Brownian Motion[1]

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1 Introduction

This expository lecture is about Brownian motion, a term I had come across for the first time in a course on physical chemistry at the Madras Christian College. Brownian motion is an outstanding example of a phenomenon / situation in another science giving rise to a rich mathematical theory. It has deep connections with other branches of mathematics as well. We look at some of these aspects.

The heuristics discussed in Section 2 lead to a description of the Wiener measure in Section 3. After a brief discussion of Markov property and heat kernel in Section 4, we take up the probabilistic way of solving the classical Dirichlet problem in Section 5; here arguments involving Brownian motion give an elegant way of solving a purely mathematical problem.

Our approach is informal and no proofs are given. Precise definitions and detailed proofs can be found in the references given at the end. The objective here is just to whet your appetite.

2 Heuristics

Brownian motion is a type of molecular movement encountered when a "solute" particle of certain size is suspended in a medium consisting of a

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very large number of "solvent" particles; the size of the solute particle is quite large compared to that of a solvent particle, but not large enough for it to sediment. All the particles are mobile and the motion of the solute particle (also called Brownian particle) is due to the fluctuations caused by the bombardment from the solvent particles. Such a movement was first observed and studied by Robert Brown, a botanist, when pollen grains were suspended in water; (hence the name: Brownian motion). This phenomenon is generally prevalent in colloidal solutions. Theoretical investigation, from the point of view of physics, was pioneered by Einstein; this resulted in an experiemental verification of the atomic theory of matter.

Painstaking experiments lead to the following observations: the motion is unceasing, haphazard; knowledge of the trajectory of the solute particle upto any specific time does not help in predicting its position with certainty at a future time. This suggests a probabilistic model as follows. We shall assume that there are no external forces acting, that the solvent medium is uniform and that there is no temperature gradient. Let $B(t)$ denote the position of the Brownian particle at time t; for convenience, $B(0) \equiv 0$ is the starting point. The natural probabilistic hypotheses are:
(i) for each $t \geq 0, B(t)$ is a random variable;
(ii) $t \mapsto B(t)$ is continuous;
(iii) for $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n < \infty$ the random variables $B(t_1) - B(0), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are independent; (note that these random variables denote the displacement over non overlapping intervals).

Observe that for any $t \geq 0, n = 1, 2, \ldots$ we can write

$$\begin{aligned} B(t) &= B(t) - B(0) \\ &= B(t) - B\left(\frac{n-1}{n}t\right) + B\left(\frac{n-1}{n}t\right) - B\left(\frac{n-2}{n}t\right) \\ &\quad + \ldots + B\left(\frac{1}{n}t\right) - B(0) \\ &= \text{ Sum of } n \text{ independent random variables.} \end{aligned}$$

Therefore central limit theorem of probability theory (together with continuity in t) leads us to believe that each $B(t)$ is a Gaussian (normal) random variable. Since the Brownian motion is taking place in a homogeneous isotropic medium we can take expectation (average value) of $B(t) \triangleq E(B(t)) = E(B(0)) = 0$. As time passes, the Brownian particle is likely to wander to longer distances about the origin; so variance of $B(t)$ will increase with time.

Let $d \geq 1$ denote the dimension. After suitable centering and scaling, *standard d-dimensional Brownian motion* can be thought of as an $\mathbb{R}^d$-valued stochastic process $\{B(t) : t \geq 0\}$ with independent Gaussian increments, having continuous sample paths, and such that for $0 \leq s < t < \infty$, the increment $B(t) - B(s)$ has the d-dimensional Gaussian distribution $N(0, (t-s)I)$ where I is $(d \times d)$ identity matrix.

