Alternating Sums of the Reciprocals of Binomial Coefficients

Hacène Belbachir and Mourad Rahmani
University of Sciences and Technology Houari Boumediene
Faculty of Mathematics
Po. Box 32, El Alia, Bab-Ezzouar 16111, Algiers
Algeria
hacenebelbachir@gmail.com
mrahmani@usthb.dz

B. Sury
Statistics & Mathematics Unit
Indian Statistical Institute
8th Mile Mysore Road
Bangalore 560059
India
sury@isibang.ac.in

Abstract

In this paper, our aim is to investigate the summations of the form $\sum_{0\leq k\leq n}(-1)^k k^m \binom{n}{k}^{-1}$. We give closed formulae in terms of Akiyama-Tanigawa matrix. Recurrence formulae, ordinary generating functions and some other results are also given.

1 Introduction and Notations

Binomial coefficients play an important role in many areas of mathematics, such as combinatorics, number theory and special functions. In 1993, Sury [16] connected the inverse of the binomial coefficients to the beta function as follows

$$\binom{n}{k}^{-1} = (n+1)\int_0^1 x^k (1-x)^{n-k}\, dx. \tag{1}$$

There are many papers dealing with sums involving inverses of binomial coefficients, see for instance [2, 7, 11, 13, 14, 15, 16, 19, 20]. For nonnegative integers n, m and p we consider the sums

$$T_n^{(m,p)} := \sum_{k=0}^{n} (-1)^k k^m \binom{p+n}{p+k}^{-1}. \tag{2}$$

These sums have been studied by many authors. Trif [18], using (1) proved for $m = 0$ that

$$T_n^{(0,p)} = \left((-1)^n + \binom{p+n+1}{p}^{-1}\right) \frac{p+n+1}{p+n+2}. \tag{3}$$

Sury, Wang and Zhao [17], studied (2) for $m = 1$ and $m = 2$, they obtain

$$\begin{aligned} T_n^{(1,p)} = \frac{p+n+1}{p+n+3}\Bigg(&\frac{(-1)^n (n+1)(p+n+3)}{p+n+2} - \\ &\binom{p+n+2}{p+1}^{-1} - (-1)^n\Bigg), \end{aligned} \tag{4}$$

and

$$\begin{aligned} T_n^{(2,p)} = (p+n+1)\Bigg(&\frac{(-1)^n (n+1)^2}{p+n+2} - \frac{(-1)^n (2n+3)}{p+n+3} \\ &+ \frac{2}{p+n+4}\left(\binom{p+n+3}{p+2}^{-1} + (-1)^n\right) \\ &- \frac{1}{p+n+3}\binom{p+n+2}{p+1}^{-1}\Bigg). \end{aligned} \tag{5}$$

Our aim is to give a closed form and recurrence relation for the sums (2). In order to investigate the summation of the form $S_n^{(m)} := T_n^{(m,0)}$ and $T_n^{(m,p)}$, we shall use the following tools [5]:

- The Stirling numbers of the first kind (signed) $\left[{n \atop k}\right]$ (see A008275 in [12]), are defined by the generating function

$$x(x-1)\cdots(x-n+1) = \sum_{k\geq 0} \left[{n \atop k}\right] x^k,$$

  and satisfy the recurrence relation

$$\left[{n+1 \atop k}\right] = \left[{n \atop k-1}\right] - n \left[{n \atop k}\right], \quad (1 \leq k \leq n),$$

  with $\left[{n \atop n}\right] = 1$, $\left[{n \atop 0}\right] = 0$ for $n \geq 1$ and $\left[{n \atop k}\right] = 0$ for $k < 0$ or $k > n$.

- The Stirling numbers of the second kind $\left\{{n \atop k}\right\}$ (see A008277 in [12]), are defined by the generating function

$$\prod_{j=1}^{k} \frac{x}{1-jx} = \sum_{n\geq k} \left\{{n \atop k}\right\} x^n,$$

and satisfy the recurrence relation

$$\left\{ {n+1 \atop k} \right\} = \left\{ {n \atop k-1} \right\} + k \left\{ {n \atop k} \right\},$$

with $\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = 1$.

