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Adelic Profinite Groups

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INTRODUCTION

Let us call a profinite group Δ *adelic*, if Δ is a subgroup of the product $\prod_p SL_n(\mathbb{Z}_p)$ for some $n \geq 2$. One of the main cases of interest is when Δ is the profinite completion of an S-arithmetic subgroup of a semisimple algebraic group. Let k be an algebraic number field, S be a finite set of places of k containing the infinite ones, G be a semisimple, connected algebraic group over k, and Γ be an S-arithmetic group. We say that Γ satisfies the congruence subgroup property if the S-congruence kernel $C^S(G)$ is a finite group (see [PR2, Chap. 9]). If $n \geq 3$, and $S = \{\infty\}$, the S-congruence kernel $C^S(SL_n)$ is trivial, and so we can identify the profinite completion $\widehat{SL_n(\mathbb{Z})}$ of $SL_n(\mathbb{Z})$ with the product $\prod_p SL_n(\mathbb{Z}_p)$.

The purpose of this note is twofold. Firstly, we prove some results on bounded generation for an arbitrary finitely generated adelic, profinite group. Then, we apply these results to the case of arithmetic groups to prove the following conjecture of A. Lubotzky.

Conjecture. *If the profinite completion $\hat{\Gamma}$ of an S-arithmetic group Γ is adelic, then Γ satisfies the congruence subgroup property.*

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It is easy to observe that conversely, the congruence subgroup property for an S-arithmetic group Γ implies the existence of an embedding of $\hat{\Gamma}$ as an adelic group for some SL_n (see Section 1). Hence this property can be thought of as an abstract topological characterisation of S-arithmetic groups with of as an abstract topological characterisation of S-arithmetic groups with the congruence subgroup property. We may assume throughout the paper that S-arithmetic groups are infinite, i.e., that the group $\prod_{v \in S} G(k_v)$ is not compact.

1. GENERALITIES ON ADELIC GROUPS

Recall that a profinite group Δ is said to have bounded generation (of degree $\leq r$), if there are elements $g_1, \ldots, g_r$ (not necessarily distinct) such that $\Delta = \overline{\langle g_1 \rangle} \cdots \overline{\langle g_r \rangle}$. It is well known (see [PR1, L1]) that for profinite completions of arithmetic groups, this property characterises those with the congruence subgroup property. Moreover, for a pro-p group Δ, this is equivalent (loc.cit) to Δ being a p-adic analytic group.

The following lemma is probably well known but we could not find an explicit reference.

LEMMA 1. *Let Δ be a finitely generated, profinite group. If Δ is nilpotent, then* any *subgroup of Δ has bounded generation.*

Proof. This can be proved by induction on the nilpotency class. If Δ is abelian, it is evident. Let us now assume that Δ is not abelian and that bounded generation holds for finitely generated profinite groups of nilpotency class smaller than that of Δ. Let us write $\Delta = \overline{A}$, where A is a finitely generated (nilpotent) group. It is well known that any subgroup (in particular $[A, A]$) of A is finitely generated. But, clearly, $[A, A]$ is dense in $D(\Delta) = \overline{[\Delta, \Delta]}$. Therefore, $D(\Delta) = \overline{[\Delta, \Delta]}$ is finitely generated (notice that the discussion shows that any subgroup of Δ is topologically finitely generated). But the induction hypothesis, it follows that $D(\Delta)$ has bounded generation. Since the abelian quotient $\Delta/D(\Delta)$ obviously has bounded generation, it is evident that Δ itself has bounded generation. Since any subgroup of Δ is finitely generated, it also has bounded generation.

COROLLARY 1. *A finitely generated, pro-p group Δ which is nilpotent, is a p-adic analytic group.*

Remark. This result is not true for solvable groups. For instance, for any prime p, the group $P = C_p \wr \mathbb{Z}_p$ which is the wreath product of a cyclic group of order p with $\mathbb{Z}_p$, is a solvable pro-p group which is not analytic. The reason is that P has as subquotients the elementary abelian groups C_p^n for arbitrary n.

LEMMA 2. *Suppose* Γ *satisfies the congruence subgroup property. Then,* $\hat{\Gamma}$ *is adelic, i.e., the profinite completion* $\hat{\Gamma} \hookrightarrow \prod_p SL_n(\mathbb{Z}_0)$ *for some* n.

