Sometimes a game has more than one Nash equilibrium and it is hard to say which is more likely to occur.When such a game is sequential it is sometimes possible to argue that one of the Nash equilibria is more likely to occur than the other.

		Player B	
		L	**R**
playerA	**U**	**(3,9)**	**(1,8)**
	D	**(0,0)**	**(2,1)**

(U,L) and **(D,R)** are both Nash equilibria when this game is played simultaneously and we have no way of deciding which equilibrium is more likely to occur.

Suppose instead that the game is played sequentially, with A leading and B following. We can rewrite the game in its extensive form.



A plays first,**B** plays second

(U,L) is a Nash equilibrium.

(D,R) is a Nash equilibrium.
Which is more likely to occur?

If **A** plays **U** then **B** plays **L**; **A** gets 3. If **A** plays **D** then **B** plays **R**; **A** gets 2.

So **(U,L)** is the likely Nash equilibrium.

One-Shot versus Repeated Games:

- ***One-shot:*** play of the game occurs once. Players likely to not know much about one another. Example - tipping on your vacation
- ***Repeated:*** play of the game is repeated with the same players. Indefinitely versus finitely repeated games
  Reputational concerns matter; opportunities for cooperative behavior may arise.
- ***Advise:*** If you plan to pursue an aggressive strategy, ask yourself whether you are in a one-shot or in a repeated game. If a repeated game, think again

Strategies:

- ***A strategy*** must be a "comprehensive plan of action", a decision rule or set of instructions about which actions a player should take following all possible *histories of play*.
- It is the equivalent of a memo, left behind on vacation, that specifies the actions you want taken in every situation which could arise during your absence.
- Strategies will depend on whether the game is one-shot or repeated. *Examples of one-shot strategies Prisoners' Dilemma:* Don't Confess, Confess
- How do strategies change when the game is repeated?

Pure Stratagies:

		Player B	
		L	R
playerA	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

This is our original example once more.Suppose again that play is simultaneous.
We discovered that the game has two Nash equilibria**; (U,L) and (D,R).**

Player A's has been thought of as choosing to play either **U or D,** but no combination of both; that is, as playing purely **U or D. U and D** are **Player A's pure strategies.** Similarly, **L and R** are **Player B's pure strategies.**

Consequently**, (U,L) and (D,R)** are **pure strategy Nash equilibria.** Must every game have at least one **pure strategy Nash equilibrium?**

		Player B	
		L	R
playerA	U	(1,2)	(0,4)
	D	(0,5)	(3,2)

Here is a new game. Are there any pure strategy Nash equilibria?

Is **(U,L)** a Nash equilibrium? No.

Is **(U,R)** a Nash equilibrium? No**.**
Is **(D,L)** a Nash equilibrium? No.
Is **(D,R)** a Nash equilibrium? No.

So the game has no Nash equilibria in pure strategies. Even so, the game does have a Nash equilibrium, but in **mixed strategies.**

Mixed Strategies:

- Instead of playing purely Up or Down, Player A selects a probability distribution $(\pi_U . 1-\pi_U)$, meaning that with probability π_UPlayer A will play Up and with probability $1-\pi_U$will play Down.
- Player A is mixing over the pure strategies Up and Down.
- The probability distribution $(\pi_U . 1-\pi_U)$ is a ***mixed strategy*** for Player A.

- Similarly, Player B selects a probability distribution $(\pi_L . 1-\pi_L)$, meaning that with probabilityπ_L Player B will play Left and with probability$1-\pi_L$ will play Right.

- Player B is mixing over the pure strategies Left and Right.

- The probability distribution $(\pi_L . 1-\pi_L)$ is a ***mixed strategy*** for Player B

		Player B	
		L	**R**
playerA	**U**	**(1,2)**	**(0,4)**
	D	**(0,5)**	**(3,2)**

This game has no pure strategy Nash equilibria but it does have a Nash equilibrium in mixed strategies. How is it computed?

		Player B	
		L(π_L)	**R($1-\pi_L$)**
playerA	**U(π_U)**	**(1,2)**	**(0,4)**
	D($1-\pi_U$)	**(0,5)**	**(3,2)**

If **B** plays **Left** her expected payoff is $2\pi_U + 5(1-\pi_U)$.

If **B** plays **Right** her expected payoff is $4\pi_U + 2(1-\pi_U)$.