3 Wiener measure

To claim that the d-dimensional Brownian motion process, as a mathematical entity, is meaningful one needs the following: a probability space $(S, \mathcal{F}, P)$ and a function $B : [0, \infty) \times S \to \mathbb{R}^d$ such that
1) $t \mapsto B(t, \omega)$ is continuous for any $\omega \in S$; $B(0, \omega) = 0$ for all ω
2) $\omega \mapsto B(t, \omega)$ is an $\mathcal{F}$-measurable function for any $t \geq 0$;
3) for $0 < t_1 < t_2 < \ldots < t_n < \infty$, the increments $B(t_1) - B(0), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are independent random variables having respectively $N(0, t_1 I), N(0, (t_2 - t_1)I), \ldots, N(0, (t_n - t_{n-1})I)$ distributions.

What can be candidates for $(S, \mathcal{F}, P)$ and $B(\cdot, \cdot)$? Are there are natural candidates?

To answer these let $\Omega_d \triangleq C([0, \infty) : \mathbb{R}^d) = \{w : [0, \infty) \to \mathbb{R}^d : w$ continuous$\}$ endowed with the metric

$$\rho(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\sup\limits_{0 \leq t \leq k} \|f(t) - g(t)\|}{1 + \sup\limits_{0 \leq t \leq k} \|f(t) - g(t)\|}.$$

(Ω_d, ρ) is a complete separable metric space; ρ gives the topology of uniform convergence on compact sets. There is a natural Borel structure associated with this topology; let $\mathcal{B}$ denote the Borel σ-algebra of Ω_d.

Now pretend that we have $(S, \mathcal{F}, P), B(\cdot, \cdot)$ satisfying 1) - 3) above. Then one can get a map $\hat{B} : S \to \Omega_d$ given by $(\hat{B}\omega)(\cdot) = B(\cdot, \omega)$ for $\omega \in S$. Since $\mathcal{B}$ is generated by the so called "finite dimensional cylinder sets" in Ω_d, it is not difficult to show that $\hat{B}$ is measurable. So $\hat{B}$ induces a probability measure $P\hat{B}^{-1}$ on $(\Omega_d, \mathcal{B})$ determined by $P\hat{B}^{-1}(E) = P(\hat{B}^{-1}(E))$ for any $E \in \mathcal{B}$. The measure $P\hat{B}^{-1}$ can be regarded as the distribution of the process B.

Let $X_t(w) \equiv X(t,w) \stackrel{\Delta}{=} w(t), t \geq 0, w \in \Omega_d$ denote the t-th coordinate projection. If $(S, \mathcal{F}, P), B(\cdot,\cdot), \hat{B}$ are as above, it is easily checked that, on the probability space $(\Omega_d, \mathcal{B}, P\hat{B}^{-1})$ the stochastic process $\{X_t : t \geq 0\}$ given by the coordinate projections has all the properties of Brownian motion starting at 0, viz. properties 1) - 3) above hold with B replaced by X.

Going back to our question, since $\Omega_d, \mathcal{B}, \{X_t : t \geq 0\}$ are already available, our purpose will be served if we can construct a probability measure P_0 on $(\Omega_d, \mathcal{B})$ such that on $(\Omega_d, \mathcal{B}, P_0)$ the family $\{X_t : t \geq 0\}$ forms a Brownian motion starting at 0. (*Note*: This is similar to constructing appropriate probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to show that random variables with certain distributions exist. For example, how will you prove mathematically that a standard normal random variable exists?)

Thanks to Norbert Wiener, such a probability measure P_0 can be constructed starting from finite dimensional Gaussian distributions; see [KS], [P], [B] for details. So the probability measure P_0 is called the *standard d-dimensional Wiener measure*. For any d-dimensional Brownian motion $B(\cdot,\cdot)$ starting at 0, with $\hat{B}$ defined as above it can be seen that $P\hat{B}^{-1} = P_0$; that is, P_0 is the *distribution* of any standard d-dimensional Brownian motion. All the probabilistic information about the d-dimensional Brownian motion is available in the probability measure P_0.

