Some Assignment Problems in Algebraic Number Theory
M.Math. Class - 2007

Q 1.
Let $[K : Q] = n$ *and* $I \neq 0$ *be an ideal in* $\mathcal{O}_K$. *Prove :*
(a) $I = \sum_{i=1}^{n} Z\alpha_i$ *for some* α_i*'s with* $K = \sum_{i=1}^{n} Q\alpha_i$.
(b) $I \cap Z \neq 0$ *and* $\mathcal{O}_K/I$ *is finite.*

Q 2.
If $I = \prod_{i=1}^{r} P_i^{\alpha_i}, J = \prod_{i=1}^{r} P_i^{\beta_i}$ *are the prime ideal decompositions of two fractional ideals in a Dedekind domain, prove that*

$$GCD(I, J) = I + J = \prod_{i=1}^{r} P_i^{min(\alpha_i,\beta_i)},$$

$$LCM(I, J) = I \cap J = \prod_{i=1}^{r} P_i^{max(\alpha_i,\beta_i)}.$$

Q 3.
If the ring of integers of a number field is a UFD, prove that it must be a PID.

Q 4. *Prove that* $\mathbf{Z}[e^{2i\pi/23}]$ *is not a PID. State precisely what you assume.*

Q 5. *Use the fact that* $\mathbf{Z}[\sqrt{-5}]$ *has class number* 2 *to show that the equation* $y^2 = x^3 - 5$ *has no solutions in integers.*

Q 6. *Show that the ring of integers of* $\mathbf{Q}[\zeta_n + \zeta_n^{-1}]$ *is* $\mathbf{Z}[\zeta_n + \zeta_n^{-1}]$ *for any* n. *You may use that the ring of integers of any cyclotomic field* $\mathbf{Q}[\zeta_n]$ *is* $\mathbf{Z}[\zeta_n]$.

Q 7.
Use the Minkowski bound to determine the class number of $\mathbf{Q}[\sqrt{-5}]$.

Q 8.
If K *is a cubic extension of* $\mathbf{Q}$ *with discriminant between* -1 *and* -49, *use Minkowski bound to prove that* K *has class number* 1.
Hint : Observe that the discriminant has the sign $(-1)^{r_2}$, *where* $2r_2$ *is the number of non-real embeddings of any number field.*

Q 9.
Show that the fundamental unit in $\mathbf{Q}(\sqrt{d^2 - 1})$ *is* $d + \sqrt{d^2 - 1}$ *for each square-free integer* $d \geq 2$.

Q 10.
In $\mathbf{Z}_5$, compute with proof the limit of the sequence $a_1 = 4, a_2 = 34, a_3 = 334, a_4 = 3334, \cdots$

Q 11.
Look up Krasner's lemma from any book. Apply it to prove that there are only finitely many p-adic fields of a given degree upto isomorphism.
Hint : Use the fact that the maximal ideal of the ring of integers of a p-adic field is compact.

Q 12.
Find a necessary and sufficient condition for a p-adic field to contain a primitive p-th root of unity.

Q 13.
Let $p \geq 3$ be prime. Let A be an $n \times n$ matrix with entries from $\mathbf{Z}_p$ such that $\det A$ is a unit. Assume that all the entries of the matrix $A - I_n$ are in the maximal ideal $p\mathbf{Z}_p$. Prove that A must have infinite order unless $A = I_n$.

Q 14.
Prove that $\mathbf{Q}_p(\zeta_p) = \mathbf{Q}_p((-p)^{1/(p-1)})$.
Hint : Use the fact that p is totally ramified in the field on the left.

Q 15.
Show that an element $a \in \mathbf{Q}_p$ is in $\mathbf{Z}_p^*$ (that is, $|a|_p = 1$) if and only if it has an m-th root for each m coprime to $p(p-1)$.
Hint : Use Hensel.

Q 16.
(i) Learn/work out a proof of the fact that the norm map from an unramified extension l/k of p-adic fields is surjective.
(ii) Learn about the exponential and logarithmis series over $\mathbf{Q}_p$ from Gouvea's book, for instance.

Q 17.
Indicate the steps used to deduce that the ray class group for a modulus of a number field, is finite, of order a multiple of the class number.

Q 18.
Use the following steps to give a new proof of the fact that for a real, non-trivial Dirichlet character χ, one has $L(1, \chi) \neq 0$:-
Assuming otherwise, consider $f(t) = \sum_n \frac{\chi(n)t^n}{1-t^n} = -\sum_n \chi(n)b_n$, where $b_n = \frac{1}{n(1-t)} - \frac{t^n}{1-t^n}$. Show that b_n is a non-increasing sequence and use $|\sum_{n \leq x} \chi(n)| \leq d$ for all x (where χ is modulo d) and use the Abel summation formula to

conclude that $\sum \chi(n) b_n$ is bounded in $[0,1)$ and derive a contradiction.

Q 19.
Look up the definition of polar density of a set of primes of a number field K defined in Marcus in terms of the analytic properties of the Dedekind zeta function $\zeta_K(s)$.
(i) If S, T are two sets of primes of K such that $P \in (S \cup T) - (S \cap T)$ implies $f(P/P \cap \mathbf{Z}) > 1$, show that the polar densities of S, T coincide (if any one of them exists).
(ii) Prove that if a set of primes in K has a polar density, then it also has Dirichlet density which is also the same.

Q 20.
Let L/K be a cyclic (Galois) extension of number fields. Show that the set of primes of K which are inert (that is, remain prime) in L has a Dirichlet density and compute it.

Q 21.
Let L/K be an abelian extension of number fields. Show that the set of primes of K which split completely in L has a Dirichlet density and compute it.