Weyl's Equidistribution Theorem

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Introduction

Consider the unit circle T in the Euclidean plane. If it is rotated like a stationary wheel, in an anti-clockwise direction by 45 degrees, then the `spoke of the wheel' joining the centre (0; 0) to the point (1; 0) (call it v) gets mapped to the spoke joining the centre to the point (Cos 45, Sin 45). We shall think of points on the plane as complex numbers when convenient.

Clearly after seven more rotations v returns to its original position at (1; 0). Notice that 45 degrees is ¼=4 radians, which is a rational multiple of ¼ A moment's thought tells us that, if instead of ¼=4 radians, we rotate by any angle µ which is a rational multiple of ¼radians, say, ¼a=b, then again v returns to its original position after a ¯nite number (at most 2b) of repetitions of this rotation. On the other hand, a rotation by an angle ® which is an irrational multiple of ¼radians never returns v to its original position. In fact, it gets arbitrarily close to any radial position and, what is more, the positions of v after a large number of repetitions of this rotation, seem to be `uniformly scattered'. This is a theorem of Hermann Weyl and will be proved in this article. Note that for °, an irrational real number, a simple application of the pigeon-hole principle shows that the sequence of fractional parts of integral multiples of ° is dense in (0; 1). This fact seems to have been known from early 14th century itself. N Oresme (1320-1382) considers two bodies moving on a circle with uniform but incommensurable velocities and writes, "No sector of a circle is so small that two such mobiles could not conjunct in it at some future time and could not have conjuncted in the past."

Weyl, a doyen of early twentieth century mathematics, presented in 1909 a result, which later came to be known as Weyl's equidistribution theorem. Weyl worked in diverse spheres of mathematics, among them, continuous groups and matrix representations. It was during his research into representation theory that Weyl discovered his theorem on equidistribution. Subsequently a vast amount of literature was devoted to the review of his proof. However, there remain to this day, several unanswered questions which arose in the aftermath of Weyl's discovery.

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Equidistribution

What is Equidistribution?

Let $(u_n)_{n>0}^{\infty}$ be a sequence of elements from the interval [0,1]. Let a, b such that $[a, b] \subset [0, 1]$. For each $n \in \mathbb{N}$, we define $s_n(a, b)$ to be number of integers k, $1 \le k \le n$ for which $u_k \in [a, b]$. Then (u_n) is said to be equidistributed in [0,1] if $\forall\, a, b : [a, b] \subset [0, 1]$

$$\lim_{n \to \infty} \frac{s_n(a, b)}{n} = b - a.$$

Denote the fractional part of any $x \in \mathbb{R}$ by $\langle x \rangle$; notice that $\langle x \rangle \in [0, 1]$ and $x - \langle x \rangle \in \mathbb{Z}$.

If we begin with any sequence (u_n) of real numbers, then we say that (u_n) is equidistributed modulo 1 if the sequence $(\langle u_n \rangle)$ of its fractional parts is equidistributed in [0,1]. Equidistribution is also known as uniform distribution.

A natural question is:

Is $\langle \sqrt{n} \rangle$ equidistributed?

The answer is 'yes' as we shall shortly show.

For a sequence (u_n) in $(0, 1)$, define its discrepancy as

$$D_N = \mathrm{Sup}\,\left\{\left|\frac{s_N(a, b)}{N} - (b - a)\right| ; 0 \le a \le b \le 1\right\}.$$

The sequence $\sqrt{n}$ of square roots of natural numbers is equidistributed modulo 1.

The property of equidistribution of (u_n) can also be expressed in terms of the discrepancy as follows. First, let us define another variant of D_N as

$$D_N^* = \sup\left\{\left|\frac{s_N(0,a)}{N} - a\right|; 0 \le a \le 1\right\}.$$

Let us compare D_N and D_N^*. It is evident $D_N^* \le D_N$.