They also verify the following important identity

$$x^n = \sum_{k=0}^{n} (-1)^{n+k} \left\{ {n \atop k} \right\} x(x+1)\cdots(x+k-1). \tag{6}$$

- The Eulerian numbers $\left\langle {n \atop k} \right\rangle$ (see A008292 in [12]) are defined by

$$\left\langle {n \atop k} \right\rangle = \sum_{i=0}^{k} (-1)^i (k-i)^n \binom{n+1}{i}, \quad (1 \le k \le n),$$

and satisfy the recursive identity

$$\left\langle {n \atop k} \right\rangle = k \left\langle {n-1 \atop k} \right\rangle + (n-k+1) \left\langle {n-1 \atop k-1} \right\rangle,$$

with $\left\langle {1 \atop 1} \right\rangle = 1$.

- The Worpitzky numbers $W_{n,k}$ (see A028246 in [12]), are defined by

$$W_{n,k} = \sum_{i=0}^{k} (-1)^{i+k} (i+1)^n \binom{k}{i}.$$

They can also be expressed through the Stirling numbers of the second kind as follows

$$W_{n,k} = k! \left\{ {n+1 \atop k+1} \right\}. \tag{7}$$

The Worpitzky numbers satisfy the recursive relation

$$W_{n,k} = (k+1) W_{n-1,k} + k W_{n-1,k-1} \quad (n \ge 1, k \ge 1). \tag{8}$$

Some simple properties are given

$$\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k = \sum_{k=0}^{n} (x-1)^{n-k} k W_{n-1,k-1}, \tag{9}$$

$$\sum_{k=0}^{n} \binom{n}{k} \left\{ {k \atop t} \right\} = \left\{ {n+1 \atop t+1} \right\}, \tag{10}$$

and

$$\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{k+1}{t} = W_{n,n-t}. \tag{11}$$

- The Bernoulli numbers B_n are defined by the exponential generating function

$$\frac{x}{1-e^{-x}} = \sum_{n\geq 0} B_n \frac{x^n}{n!}.$$

  The recursive relation is

$$B_0 = 1,$$
$$B_n = 1 - \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{n-k+1}, \quad (n \geq 1).$$

  Thus we have $B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0$, and so on, they can also be expressed through the Worpitzky numbers

$$B_n = \sum_{k=0}^{n} (-1)^k \frac{W_{n,k}}{k+1}.$$

- The Akiyama-Tanigawa matrix $(A_{n,k})_{n,k\geq 0}$ associated with initial sequence $A_{0,k} = \frac{1}{k+1}$ is defined by (see [1, 4, 8, 10])

$$A_{n,k} = (k+1)\left(A_{n-1,k} - A_{n-1,k+1}\right),$$

  or equivalently by [6]

$$\begin{aligned} A_{n,k} &= \frac{1}{k!} \sum_{i=0}^{k} (-1)^k \begin{bmatrix} k+1 \\ i+1 \end{bmatrix} B_{n+i}, \\ &= \sum_{i=1}^{n} (-1)^{i-1} \binom{k+i+1}{k+1}^{-1} W_{n,i}. \end{aligned} \tag{12}$$

  The Akiyama-Tanigawa matrix $A_{n,k}$ is then

$$A_{n,k} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{6} & \frac{1}{6} & \frac{3}{20} & \frac{2}{15} & \frac{5}{42} & \cdots \\ 0 & \frac{1}{30} & \frac{1}{20} & \frac{2}{35} & \frac{5}{84} & \cdots \\ -\frac{1}{30} & -\frac{1}{30} & -\frac{3}{140} & -\frac{1}{105} & 0 & \cdots \\ 0 & -\frac{1}{42} & -\frac{1}{28} & -\frac{4}{105} & -\frac{1}{28} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