For $x \in \mathbb{R}^d$ let w_x be the continuous function on $[0, \infty)$ taking the constant value x. Define the probability measure P_x on $(\Omega_d, \mathcal{B})$ by $P_x(E) = P_0(E - w_x), E \in \mathcal{B}$. Under P_x note that $X_t \sim N(x, tI), (X_t - X_s) \sim N(0, (t-s)I)$ and that the increments $X_{t_1} - X_0, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent for $0 \leq t_1 < t_2 < \ldots < t_n < \infty$; that is, under P_x, the process $\{X_t : t \geq 0\}$ is a d-dimensional Brownian motion starting at $x \in \mathbb{R}^d$; P_x is called the *d-dimensional Wiener measure starting at x*.

Brownian motion has several interesting properties. One may recall that Weierstrass had shown the existence of a continuous but nowhere differentiable function. It can be proved that almost all sample paths of a Brownian motion process are nowhere differentiable. In other words, nowhere differentiable functions are "typical" rather than "exceptional". See [KS].

4 Markov property, Heat kernel

An intuitively obvious consequence of the independent increment property is the *Markov (or memoryless) property*: Given the knowledge of "past and present", to predict the probabilistic behaviour at a "future time" the past is irrelevant; that is, for $0 \leq s < t$, Borel set $A \subseteq I\!R^d$,

$$P_x(X(t) \in A | \{X(r) : 0 \leq r \leq s\}) = P_x(X(t) \in A | X(s)). \tag{1}$$

The conditional probability on l.h.s. of (1) is the prediction based on "past and present", whereas r.h.s. of (1) gives the prediction based on "present alone".

We know that the probability density function of $N(x, (t-s)I)$ is

$$p(s, x; t, z) = \left(\frac{1}{2\pi(t-s)} \right)^{d/2} \exp \left\{ -\frac{1}{2(t-s)} \sum_{i=1}^{d} (z_i - x_i)^2 \right\} \tag{2}$$

for $z \in I\!R^d$, where $x \in I\!R^d, 0 \leq s < t < \infty$. It can be shown that

$$P_x(X(t) \in A | X(s) = y) = \int_A p(s, y; t, z) dz \tag{3}$$

so that $p(s, x; t, z)\ dz$ can be interpreted as the "infinitesimal probability" that $X(t) \in dz$ given $X(s) = x$. Thus $p(\cdot, \cdot; \cdot, \cdot)$ given by (2) is the *transition probability density function* of the d-dimensional Brownian motion.

Now, simple differentiation shows

$$\begin{aligned} \frac{\partial p}{\partial t}(s, x; t, z) &= \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2}{\partial z_i^2}\ p(s, x; t, z) \\ &\equiv \frac{1}{2}\ \Delta_z p(s, x; t, z) \end{aligned} \tag{4}$$

which is the *d-dimensional heat (or diffusion) equation;* Δ_z denotes the *d-dimensional Laplacian* in z-variables. Not only does p satisfy the heat equation; but it is also the *fundamental solution to heat equation*; that is solution to the initial value problem for heat equation with initial value f can be expressed as

$$u(t, z) = \int f(x)\ p(0, x; t, z) dx.$$

(This is basically an exercise using dominated convergence theorem and weak convergence of probability measures.) The function p is also called the *heat kernel.* The above discussion indicates that solutions to heat equation can be given in terms of Brownian motion. Indeed this is so, and the celebrated Feynman-Kac formula is an offshoot of this circle of ideas. An interested reader can look up [KS], [V] for details.

5 Brownian motion and the Dirichlet problem

For $d \geq 3$ a simple change of variables gives

$$G(x,z) \triangleq \int_0^\infty p(0,x;t,z)dt = C(d)\left(\frac{1}{|z-x|}\right)^{d-2} \tag{5}$$

for $x \neq z$, where $C(d)$ is a constant depending only on the dimension $d; G$ is the so called Newtonian potential. It may be noted that $z \mapsto G(x,z)$ is harmonic on $\{z \neq x\}$. For $d = 1, 2$ also similar radial harmonic functions can be obtained from p, but will require appropriate centering.

This suggests that there could be connection between Brownian motion and the Laplacian. In fact the connection is very deep. We look at just one aspect of this, where probabilistic arguments are very elegant, viz. probabilistic solution to the classical Dirichlet problem for the Laplacian $\Delta = \sum_{i=1}^{d} \partial^2/\partial x_i^2$.