On the other hand, let $\epsilon > 0$ and $(a,b) \subset (0,1)$. Then,

$$s_N(a,b) \le s_N(0,b) - s_N(0,a-\epsilon).$$

Therefore, as $\epsilon \to 0$, we get $D_N \le 2D_N^*$. In other words,

$$D_N^* \le D_N \le 2D_N^*.$$

Therefore, $D_N \to 0 \Leftrightarrow D_N^* \to 0$ as $N \to \infty$. If $D_N \to 0$, then (u_n) is equidistributed in $(0,1)$ by definition. The converse is also true but we do not need it. Thus we may use $s_n(0,\alpha)$ instead of $s_n(a,b)$ as we have proved the equivalence of the two definitions.

Let us get back to the problem of equidistribution of $\langle\sqrt{n}\rangle$.

If $\alpha \in (0,1)$, let us now evaluate the number of integers n such that

$$\langle\sqrt{n}\rangle \in [0,\alpha].$$

For any n, let $d = [\sqrt{n}]$, the greatest integer less than or equal to $\sqrt{n}$. Now, $0 \le \langle\sqrt{n}\rangle \le \alpha$ implies that $d \le \sqrt{n} \le d+\alpha$. So, $d^2 \le n \le (d+\alpha)^2 = d^2 + 2d\alpha + \alpha^2$. For a given d, there are $1 + [2d\alpha + \alpha^2]$ such n. Moreover, for any other d, these are disjoint since $(d+\alpha)^2 < (d+1)^2$.

In other words, for any d, the cardinality $s_{d^2}(0,\alpha)$ of $\{k : 0 \le k \le d^2; \langle\sqrt{k}\rangle \le \alpha\}$ equals $\sum_{i=0}^{d-1}(1 + [2i\alpha + \alpha^2])$. Therefore, for any n and for $d = [\sqrt{n}]$, we have

$$|s_n(0,\alpha) - n\alpha| = |s_n(0,\alpha) - s_{d^2}(0,\alpha) + s_{d^2}(0,\alpha) - n\alpha|$$

$$\le |s_n(0,\alpha) - s_{d^2}(0,\alpha)| + |s_{d^2}(0,\alpha) - n\alpha|$$

$$\leq n - d^2 + \left|\sum_{i=0}^{d-1}(1 + [2i\alpha + \alpha^2]) - n\alpha\right| < 2d + 1 + \left|\sum_{i=0}^{d-1}(2i\alpha + 2]) - n\alpha\right|;$$

which gives easily that

$$|s_n(0;\alpha) - n\alpha| < 7d + 2 \leq 7\sqrt{n} + 2.$$

In other words, $\left|\frac{s_n(0;\alpha)}{n} - \alpha\right| \to 0$ as $n \to \infty$. We have shown that $D_N^* \to 0$ as $N \to \infty$. Therefore, $\langle\sqrt{n}\rangle$ is equidistributed in $(0,1)$ by the remark below.

A similar argument with $(\log(n+1))_{n \in \mathbb{Z}^+}$ tells us that $s_n(0;\frac{1}{2})$ for n of the form $[e^{k+\frac{1}{2}}]$, $k \in \mathbb{Z}$ fails to converge to $1/2$. So $(\log(n+1))_n$ is not equidistributed modulo 1.

The sequence log n; $n \geq 2$ is equidistributed modulo 1.

Weyl's Criterion

A sequence (u_n) of real numbers is equidistributed modulo 1 if, and only if, for all $k \in \mathbb{N}$, $\frac{1}{N}\sum_{n=0}^{N} e^{2i\pi k u_n} \to 0$ as $N \to \infty$.

A special case of this is already very interesting:

Let γ be an irrational, real number. Then

$$\lim_{n\to\infty} \frac{1}{n}|\{k : 1 \leq k \leq n : \langle k\gamma\rangle \in [a,b]\}| = b - a$$

for each pair a, b such that $[a,b] \subset [0,1]$.

In other words, the sequence $(n\gamma)$ is equidistributed modulo 1.

The proof is constructive and one can check how the techniques work, using a particular γ, say, $\sqrt{2}$.

Proof of Weyl's Criterion

The crux of the proof lies in finding a suitable upper bound for the discrepancy. Set $\sigma_r(N) = \sum_{n<N} e^{2i\pi r u_n}$.