2 Explicit formula for $\mathbf{S}_n^{(m)}$

For any nonnegative integer m, we consider the sums

$$S_n^{(m)} := \sum_{k=0}^{n} (-1)^k k^m \binom{n}{k}^{-1}. \tag{13}$$

The following result holds

Theorem 1. *For any nonnegative integers n and m, we have*

$$S_n^{(m)} = (n+1) \sum_{j=0}^{m} \frac{(-1)^{m+j}}{n+j+2} \left(1 + (-1)^n \binom{n+j+1}{j}\right) W_{m,j}. \tag{14}$$

Proof. We can write $S_n^{(m)}$ as follows

$$\begin{aligned} S_n^{(m)} &= \sum_{k=0}^{n} (-1)^k \binom{n}{k}^{-1} ((k+1)-1)^m \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k}^{-1} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} (k+1)^i, \end{aligned}$$

and with (6), we obtain

$$\begin{aligned} S_n^{(m)} &= \sum_{k=0}^{n} (-1)^k \binom{n}{k}^{-1} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \sum_{j=0}^{i} (-1)^{i+j} \left\{ {i \atop j} \right\} (k+1) \cdots (k+j) \\ &= \sum_{k=0}^{n} \sum_{i=0}^{m} \sum_{j=0}^{i} \frac{(-1)^k}{n!} (-1)^{m+j} \binom{m}{i} \left\{ {i \atop j} \right\} k!\,(k+1) \cdots (k+j)\,(n-k)! \\ &= \sum_{i=0}^{m} \sum_{j=0}^{i} (-1)^{m+j} \binom{m}{i} \left\{ {i \atop j} \right\} (n+1) \cdots (n+j) \sum_{k=0}^{n} (-1)^k \binom{n+j}{k+j}^{-1}. \end{aligned}$$

Now, from (3) and after some rearrangement, we get

$$S_n^{(m)} = (n+1) \sum_{j=0}^{m} \frac{(-1)^{m+j}}{n+j+2} \left(1 + (-1)^n \binom{n+j+1}{j}\right) j! \sum_{i=0}^{m} \binom{m}{i} \left\{ {i \atop j} \right\}.$$

From (7) and (10), the result holds.

□

3 Recurrence relation for $S_n^{(m)}$

Theorem 2. *For any nonnegative integers m and n, we have*

$$S_{n+1}^{(m)} = \delta_{0m} - \frac{1}{n+1} \sum_{i=0}^{m+1} \binom{m+1}{i} S_n^{(i)}, \tag{15}$$

where δ_{ij} is the Kronecker symbol.

Proof. Recall that $\binom{n+1}{k} = \frac{n+1}{k}\binom{n}{k-1}$, we have

$$\begin{aligned} S_{n+1}^{(m)} &= \delta_{0m} + \sum_{k=1}^{n+1} (-1)^k k^m \binom{n+1}{k}^{-1} \\ &= \delta_{0m} - \frac{1}{n+1}\sum_{k=0}^{n} (-1)^k (k+1)^{m+1} \binom{n}{k}^{-1} \\ &= \delta_{0m} - \frac{1}{n+1}\sum_{i=0}^{m+1}\binom{m+1}{i}\sum_{k=0}^{n} (-1)^k k^i \binom{n}{k}^{-1}. \end{aligned}$$

This proves the theorem. □

The recurrence relation for $S_n^{(m)}$ is given in the following

Theorem 3. *For any nonnegative integers m and n, we have*

$$\begin{aligned} S_{n+1}^{(m)} = \delta_{0m} - \frac{m+1}{n+1}S_n^{(m)} + \sum_{j=0}^{m+1}\frac{(-1)^{m+j}}{n+j+2}\left(1+(-1)^n\binom{n+j+1}{j}\right)W_{m+1,j} \qquad (16) \\ - \sum_{0\le i\le j\le m-1}\binom{m+1}{i}\frac{(-1)^{i+j}}{n+j+2}\left(1+(-1)^n\binom{n+j+1}{j}\right)W_{i,j}. \end{aligned}$$

