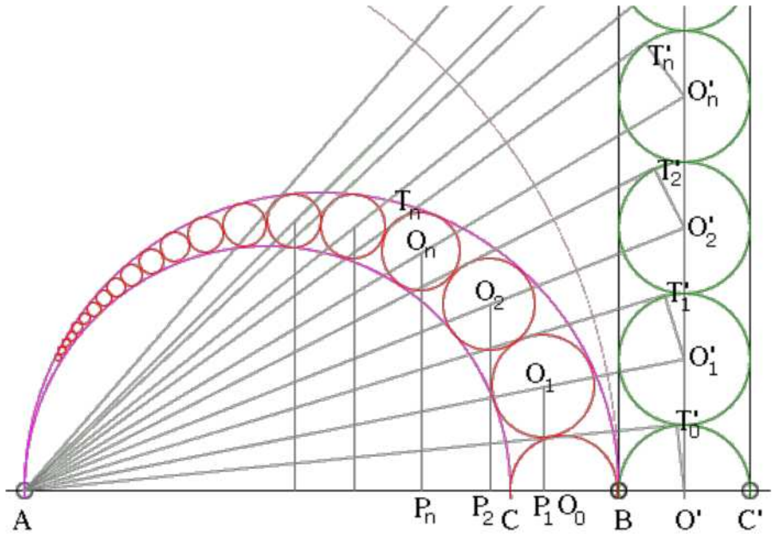

Figure 1: Circle of Inversion

Solution 1 (Ashay Burungale).
More generally, look at the configuration in the figure here. Let C_n be the n-th circle with center O_n and radius r_n. Let P_n denote the projection of O_n on AB. We claim that $r_n = \frac{rab}{n^2b^2+ab+a^2}$.
The key idea is to use inversion. Look at the circle of inversion (see Figure) with radius $= 2a + 2b = 2r$. Now, the vertical line on the left which corresponds to the big semicircles is tangent to both the big

semicircle as well as to the semicircle on the right. Note that the common point of tangency lies on the circle of inversion and is, therefore, fixed. Similarly, the vertical line on the right corresponds to the left semicircles. The point A is the centre of inversion and C' is the inversion of C. The points of tangency of the common tangents through A of a pair of corresponding circles (like T_n and T'_n in the figure) play a double role. On the one hand, they are homologous, corresponding under the implied homothety and, on the other hand, they are antihomologous mapping to each other under inversion. In other words, denoting $\frac{b}{a}$ by t, we have the equalities

$$\frac{AT'_n}{AT_n} = \frac{r'_n}{r_n} = \frac{rt}{r_n}$$

and

$$AT'_n AT_n = (2r)^2.$$

Therefore, $(AT'_n)^2 = \frac{4r^3t}{r_n}$.
But, we also have $(AT'_n)^2 = (2r + tr)^2 + (2ntr)^2 - (tr)^2$.
Combining these, we get $r_n = \frac{rt}{n^2t^2+t+1}$ which proves the claim.
The original problem is the case $n = 1$.
Also solved by Santosh Nadimpalli, Prithwijit De, Sahil Mhaskar.

Solution 2.
It is easy to see that $r = cos(n\theta)$ has n petals or $2n$ petals according as to whether n is odd or even.
Let P denote the point (x_n, y_n) where $2y_n$ is the 'width' of the rose petal. Consider the obtuse angle α made by the line OP with the tangent at P. Then $tan\alpha = \frac{r}{\frac{dr}{d\alpha}}$. As y_n is maximal, the tangent at P is horizontal; that is, $\alpha + \theta_n = \pi$. Hence $r + tan\theta_n \frac{dr}{d\alpha} = 0$. Thus,

$$ntan\theta_n = cot(n\theta_n).$$

When $n = 2$, this gives $tan\theta_2 = \frac{1}{\sqrt{5}}$ and therefore,

$$y_2 = rsin\theta_2 = cos(2\theta_2)sin\theta_2 = \frac{\sqrt{6}}{9}.$$

For general n, one can either express $cot(n\theta_n)$ as a rational function of $tan\theta_n$ and use $ntan\theta_n = cot(n\theta_n)$ or do the following which is easier :
Differentiating $y = rsin\theta$ with respect to θ and set $\frac{dy}{d\theta}$ equal to 0. If θ_n is the smallest critical value, then

$$cos\theta_n cos(n\theta_n) = nsin\theta_n sin(n\theta_n).$$

Adding $ncos\theta_n cos(n\theta_n)$, we get

$$(n + 1)cos\theta_n cos(n\theta_n) = ncos(n - 1)\theta_n.$$

The Chebychev polynomials $T_n(X)$ defined by $T_n(cos\theta) = cos(n\theta)$ give us

$$cos\theta_n T_n(cos\theta_n) = \frac{n}{n+1} T_{n-1}(cos\theta_n).$$

Note that the T_n's can be obtained easily by comparing the real parts of

$$(cos\theta + isin\theta)^n = cos(n\theta) + isin(n\theta).$$

Also solved by Santosh Nadimpalli.