Proof. We are given that the S-congruence kernel $C = C^S(G)$ is finite and central in the exact sequence

$$1 \to C \to \hat{\Gamma} \xrightarrow{\pi} \overline{\Gamma} \to 1,$$

where $\overline{\Gamma}$ is the completion of Γ in the S-congruence topology. Now, since C is finite, there is an open subgroup of $\hat{\Gamma}$ which intersects C trivially. In other words, there is a subgroup Γ_0 of finite index in Γ such that the profinite completion $\widehat{\Gamma_0} \cong \overline{\Gamma_0}$. Now, the S-congruence completion $\overline{\Gamma_0}$ is isomorphic to $\prod_{v \notin S} U_v$, where the U_v are open compact in $G(O_v)$ and $U_v = G(O_v)$ for almost all v. Clearly, the product $\prod_{all\ v} G(O_v)$ can be embedded in $\prod_p SL_n(\mathbb{Z}_p)$ for some large n (we can first go to the $\mathbb{Q}$-group $R_{k/\mathbb{Q}}G$, and then go to GL_n via a $\mathbb{Q}$-representation, and finally put GL_n in SL_{n+1}). We have shown that $\widehat{\Gamma_0}$ is an adelic group. Since it is of finite index in $\hat{\Gamma}$, the latter can be embedded as an adelic group by means of the induced representation.

Remark. (i) It is not difficult to show (see [L2]) that the profinite completion of a free nonabelian group cannot be embedded in the product $\Pi_p GL_n(\mathbb{Z}_p)$ for any $n \geq 1$. In particular, the profinite completion of $SL_2(\mathbb{Z})$ is not adelic.

(ii) It is known (see [PR1, theorem 2]) that for a simple, simply connected $\mathbb{Q}$-algebraic group M, the adelic group $\prod_p M(\mathbb{Z}_p)$ has bounded generation.

It seems natural to pose:

QUESTION. *Does every finitely generated, profinite, adelic group have bounded generation*?

We show that this is true for pro-p groups. In fact, we prove the following general result. Before stating it, we recall that for any set of Π of primes, a finite Π-group is a group whose order is a product of primes from Π. Define a pro-Π group to be a projective limit of finite Π-groups.

THEOREM 1. *Let* Π *be a finite set of primes. Then, any finitely generated pro-*Π *group* H *which is adelic has bounded generation.*

Proof. Write $H \leq \prod_p SL_n(\mathbb{Z}_p)$ or some $n \geq 2$. Let π: $\prod_p SL_n(\mathbb{Z}_p) \to \prod_{p \notin \Pi} SL_n(\mathbb{Z}_p)$ be the natural projection. Then, $\mathrm{Ker}(\pi) \cong \prod_{p \in \Pi} SL_n(\mathbb{Z}_p)$. Also, $H/H \cap \mathrm{Ker}(\pi) \cong \pi(H) \leq \prod_{p \notin \Pi} SL_n(\mathbb{Z}_p)$. Now, since $\pi(H)$ is a pro-Π group, while $SL_n(\mathbb{Z}_p)$ is virtually a pro-p group, $\pi(H) \leq \prod_{p \notin \Pi} S_p$, where S_p are finite. Moreover, by Jordan's theorem, there is an abelian normal subgroup A_p and S_p of index at the most $(49n)^{n^2}$ (see [D, p. 98]). Let us call $\pi(H)$ as P for simplicity; then P is a finitely generated profinite group. Also $P/P \cap \prod_{p \notin \Pi} A_p \leq \prod_{p \notin \Pi} S_p/A_p$ is a finitely generated profinite group of finite exponent. By Zelmanov's solution of the restricted Burnside problem (see [Z]), such a group is actually finite. Hence $P \cap \prod_{p \notin \Pi} A_p$, being of finite index in P, is also finitely generated. But, this intersection is an abelian group. Hence, this intersection has bounded generation. Hence P has bounded generation as well. If we show that $H \cap \operatorname{Ker}(\pi) \leq \operatorname{Ker}(\pi) \cong \prod_{p \in \Pi} SL_n(\mathbb{Z}_p)$ has bounded generation, it would follow that H itself has bounded generation. Now, $H \cap \operatorname{Ker}(\pi)$ is a pro-Π subgroup of $\prod_{p \in \Pi} SL_n(\mathbb{Z}_p)$. Hence, $H \cap \operatorname{Ker}(\pi)$ has an open subgroup Δ of finite index such that $\Delta \leq \prod_{p \in \Pi} SL_n(p, \mathbb{Z}_p)$. Here $SL_n(p, \mathbb{Z}_p)$ denotes the subgroup consisting of matrices congruent to the identity modulo p, and is, therefore, a pro-p group. Hence, it suffices to show that Δ has bounded generation. Now, Δ is pronilpotent. So, it is a direct product of its Sylow pro-p subgroups Δ_p and $\Delta_p \leq SL_n(\mathbb{Z}_p)$. Therefore Δ_p, $p \in \Pi$ are analytic; hence, they have bounded generation, and so does the finite product.