Proof. This follows immediately from (14) and (15). □

Setting $m=1$ in (16), we have the following

Corollary 4. *If n is nonnegative integer, then*

$$S_{n+1}^{(1)} = -\frac{2}{n+1}S_n^{(1)} - \frac{(-1)^n(n^4+7n^3+15n^2+14n+12)+(n^2+6n+12)}{(n+2)(n+3)(n+4)}.$$

Our next goal is to calculate the ordinary generating functions of $S_n^{(m)}$.

4 Ordinary generating functions of $S_n^{(m)}$

In 2002, Mansour [9], generalized the idea of Sury [16] and gave an approach based on calculus to obtain the generating function for some combinatorial identities.

Theorem 5 (Mansour [9])**.** *Let $r, n \ge k$ be any nonnegative integer numbers, and let $f(n,k)$ be given by*

$$f(n,k) = \frac{(n+r)!}{n!}\int_{u_1}^{u_2} p^k(t)\, q^{n-k}(t)\, dt, \qquad (17)$$

where $p(t)$ and $q(t)$ are two functions defined on $[u_1, u_2]$. Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be any two sequences, and let $A(x), B(x)$ be the corresponding ordinary generating functions. Then

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} f(n,k)\, a_k b_{n-k}\right] x^n = \frac{d^r}{dx^r}\left[x^r \int_{u_1}^{u_2} A(xp(t))\, B(xq(t))\, dt\right]. \tag{18}$$

We apply Theorem 5, for $a_n = (-1)^n n^m$ $(m \geq 1)$ and $b_n = 1$, we have for $|x| < 1$

$$\begin{aligned} A(x) &= \frac{1}{(1+x)^{m+1}} \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle (-x)^{k+1} \\ &= \sum_{k=0}^{m} \frac{(-1)^{m+k}}{(1+x)^{k+1}} W_{m,k}, \\ B(x) &= \sum_{n\geq 0} x^n = \frac{1}{1-x}. \end{aligned}$$

From (18) we get

$$\sum_{n\geq 0} S_n^{(m)} x^n = \frac{d}{dx}\left[x \int_0^1 \frac{\sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle (-xt)^{k+1}}{(1+xt)^{m+1}(1-x+xt)} dt\right]. \tag{19}$$

Making the substitution $xt = y$ in the right-hand side of (19), we obtain

$$\sum_{n\geq 0} S_n^{(m)} x^n = \frac{d}{dx}\left[\int_0^x \frac{\sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle (-y)^{k+1}}{(1+y)^{m+1}(1-x+y)} dy\right],$$

Since the degree of the denominator is at least one higher than that of the numerator, this fraction decomposes into partial fractions of the form

$$\frac{\sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle (-y)^{k+1}}{(1+y)^{m+1}(1-x+y)} = \frac{\alpha^{(m)}(x)}{1-x+y} + \sum_{s=0}^{m} \frac{\alpha_s^{(m)}(x)}{(1+y)^{m-s+1}}, \tag{20}$$

We note in passing that (20) is equivalent to

$$\begin{aligned} \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle (-y)^{k+1} &= (1+y)^{m+1} \alpha^{(m)}(x) + (1-x+y) \sum_{s=0}^{m} (1+y)^s \alpha_s^{(m)}(x) \\ &= \sum_{k=0}^{m} (-1)^{m+k+1} y (1+y)^{m-k} W_{m-1,k-1}. \end{aligned} \tag{21}$$