We claim that $\forall R \geq 1$, and $(a,b) \subset (0,1)$,

$$|s_N(a,b) - N(b-a)| < 2\sum_{r<R} |\sigma_r(N)| + \frac{4N}{\pi}\sqrt{\sum_{r>R} \frac{|\sigma_r(N)|}{r^2 N}}.$$

Let us first show that the claim proves the criterion.

Now, clearly $|\frac{\sigma_r(N)}{N}| \leq 1$. Also, $\sum_{r>R}\frac{1}{r^2} \leq \int_R^{\infty}\frac{dx}{x^2} = \frac{1}{R}$.

So, $D_N \leq 2\sum_{1\leq r\leq R} |\frac{\sigma_r(N)}{N}| + \frac{4}{\pi\sqrt{R}}$. By the hypothesis, for all r, the first term tends to zero as $N \to \infty$. Therefore, limit superior $\limsup D_N \leq \frac{4}{\pi\sqrt{R}}$. Since R is arbitrary, $D_N \to 0$ i.e., (u_n) is equidistributed modulo 1.

Let us now prove the claim made.

Let $(a,b) \subset (0,1)$ and $\varepsilon > 0$. If $b - a + 2\varepsilon < 1$, we define a function F as a periodic function with period 1, which is linear on each of the intervals $[a-\varepsilon, a]$ and $[b, b+\varepsilon]$ and is the constant 1 on $[a,b]$ and vanishes on $[b+\varepsilon, a+1-\varepsilon]$. Such a periodic function has a Fourier series expansion $F(x) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi kx}$.

Recall that we have defined F above in case $b-a+2\varepsilon < 1$. When $b-a+2\varepsilon \geq 1$, we define a function G just like F but with a, b replaced by $a+\varepsilon$ and $b-\varepsilon$, respectively.

Let us consider the case $b - a + 2\varepsilon < 1$ first.

Note that $s_N(a,b) = \sum_{n<N} F(u_n) = \sum_{k\in\mathbb{Z}} c_k\sigma_k(N) \leq (b-a+\varepsilon)N + 2\sum_{r\geq 1}|c_r||\sigma_r(N)|$ since $c_0 = b-a+\varepsilon$.

Thus, $s_N(a,b) - (b-a)N \leq \varepsilon N + 2\sum_{r\geq 1}|c_r||\sigma_r(N)|$.

Now, if $b-a+2\varepsilon \geq 1$, then $N \leq (b-a+2\varepsilon)N$; so $s_N(a,b) \leq N \leq (b-a+2\varepsilon)N$.

Hence, in either case,

$$s_N(a,b) - (b-a)N \leq 2\varepsilon N + 2\sum_{r\geq 1}|c_r||\sigma_r(N)|.$$

On the other hand, similarly, $s_N(a,b) \geq (b-a+2\varepsilon)N - 2\sum_{r\geq 1}|c_r||\sigma_r(N)|$.

Therefore, for any $\varepsilon > 0$ and any $N \geq 1$, we get by our assumption, that $D_N \leq 2\varepsilon + 2\sum_{r\geq 1} |c_r| |\frac{\sigma_r(N)}{N}|$.

It is easy to see from the expression

$c_k = \int_{a-\varepsilon}^{a+1-\varepsilon} e(-kx)F(x)dx$ c_k for $k \neq 0$ that $|c_k| \leq \frac{1}{\pi^2 k^2 \varepsilon}$.

Using this, and taking $\varepsilon = \frac{1}{\pi}\sqrt{\sum_{r<R} \frac{|\sigma_r(N)/N|}{r^2}}$, the claim follows. This completes the proof of Weyl's criterion.

Application to Prime Number Theory

Most of the deep, exciting applications of Weyl's theorem require a knowledge of abstract measure theory (see [1]) or of number theory. We discuss one application to number theory.

Let p_n denote the nth prime number. We investigate the behaviour of the sequence $(\log p_n)_{n\in N}$.

The formula $\lim_{n\to\infty} \left(\frac{p_n}{n\log n}\right) = 1$ is equivalent to the so-called prime number theorem (see [2]).

Suppose now that the sequence $(\log p_n)$ has equidistribution modulo 1.