Remark. The above proof works without using the theorem of Zelmanov if all the primes in Π are greater than n. The reason is as follows. As before, we have $P := \pi(H) \leq \prod_{p \notin \Pi} S_p$. We claim that the finite groups S_p are actually abelian. For this consider $S_p \leq GL(n, \mathbb{Z}_p) \leq GL(n, \overline{\mathbb{Q}_p})$. This representation of S_p is completely reducible. If an irreducible component has degree d, then d divides the order of S_p, so d is of the form $\prod_{p \in \Pi} p^{a_p}$. But, $d \leq n < p$ for each $p \in \Pi$, which means that $d = 1$. In other words, for each $p \notin \Pi$, the group S_p is conjugate (in $GL(n, \overline{\mathbb{Q}_p})$) to the group of diagonal matrices. This shows that every S_p, $p \notin \Pi$ is abelian. Now, P has bounded generation, as it is a finitely generated abelian profinite group. Therefore, H also has bounded generation as before.

2. CONGRUENCE SUBGROUP PROPERTY

In this section, we first show that for any set Π of primes, any pro-Π quotient of a finitely generated profinite group Δ is the quotient of a pro-Π subgroup of Δ. This is an easy consequence of a result of Huppert [H, Satz 11] for finite groups (see also [CKK][1]). We use the above results along with results of [PR1] to show that adelic arithmetic groups are precisely the arithmetic groups with the congruence subgroup property

[1]Thanks are due to A. Lubotzky for this reference.

(abbreviated CSP). Recall that the Frattini subgroup $\Phi(\Delta)$ of a profinite group Δ is the intersection of all maximal open subgroups. It is pro-nilpotent.

PROPOSITION 1. *Let Δ be a finite generated profinite group. Let ϕ: $\Delta \to P$ be a continuous, surjective homomorphism with kernel K. Then, there exists a subgroup H of Δ such that $\phi(H) = P$ and $K \cap H \leq \Phi(H)$.*

Further, H has the property that if Π is a set of primes such that P is pro-Π, then H is also pro-Π.

Proof. We first choose a closed subgroup H of Δ that is minimal among closed subgroups with the property $\phi(H) = P$. The existence of such an H is guaranteed by Zorn's lemma as follows. Let $\mathcal{C}$ be a chain of closed subgroups with the above property and if the intersection $C_0 = \bigcap \mathcal{C}$ does not satisfy $\phi(C_0) = P$, there exists $x \in P$ such that the inverse image $\phi^{-1}(x)$ misses C_0. But, since Δ is compact, this inverse image misses a finite intersection. But, this is a contradiction. Thus, we do have a minimal closed subgroup H of Δ with the property $\phi(H) = P$. Now, we show that $K \cap H \leq \Phi(H)$. Suppose not. Then, some maximal open subgroup M of H does not contain $K \cap H$. By maximality of M, the subgroup $(K \cap H)M = H$. But, then $\phi(M) = P$ and M is a closed subgroup smaller than H. This contradiction proves the first part of the proposition.

Assume now that P is pro-Π. Obviously, H is pro-$\tilde{\Pi}$ for some set $\tilde{\Pi}$ containing Π. Suppose, if possible, that there is a prime p in $\tilde{\Pi}$ but not in Π. Let S_p be a Sylow pro-p subgroup of H. So, $S_p \leq K \cap H$. We know that $K \cap H \leq \Phi(H)$ and is, therefore, pro-nilpotent. Since S_p is the unique Sylow pro-p subgroup of $K \cap H$, it is normal in H. By the analogue of the Schur−Zassenhaus theorem for profinite groups, S_p has a complement L in H. But, then $\phi(L) = \phi(H) = P$, which contradicts the minimality of H. Hence we must have $S_p = \{e\}$. This proves the proposition.