For $y=-1$ and using the fact that $W_{p,p}=p!$ for $p\geq 0$, we immediately obtain the well-known identity

$$\sum_{k=0}^{m}\left\langle{m \atop k}\right\rangle = m!.$$

Next, if we set $y=0$ in (21) then we obtain a relation between $\alpha^{(m)}(x)$ and $\alpha_s^{(m)}(x)$ for $|x|<1$

$$\sum_{s=0}^{m}\alpha_s^{(m)}(x)=\frac{\alpha^{(m)}(x)}{x-1}. \tag{22}$$

Proposition 6. *For $m\geq 1$, we have*

$$\begin{aligned}\alpha_s^{(m)}(x) &= \sum_{i=0}^{s}\frac{(-1)^{i+s+1}}{x^{i+1}}\sum_{k=0}^{m}\left\langle{m \atop k}\right\rangle\binom{k+1}{s-i} \\ &= \sum_{j=m-s}^{m}\frac{(-1)^{m+j+1}}{x^{s-m+1+j}}W_{m,j},\end{aligned} \tag{23}$$

and

$$\begin{aligned}\alpha^{(m)}(x) &= \frac{1}{x^{m+1}}\sum_{k=0}^{m}\left\langle{m \atop k}\right\rangle(1-x)^{k+1} \\ &= \sum_{j=0}^{m}\frac{(-1)^{m+j}}{x^{j+1}}W_{m,j}, \\ &= -\alpha_m^{(m)}(x).\end{aligned} \tag{24}$$

Proof. We verify that (23) and (24) satisfy (21). Denote the right-hand side of (21) by $R^{(m)}(y)$After some rearrangement, we get

$$R^{(m)}(y)=\sum_{k=0}^{m}\left\langle{m \atop k}\right\rangle\left[\frac{(1+y)^{m+1}}{x^{m+1}}(1-x)^{k+1} + (1-x+y)\sum_{s=0}^{m}(1+y)^{s}\sum_{j=0}^{s}\frac{(-1)^{j+1}}{x^{s-j+1}}\binom{k+1}{j}\right],$$

using binomial formula and for $k\leq m$, we obtain

$$R^{(m)}(y)=\sum_{k=0}^{m}\left\langle{m \atop k}\right\rangle\left[\sum_{s=m+1}^{m+1}\frac{(1+y)^{s}}{x^{s}}\sum_{j=0}^{s}\binom{k+1}{j}(-1)^{j}x^{j} - \frac{(1-x+y)}{x}\sum_{s=0}^{m}\frac{(1+y)^{s}}{x^{s}}\sum_{j=0}^{s}(-1)^{j}x^{j}\binom{k+1}{j}\right]$$

$$= \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \left[\sum_{s=m+1}^{m+1} \frac{(1+y)^s}{x^s} \sum_{j=0}^{s} \binom{k+1}{j} (-1)^j x^j + \sum_{s=0}^{m} \frac{(1+y)^s}{x^s} \sum_{j=0}^{s} (-1)^j x^j \binom{k+1}{j} - \sum_{s=0}^{m} \frac{(1+y)^{s+1}}{x^{s+1}} \sum_{j=0}^{s} (-1)^j x^j \binom{k+1}{j} \right]$$

$$= \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \left[\sum_{s=0}^{m+1} \frac{(1+y)^s}{x^s} \sum_{j=0}^{s} (-1)^j x^j \binom{k+1}{j} - \sum_{s=0}^{m} \frac{(1+y)^{s+1}}{x^{s+1}} \sum_{j=0}^{s} (-1)^j x^j \binom{k+1}{j} \right]$$

$$= \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \left[\sum_{s=0}^{m+1} \frac{(1+y)^s}{x^s} \left(\sum_{j=0}^{s} (-1)^j x^j \binom{k+1}{j} - \sum_{j=0}^{s-1} (-1)^j x^j \binom{k+1}{j} \right) \right].$$

Finally,

$$\begin{aligned} R^{(m)}(y) &= \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \left[\sum_{s=0}^{k+1} (1+y)^s \left((-1)^s \binom{k+1}{s} \right) \right] \\ &= \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle (-y)^{k+1}. \end{aligned}$$