Let D be a bounded domain with smooth boundary ∂D; let g be a given bounded continuous function on ∂D. Classical Dirichlet problem consists in finding a function $u(\cdot)$ defined on $\bar{D}$ satisfying

$$\left.\begin{array}{ll} \text{(i)} & u \text{ is harmonic in } D; \text{that is} \\ & \Delta u(x) = 0, x \in D; \\ \text{(ii)} & u \text{ is continuous on } \bar{D}; \\ \text{(iii)} & u(x) = g(x), x \in \partial D. \end{array}\right\} \tag{6}$$

Ingredients for solving (6) in terms of Brownian motion are:
(a) a function is harmonic in $D \Leftrightarrow$ it has the mean value property in D;
(b) rotational invariance of Brownian motion (about the starting point); this

enables mean value property to be characterized in probabilistic language; (c) strong Markov property of Brownian motion, viz. analogue of equation (1) still holds if deterministic time s, t are replaced by certain random times called "stopping times".

For $x \in I\!R^d, r > 0$ let $S(x : r) = \{y : \|x - y\| < r\}$ be the open ball with centre x and radius r. Recall that a function f has the *mean value property in D* if for $x \in D, r > 0$ satisfying $\overline{S(x : r)} \subset D$ one has

$$f(x) = \int_{\partial S(x:r)} f(y) \; d\mu_{x,r}(y) \tag{7}$$

where $\mu_{x,r}$ denotes the normalized surface area measure on $\partial S(x : r)$; (see [S] for integration with respect to surface area measure). It is well known that (a) above is a consequence of Green's formula.

As $p(s, x; t, z)$ is a function of $\|z - x\|$ it is clear that the heat kernel is invariant under orthogonal transformations about the initial point x. It can then be shown that if Brownian trajectories are rotated about the starting point, the distribution of Brownian motion does not change.

Now consider $\{X(t) : t \geq 0\}$ under P_0; that is, Brownian motion starting at 0. For $r > 0$ let

$$\begin{aligned} \tau_r(w) &= \inf\{t \geq 0 : X(t, w) = w(t) \notin S(0 : r)\} \\ &= \text{ first exit time of BM from } S(0 : r), \\ X_{\tau_r}(w) &= X(\tau_r(w), w) = w(\tau_r(w)) \\ &= \text{ place at which BM exits for the first time from } S(0 : r). \end{aligned}$$

Continuity of Brownian paths implies that $X_{\tau_r}(\cdot)$ is a random variable taking values in $\partial S(0 : r)$ with distribution $P_0 X_{\tau_r}^{-1}$. Now rotational invariance of Brownian motion shows that $P_0 X_{\tau_r}^{-1}$ is a probability measure on $\partial S(0 : r)$ which is invariant under orthogonal transformations, and hence $P_0 X_{\tau_r}^{-1} = \mu_{0,r}$. See [PS] for a proof.

Similarly if the starting point of the Brownian motion is x, and

$$\tau_r(w) = \inf\{t \geq 0 : X(t, w) \notin S(x : r)\}$$

then $P_x X_{\tau_r}^{-1} = \mu_{x,r}$.

To get a candidate for solution to (6), pretend that $u(\cdot)$ is a solution. Then for $x \in D$,

$$\begin{aligned}
u(x) &= \int\limits_{\partial S(x:r)} u(z)\ d\mu_{x,r}(z) \text{ (by (a) and (7))} \\
&= \int\limits_{\partial S(x:r)} u(z)\ dP_x X_{\tau_r}^{-1}(z) \text{ (since } P_x X_{tau_r}^{-1} = \mu_{x,r}) \\
&= \int\limits_{\Omega_d} u(X_{\tau_r}(w)) dP_x(w) \text{ (by change of variables)} \\
&= E_x[u(X(\tau_r))] \qquad (8)
\end{aligned}$$

where E_x denotes integration (expectation) with respect to P_x. Observe that (8) is a probabilistic version of the mean value property. Also the extremes of (8) now suggest a candidate for solution to (6).