Define N_k and M_k as follows:

$$\begin{aligned} N_k &= \inf\{n : p_n > e^k\} \\ M_k &= \inf\{n : p_n > e^{k-1/2}\}. \end{aligned}$$

Let χ be the periodic function with period one, defined by

$$\chi(x) = \begin{cases} 1 & \forall x \in [0; \frac{1}{2}) \\ 0 & \forall x \in [\frac{1}{2}; 1) \end{cases}$$

$$\sum_{n<M_k} \chi(\log p_n) = \sum_{n<N_k} \chi(\log p_n).$$

By our hypothesis,

$$\frac{1}{M_k}\sum_{n\leq M_k} \chi(\log p_n) \text{ and } \frac{1}{N_k}\sum_{n\leq N_k} \chi(\log p_n)$$

The fractional parts of log *p* as *p* runs over prime numbers, is not equidistributed modulo 1.

For almost all $\alpha>1$ (in the sense of the Lebesgue measure), the sequence (α^n) is equidistributed modulo 1.

have the same limit, say l, as $k \to \infty$. If this limit is not zero, then

$$\frac{N_k}{M_k} \to 1 \text{ as } k \to \infty .$$

Let $\pi(x)$ be the member of prime numbers less than or equal to x. The famous prime number theorem asserts (see [2]) that

$$\pi(x) \sim \frac{x}{\log x} \text{ as } x \to \infty .$$

Therefore, as $k \to \infty$,

$$N_k = \pi(e^k) \sim \frac{e^k}{k} \sim \frac{e^k}{k-\frac{1}{2}} \sim M_k \sqrt{e}.$$

Thus gives a contradiction to the assumption of equidistributivity of $(\log p_n)$ modulo 1 if we can show that the limit of $\frac{1}{M_k}\sum_{n<M_k} \chi(\log p_n)$ as $k \to \infty$, if it exists, is non-zero.

Now $\sum_{n<M_k} \chi(\log p_n) \geq |\{p : k-1 \leq \log p < k-1/2\}| = \pi(e^{k-1/2}) - \pi(e^{k-1})$.

So, $\lim_{k\to\infty} \frac{1}{M_k}\sum_{n<M_k} \chi(\log p_n) \geq 1 - e^{-1/2} > 0$.

An Unsolved Question

Here, we present one of the simpler problems from [1]. The problem of characterising those γ with $\langle n\gamma\rangle$ equidistributed was solved completely by the condition that γ is irrational. However, we have still not succeeded in characterising those α for which $\langle \alpha^n\rangle$ is equidistributed.

A result due to Koksma asserts:

For almost all $\alpha > 1$ (in the sense of the Lebesgue measure), the sequence (α^n) is equidistributed modulo 1.

For example let $\alpha = \frac{1+\sqrt{5}}{2}$. By solving the difference equation $u_{r+1} = u_r + u_{r-1}$ with initial conditions $u_0 =$

$2 \; u_1 = 1$ or, simply by induction, we see that

$$u_r = \left(\frac{1+\sqrt{5}}{2}\right)^r + \left(\frac{1-\sqrt{5}}{2}\right)^r$$

is a solution and that u_r is always an integer. But

$$\left(\frac{1-\sqrt{5}}{2}\right)^r \quad < \; 0 \text{ for } r \text{ odd}$$
$$\geq \; 0 \text{ for } r \text{ even}.$$

Moreover, $\left(\frac{1-\sqrt{5}}{2}\right)^r \to 0$ as $r \to \infty$.

Therefore

$$\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{2r+1}\right\} \to 0 \text{ as } r \to \infty$$
$$\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{2r}\right\} \to 1 \text{ as } r \to \infty;$$

Hence

$$\left|\frac{1}{n}\left\{1 \leq r \leq n : \left\{\left(\frac{1+\sqrt{5}}{2}\right)^r\right\} \in \left(\frac{1}{4},\frac{3}{4}\right)\right\}\right| \to 0 \text{ as } n \to \infty;$$

which shows that the sequence $\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$ is not equidistributed modulo 1.

Suggested Reading

[1] **D Parent, *Exercises in Number Theory*, Springer-Verlag, 1978.**

[2] **B Sury, Bertrand's postulate, *Resonance*, Vol.7, No.6, p.77-87, 2002.**

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