According to (7) and (11), we have

$$\begin{aligned} \alpha_s^{(m)}(x) &= \sum_{i=0}^{s} \frac{(-1)^{i+s+1}}{x^{i+1}} \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \binom{k+1}{s-i} \\ &= \sum_{i=0}^{s} \frac{(-1)^{i+s+1}(m-s+i)!}{x^{i+1}} \left\{ {m+1 \atop m-s+i+1} \right\} \\ &= \sum_{i=0}^{s} \frac{(-1)^{i+s+1}}{(x)^{i+1}} W_{m,m-s+i} \\ &= \sum_{j=m-s}^{m} \frac{(-1)^{m+j+1}}{x^{s-m+1+j}} W_{m,j}. \end{aligned}$$

It follows from (9) that

$$\begin{aligned}
\alpha^{(m)}(x) &= \frac{1}{x^{m+1}}\sum_{k=0}^{m}\left\langle {m \atop k} \right\rangle (1-x)^{k+1} \\
&= \frac{1-x}{x^{m+1}}\sum_{k=1}^{m}(-1)^{m+k} x^{m-k} k W_{m-1,k-1} \\
&= (1-x)\sum_{k=0}^{m}(-1)^{m+k}\frac{k}{x^{k+1}}W_{m-1,k-1} \\
&= \sum_{k=0}^{m}(-1)^{m+k}\frac{k}{x^{k+1}}W_{m-1,k-1} - \sum_{k=0}^{m}(-1)^{m+k}\frac{k}{x^{k}}W_{m-1,k-1} \\
&= \sum_{k=0}^{m}(-1)^{m+k}\frac{k}{x^{k+1}}W_{m-1,k-1} + \sum_{k=0}^{m-1}(-1)^{m+k}\frac{k+1}{x^{k+1}}W_{m-1,k}.
\end{aligned}$$

Using (8), we get $\alpha^{(m)}(x)$ as desired. This completes the proof. □

Now, integrating the right-hand side of (20) over y, we obtain

$$\int_0^x \frac{\sum_{k=0}^{m}\left\langle {m \atop k} \right\rangle (-y)^{k+1}}{(1+y)^{m+1}(1-x+y)}dy = \alpha_m^{(m)}(x)\ln(1-x^2) + \sum_{s=0}^{m-1}\frac{\alpha_s^{(m)}(x)}{m-s}\left[1-(1+x)^{s-m}\right]. \quad (25)$$

By differentiating (25) we get the ordinary generating function of $S_n^{(m)}$

$$\begin{aligned}
\sum_{n\geq 0} S_n^{(m)} x^n &= \ln(1-x^2)\frac{d}{dx}\alpha_m^{(m)}(x) + \sum_{s=0}^{m-1}\frac{\frac{d}{dx}\alpha_s^{(m)}(x)}{m-s}\left(\left[1-(1+x)^{s-m}\right]\right) \\
&\quad + \sum_{s=0}^{m-1}\left((1+x)^{s-m-1}\alpha_s^{(m)}(x)\right) - \frac{2x}{1-x^2}\alpha_m^{(m)}(x), \quad (26)
\end{aligned}$$

with

$$\frac{d}{dx}\alpha_s^{(m)}(x) = \sum_{j=m-s}^{m}\frac{(s-m+1+j)(-1)^{m+j}}{x^{s-m+2+j}}W_{m,j}.$$

With Proposition 6, we can now rewrite (26) as follows

Theorem 7. *For any real numbers x such that $|x|<1$ and for all nonnegative integer m, we have*

$$\sum_{n\geq 0} S_n^{(m)} x^n = \left(\sum_{j=0}^{m} \frac{(1+j)(-1)^{m+j}}{x^{2+j}} W_{m,j}\right) \ln(1-x^2)$$
$$+ \sum_{0\leq j\leq s\leq m-1} \frac{(-1)^j}{x^{s-j+2}} W_{m,m-j} \left(\frac{s-j+1}{m-s}\left(1-(1+x)^{s-m}\right) - x(1+x)^{s-m-1}\right)$$
$$+ \frac{2}{1-x^2} \sum_{j=0}^{m} \frac{(-1)^{m+j}}{x^j} W_{m,j}. \quad (27)$$