Let $\tau(w) = \inf\{t \geq 0 : X(t, w) \notin D\}$ = first exit time of X from D. Since D is bounded, for any starting point $x \in D$ it can be shown that the Brownian motion makes a first exit from D in finite time; that is, $P_x(\tau < \infty) = 1$ for all $x \in D$. By continuity of trajectories $X(\tau) \in \partial D$. Define

$$u(x) = E_x[g(X((\tau))], x \in \bar{D}. \qquad (9)$$

Note that u is well defined and bounded.

Let $x \in D, r > 0$ be such that $\overline{S(x:r)} \subset D$; let τ_r be as before. Now the random times τ, τ_r are examples of stopping times alluded to earlier; (for a precise definition, etc. see [KS], [PS]); also $\tau_r(w) < \tau(w)$ for a.a. w with respect to P_x, for any $x \in D$; that is, $P_x(\tau_r < \tau) = 1$ for $x \in D$. So using the strong Markov property it can be proved that

$$u(x) = E_x[E_{X(\tau_r)}(g(X(\tau)))] = E_x[u(X(\tau_r))] \qquad (10)$$

where $E_{X(\tau_r)}[\dots]$ denotes $E_y[\dots]$ with $y = X(\tau_r)$. Thus u has the mean value property and hence is harmonic in D. It is, of course, clear that $u = g$ on ∂D.

To prove $u(\cdot)$ is continuous on $\bar{D}$ it is now enough to show that $u(x_n) \to u(z)$ as $x_n \to z$ where $x_n \in D, z \in \partial D$. As g is bounded continuous, this basically amounts to showing $P_{x_n} X_\tau^{-1} \xrightarrow{d} \delta_z$, where δ_z is the degenerate measure given by $\delta_z(H) = 1$ or 0 according as $z \in H$ or $z \notin H$, and $\xrightarrow{d}$ denotes "weak convergence of probability measures" or "convergence in distribution". This can be done as ∂D is smooth. See [B], [P] for information on weak convergence of probability measures, and [KS] for details concerning continuity of u. Thus u defined by (9) solves the Dirichlet problem (6).

Uniqueness can also be established using probabilistic arguments; see [KS].

The strength of the probabilistic method, apart from readily giving an expression for solution, is that it effortlessly takes care of complicated domains and even certain domains with nonsmooth boundary. Continuity of u on $\bar{D}$ is a delicate point which depends on the regularity of the domain. In fact, domains for which the Dirichlet problem is solvable can be characterized in terms of Brownian motion. See [KS], [PS].

Remark: If $d \geq 2$ and $D = S(0 : R)$ for some $R > 0$, it is known that the solution to (6) can be written as

$$u(x) = \int\limits_{\partial S(0:R)} g(z)\ \pi(x,z)\ d\mu_{0,R}(z), x \in D$$

where

$$\pi(x,z) = \frac{R^{d-2}(R^2 - \|x\|^2)}{\|x-z\|^2}, x \in D, z \in \partial D$$

is the Poisson kernel. Comparison with (8) and (9) shows that

$$P_x X_\tau^{-1}(A) = \int\limits_A \pi(x,z) d\mu_{0,R}(z), A \subset \partial S(0:R);$$

that is, the Poisson kernel is the Radon-Nikodym derivative of the hitting measure $P_x X_\tau^{-1}$ with respect to the normalized surface area measure. □

As indicated in the introduction we have barely touched upon some aspects of Brownian motion. An interested reader can consult [D], [KS], [V] and the references therein to know more about the connections between Brownian motion and analysis.

Brownian motion itself is a prototype of a class of Markov processes called diffusion processes, having applications in diverse fields. Study of diffusion processes has produced a plethora of tools and theories like stochastic calculus, stochastic differential equations, martingale problems, ... Prominent among those who have made seminal contributions to the subject is S.R.S. Varadhan, an illustrious alumnus of this college.

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