If $2\pi_U + 5(1-\pi_U) > 4\pi_U + 2(1-\pi_U)$ then **B** would play only **Left.** But there are noNash equilibria in which B plays only **Left.**

If $2\pi_U + 5(1-\pi_U) < 4\pi_U + 2(1-\pi_U)$ then **B** would play only **Right.** But there are no Nash equilibria in which **B** plays only **Right.**

So for there to exist a Nash equilibrium**, B** must be indifferent between playing **Left or Right**; i.e.

$$2\pi_U + 5(1-\pi_U) = 4\pi_U + 2(1-\pi_U)$$
$$\Rightarrow \pi_U = 3/5.$$

If **A** plays **Up** his expected payoff is $0 \times \pi_L + 3 \times (1-\pi_L) = 3(1-\pi_L)$.

If **A** plays **Down** his expected payoff is $1 \times \pi_L + 0 \times (1-\pi_L) = \pi_L$.

If $\pi_L > 3(1-\pi_L)$ then **A** would play only **Up.** But there are no Nash equilibria in which **A** plays only **Up.**

If $\pi_L < 3(1-\pi_L)$ then **A** would play only **Down.** But there are no Nash equilibria in which **A** plays only **Down.**

So for there to exist a Nash equilibrium, **A** must be indifferent between playing **Up or Down;** i.e. $\pi_L = 3(1-\pi_L) \Rightarrow \pi_L = 3/4.$

So the game's only **Nash equilibrium** has **A** playing the **mixed strategy (3/5, 2/5)** and has
B playing the **mixed strategy (3/4, 1/4).**

		Player B	
		L(3/4)	**R(1/4)**
playerA	**U(3/5)**	**(1,2)**	**(0,4)**
	D(2/5)	**(0,5)**	**(3,2)**

The payoffs will be **(1,2)** with probability $\frac{3}{5} \times \frac{3}{4} = \frac{9}{20}$

The payoffs will be **(0,4)** with probability $\frac{3}{5} \times \frac{1}{4} = \frac{3}{20}$

The payoffs will be **(0,5)** with probability $\frac{2}{5} \times \frac{3}{4} = \frac{6}{20}$

The payoffs will be **(3,2)** with probability $\frac{2}{5} \times \frac{1}{4} = \frac{2}{20}$

A's expected Nash equilibrium payoff is

$$1 \times \frac{9}{20} + 0 \times \frac{3}{20} + 0 \times \frac{6}{20} + 3 \times \frac{2}{20} = \frac{3}{4}.$$

B's expected Nash equilibrium payoff is

$$2\times\frac{9}{20}+4\times\frac{3}{20}+5\times\frac{6}{20}+2\times\frac{2}{20}=\frac{16}{5}.$$

How Many Nash Equilibria?

- A game with a finite number of players, each with a finite number of pure strategies, has at least one Nash equilibrium.
- So if the game has no pure strategy Nash equilibrium then it must have at least one mixed strategy Nash equilibrium.

Repeated Game Strategies

- In repeated games, the sequential nature of the relationship allows for the adoption of strategies that are contingent on the actions chosen in previous plays of the game.
- Most contingent strategies are of the type known as "trigger" strategies.
  Example ***trigger strategies*** :In prisoners' dilemma: Initially play Don't confess. If your opponent plays Confess, then play Confess in the next round. If your opponent plays Don't confess, then play Don't confess in the next round. This is known as the "tit for tat" strategy

Information:

- Players have perfect information if they know exactly what has happened every time a decision needs to be made, e.g. in Chess.
- Otherwise, the game is one of imperfect information
- Example: In the repeated investment game, the sender and receiver might be differentially informed about the investment outcome. For example, the receiver may know that the amount invested is always tripled, but the sender may not be aware of this fact.

- Payoffs are known and fixed. People treat expected payoffs the same as certain payoffs (they are risk neutral).

  - Example: a risk neutral person is indifferent between $25 for certain or a 25% chance of earning $100 and a 75% chance of earning 0.

  - We can relax this assumption to capture risk averse behavior.

- All players behave rationally.
    - They understand and seek to maximize their own payoffs.
    - They are flawless in calculating which actions will maximize their payoffs.
- The rules of the game are common knowledge:
    - Each player knows the set of players, strategies and payoffs from all possible combinations of strategies: call this information “X.”
    - Each player knows that all players know X, that all players know that all players know X, that all players know.., ad infinitum.

Equilibrium:

- The interaction of all (rational) players' strategies results in an outcome that we call "equilibrium."
- In equilibrium, each player is playing the strategy that is a "best response" to the strategies of the other players. No one has an incentive to change his strategy given the strategy choices of the others.
- Equilibrium is not:
    - The best possible outcome. Equilibrium in the one-shot prisoners' dilemma is for both players to confess.
    - A situation where players always choose the same action. Sometimes equilibrium will involve changing action choices (known as a mixed strategy equilibrium).

References