In particular for $m = 0$ and $m = 1$, we get

$$\sum_{n\geq 0} S_n^{(0)} x^n = \frac{2}{1-x^2} + \frac{\ln(1-x^2)}{x^2},$$

and

$$\sum_{n\geq 0} S_n^{(1)} x^n = \frac{2+3x}{x(1+x)^2} + \frac{2-x}{x^3} \ln\left(1-x^2\right).$$

5 The asymptotic expansion

In the previous sections, $S_n^{(m)}$ becomes more complex when, m grows, so it is important to have asymptotic expansion of $S_n^{(m)}$.

Theorem 8. *For $m > 0$, we have*

$$S_{2n}^{(m)} \sim (2n)^m \text{ and } S_{2n+1}^{(m)} \sim -(2n+1)^m.$$

Proof. Write

$$k^m = c_0 + c_1(k+1) + c_2(k+1)(k+2)\cdots + c_m(k+1)\cdots(k+m),$$

where c_i's depnd on m $(c_m = 1)$. we immediately have

$$S_n^{(m)} = c_0 \sum_{k=0}^{n} (-1)^k \binom{n}{k}^{-1} + c_1 \sum_{k=0}^{n} (-1)^k (k+1) \binom{n}{k}^{-1} + \cdots + \sum_{k=0}^{n} (-1)^k (k+1)\cdots(k+m) \binom{n}{k}^{-1}.$$

After some rearrangement, we have

$$S_n^{(m)} = c_0 T_n^{(0,0)} + c_1(n+1) T_n^{(0,1)} + c_2(n+1)(n+2) T_n^{(0,2)} + \cdots + (n+1)\cdots(n+m) T_n^{(0,m)}.$$

Since $T_{2n}^{(0,p)} \to 1$ and $T_{2n+1}^{(0,p)} \to -1$, the result holds. □

6 A connection to Akiyama-Tanigawa matrix

In this section we consider $T_n^{(m,p)}$. The following lemma will be useful in the proof of the main theorem of this section.

Lemma 9. *For $m \geq 1$, we have*

$$\sum_{k=0}^{n} k^m z^k = \sum_{k=0}^{m} W_{m,k} \left(\frac{z}{1-z}\right)^{k+1} - z^{n+1} \sum_{s=0}^{m} \binom{m}{s} (n+1)^{m-s} \sum_{k=0}^{s} (-1)^{s+k} W_{s,k} (1-z)^{-k-1}.$$

Proof. Recall that, for $m \geq 1$

$$\sum_{k=0}^{\infty} k^m z^k = \sum_{k=0}^{m} W_{m,k} \left(\frac{z}{1-z}\right)^{k+1} = \sum_{k=0}^{m} (-1)^{m+k} W_{m,k} (1-z)^{-k-1},$$

we have

$$\begin{aligned}
\sum_{k=0}^{n} k^m z^k &= \sum_{k=0}^{\infty} k^m z^k - \sum_{k=n+1}^{\infty} k^m z^k \\
&= \sum_{k=0}^{m} W_{m,k} \left(\frac{z}{1-z}\right)^{k+1} - z^{n+1} \sum_{i=0}^{\infty} (i+n+1)^m z^i \\
&= \sum_{k=0}^{m} W_{m,k} \left(\frac{z}{1-z}\right)^{k+1} - z^{n+1} \sum_{i=0}^{\infty} \sum_{s=0}^{m} \binom{m}{s} (n+1)^{m-s} i^s z^i \\
&= \sum_{k=0}^{m} W_{m,k} \left(\frac{z}{1-z}\right)^{k+1} - z^{n+1} \sum_{s=0}^{m} \binom{m}{s} (n+1)^{m-s} \sum_{i=0}^{\infty} i^s z^i,
\end{aligned}$$

as desired. □

For an alternative proof see Boyadzhiev [3]. The main result of this section is to prove the following theorem which expresses explicitly the alternating sums of the reciprocals of binomial coefficients, $T_n^{(m,p)}$, in terms of Akiyama-Tanigawa matrix $A_{n,k}$.