COROLLARY 2. *Let Δ be a finitely generated profinite group. Let Π be any set of primes. Then, any pro-Π quotient of Δ is a quotient of a pro-Π subgroup of Δ.*

Putting together Theorem 1 and Corollary 2 we have:

COROLLARY 3. *If H is a finitely generated profinite group which is adelic, and if Π is a finite set of primes, then pro-Π quotients of H have bounded generation.*

Let k be a number field, S be a finite set of places containing the archimedean ones, and G be a simple, simply connected algebraic k-group. Let T be the (finite) set of places of k, defined by

$$T = \{v \quad \textit{finite} : G(K_v) \textit{ is compact}\}.$$

Let Γ be an S-arithmetic group. Assume that the profinite completion $\hat{\Gamma}$ is adelic, i.e., for some $n \geq 2$, $\hat{\Gamma} \hookrightarrow \prod_p SL_n(\mathbb{Z}_p)$. Then, a special case of Corollary 3 is:

COROLLARY 4. *For any finite set Π of primes, pro-Π quotients of $\hat{\Gamma}$ have bounded generation. In particular, for any prime p, the pro-p completion of Γ is an analytic pro-p group.*

Let us recall the so-called standard description of normal subgroups of $G(k)$. Let T be, as before, the finite set of places of k, defined by $T = \{v \ \textit{finite} : G(k_v) \textit{ is compact}\}$. Then, we say that $G(k)$ has the standard description of normal subgroups if each non-central normal subgroup of $G(k)$ is open in the T-adic topology given by diagonally embedding $G(k)$ in $\prod_{v \in T} G(k_v)$. This is the analogue of the congruence subgroup property for the group $G(k)$ itself. This was conjectured by Platonov and Margulis and has been proved for all the groups except for those of type A_n where it is known for some n and open for all odd $n \geq 5$ (see [PR2, Chap. 9]).

It should be noted (loc.cit.) that for the CSP to hold for S-arithmetic groups, it is necessary that $G(k)$ has a standard description of normal subgroups, and $S \cap T = \emptyset$.

THEOREM 2. *Let Γ be an S-arithmetic subgroup of G. Suppose the profinite completion $\hat{\Gamma} \hookrightarrow \prod_p SL_n(\mathbb{Z}_p)$ for some $n \geq 2$. Assume that $G(k)$ has the standard description of normal subgroups and also that $S \cap T = \emptyset$. Then, Γ has the congruence subgroup property.*

Proof. Look at the S-congruence kernel C. We have exact sequences

$$1 \to C \to \widehat{G(k)} \xrightarrow{\pi} \overline{G(k)} = G(\mathbb{A}_S) \to 1$$

$$1 \to C \to \widehat{G(O_S)} \xrightarrow{\pi} \overline{G(O_S)} = \prod_{v \notin S} G(O_v) \to 1,$$

where, for any subgroup H of $G(k)$, $\hat{H}$ and $\overline{H}$ denote its S-arithmetic and S-congruence completions, respectively. Now, it is well known from the work of Prasad-Raghunathan (see [PR2, Theorem 9.15]) that the finiteness of C is equivalent to its centrality in the above sequences. Further, it is easy to see [PR1, Proposition 2] that if C is not central, it is not of type (F) (recall that a profinite group is of type (F) if the number of continuous homomorphisms to any finite group of finite). Let us suppose, if possible, that C is not of type (F). Then, using this and the congruence sequence above [PR1], construct an exact sequence of the type

$$1 \to \prod_I F \to P \xrightarrow{\delta} G_v \to 1.$$

Here F is a finite simple group, I is an infinite set, v is some place of k, G_v is a finite central extension of $G(k_v)$, and, most importantly, the inverse image $\delta^{-1}(W)$ of any open compact subgroup of G_v is a quotient of an open subgroup of $\hat{\Gamma}$. Note also that $\delta^{-1}(W)$ is a pro-Π group, where Π is the finite set consisting of the primes dividing the order of F and the prime q corresponding to v. By our assumption, Γ is adelic, and by Corollary 3, $\delta^{-1}(W)$ has bounded generation for any open compact subgroup W of G_v. But, as proved at the end of Sections 4 and 5 of [PR1], if W is deep enough, the inverse image $\delta^{-1}(W)$ does not have bounded generation. This contradiction proves the theorem.

COROLLARY 5. *With the assumptions as in the theorem, the profinite completion of* Γ *has bounded generation.*

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