Theorem 10. *For nonnegative integers n, m and p, we have*

$$T_n^{(m,p)} = \binom{n+p}{p}^{-1} \delta_{0m} + (n+p+1) \sum_{s=0}^{m} (-1)^{n+s} \binom{m}{s} (n+1)^{m-s} A_{s,n+p+1} - \frac{n+p+1}{n+1} \sum_{s=0}^{m} (-1)^s \binom{n+s+p+2}{p+s+1}^{-1} W_{m,s}, \quad (28)$$

where $A_{i,j}$ is the Akiyama-Tanigawa matrix.

Proof. By the Beta function we can write

$$\begin{aligned} T_n^{(m,p)} &= \sum_{k=0}^{n} (-1)^k k^m (p+n+1) \int_0^1 x^{p+k} (1-x)^{n-k}\, dx \\ &= (p+n+1) \int_0^1 x^p (1-x)^n \sum_{k=0}^{n} k^m \left(\frac{-x}{1-x}\right)^k dx. \end{aligned}$$

Using the lemma, we get

$$\begin{aligned} T_n^{(m,p)} &= (p+n+1) \int_0^1 x^p (1-x)^n \Bigg(\sum_{k=0}^{m} W_{m,k} (-x)^{k+1} \\ &\quad - \sum_{s=0}^{m} \binom{m}{s} (n+1)^{m-s} \sum_{k=0}^{s} (-1)^{s+k} W_{s,k} (-x)^{n+1} (1-x)^{k-n} \Bigg) dx \\ &= \frac{(p+n+1)}{n+1} \sum_{k=0}^{m} (-1)^{k+1} \binom{n+k+p+2}{p+k+1}^{-1} W_{m,k} \\ &\quad - (p+n+1) \sum_{s=0}^{m} \sum_{k=0}^{s} \binom{m}{s} (n+1)^{m-s} \frac{(-1)^{n+s+k+1}}{k+1} \binom{n+k+2+p}{k+1+p}^{-1} W_{s,k}. \end{aligned}$$

Finally, from (12) we obtain

$$\begin{aligned} T_n^{(m,p)} &= \frac{(p+n+1)}{n+1} \sum_{k=0}^{m} (-1)^{k+1} \binom{n+k+p+2}{p+k+1}^{-1} W_{m,k} \\ &\quad + (n+p+1) \sum_{s=0}^{m} \binom{m}{s} (n+1)^{m-s} \frac{(-1)^{n+s}}{(n+p+1)!} \sum_{k=0}^{n+p+1} (-1)^k \begin{bmatrix} n+p+2 \\ k+1 \end{bmatrix} B_{s+k}. \end{aligned}$$

As desired, this completes the proof. □

Setting $p = 0$ in (28) we can rewrite (14) as follows

Corollary 11.

$$S_n^{(m)} = \delta_{0m} - A_{m+1,n} + \sum_{s=0}^{m} (-1)^{n+s} \binom{m}{s} (n+1)^{m-s+1} A_{s,n+1}.$$

7 Recurrence Relation For $T_n^{(m,p)}$

Theorem 12. *For any nonnegative integers m, n and p*

$$T_{n+1}^{(m,p)} = \binom{n+p+1}{p}^{-1} \delta_{0m} - \frac{1}{n+p+1} \sum_{i=0}^{m+1} \left(\binom{m+1}{i} + p \binom{m}{i} \right) T_n^{(i,p)}.$$

Proof. The proof is similar to that of Theorem 2. □

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