Solution 3 (Ashay Burungale).
We claim a^3 is in the center for each a. Indeed by computation, since $a^4 = a$, we get $(a^3b - a^3ba^3)^2 = 0 = (ba^3 - a^3ba^3)^2$.
Thus, $a^3b - a^3ba^3 = (a^3b - a^3ba^3)^4 = 0 = (ba^3 - a^3ba^3)^4 = ba^3 - a^3ba^3$.
Thus, $a^3bc = a^3ba^3c$ and so a^3R is a ring with unity. Note that if these rings are commutative, then so is R because $a^3bc = a^3cb$ gives with $b = a$ that $ac = ca$. So, we may assume R has a unity, say 1.
Note that $2^4 = 2$ and $3^4 = 3$ gives $14 = 0 = 78$ which implies $2 = 0$.
Now $(1+a)^3 = 1 + a + a^2 + a^3$ means $a + a^2$ is in the center. Thus, $a + b + (a+b)^2 = a + b + a^2 + b^2 + ab + ba$ is in the center. So, $ab + ba$ is central. In particular, $a(ab + ba) = (ab + ba)a$; that is, a^2 is central. As $a + a^2$ is already in the center, so is a.

Solution 4 (Ashay Burungale).
More generally, we prove that if $b > 1$ and $b^n - 1$ divides a, then the base b expression of a has at least n non-zero digits. Let m be the minimal number of non-zero digits of any non-zero multiple of $b^n - 1$. Among all multiples with m non-zero digits, suppose $A = a_1b^{k_1} + \cdots + a_mb^{k_m}$ has the smallest digit-sum. Here $0 \leq a_i < b$ and $k_1 > k_2 > \cdots > k_m$. The key claim is that the powers k_i are all distinct mod n. If this is proved, then it would follow that the number $C = a_1b^{r_1} + \cdots + a_mb^{r_m}$ where $r_i < n$ and $r_i \equiv k_i$ mod n, is a multiple of $b^n - 1$ but is less than or equal to $(b-1)(1 + b + \cdots + b^{n-1} < b^n$. Thus, $C = b^n - 1$ and so $m = n$. Let us prove now that k_i's are distinct mod n. Suppose $i < j$ and $k_i \equiv k_j$ mod n. Choosing d large enough such that $k_j + dn > k_1$, consider the number $B = A - a_ib^{k_i} - a_jb^{k_j} + (a_i + a_j)b^{k_j+dn}$. This is a multiple of $b^n - 1$ as $B - A = a_ib^{k_i}(b^{k_j-k_i+dn} - 1) + a_jb^{k_j}(b^{dn} - 1)$. Note that by minimality of the number m of non-zero digits, the number $a_i + a_j$ must be $\geq b$. But, then the digit-sum of B is clearly (digit-sum for

$A) - a_i - a_j + 1 + (a_i + a_j - b)$ which is less than the digit-sum for A. This contradicts the choice of A. Therefore, the claim is proved and the main assertion follows.

Solution 5.
If $t < 4$, there is nothing to prove. Suppose $t \geq 4$. Solving for t_1, t_2 with $t = t_1 + t_2 = t_1 t_2$, we get

$$t_1 = \frac{t - \sqrt{t^2 - 4t}}{2}\ ,\ t_2 = \frac{t + \sqrt{t^2 - 4t}}{2}.$$

Note that $1 < t_1 \leq 2$ from $t \geq 4$. If $t_2 < 4$, then we are through. Otherwise, work with t_2 in place of t. As $t_2 + 1 < t_2 + t_1 = t$, the size decreases by more than 1 and this process will lead to a number less than 4 after finitely many steps. Note that this algorithm produces a set $t_1, t_2, \cdots, t_k$ of numbers, all less than 4, having the stronger property that

$$t_1 + \cdots + t_k = t_1 t_2 \cdots t_k;$$

$$t_2 + \cdots + t_k = t_2 \cdots t_k;$$

$$\vdots$$

$$t_{k-1} + t_k = t_{k-1} t_k.$$

Solution 6.
Consider $p(S) - q(T) \in \mathbf{Q}[S, T]$. Writing the decomposition

$$p(S) - q(T) = f_1(S, T) \cdots f_r(S, T)$$

into irreducible polynomials and using the 'Hilbertian' property of $\mathbf{Q}$ stated in the problem, there are infinitely many rational numbers t such that for each $i \leq r$, $f_i(S, t) \in \mathbf{Q}[S]$ is irreducible. Now, the hypothesis implies that for each $t \in \mathbf{Q}$, there is some $s \in \mathbf{Q}$ with $p(s) = q(t)$; that is, there is some $i \leq r$ (depending on t) such that $f_i(s, t) = 0$. Therefore, there is some $i \leq r$ such that for infinitely many $t \in \mathbf{Q}$, the polynomial $f_i(S, t) \in \mathbf{Q}[S]$ has a root in $\mathbf{Q}$. Being irerducible, this means that f_i must have degree 1 in S. In other words, $f_i(S, T) = f(T)S + g(T)$ with $f, g \in \mathbf{Q}[T]$ and f non-zero. We may write

$$p(S) - q(T) = h(S, T)(f(T)S + g(T))$$

for some $h(S,T) \in \mathbf{Q}[S,T]$.
Specializing $S = -\frac{g(T)}{f(T)}$, we get $p(-\frac{g(T)}{f(T)}) = q(T)$.
So $-\frac{g(T)}{f(T)}$ is integral (i.e., satisfies a monic polynomial) over $\mathbf{Q}[T]$. But then it must be a polynomial itself.

Solution 7.
As C is closed and bounded, there exists $P \in C$ with OP maximal. IF P coincides with O, then evidently C is just the single point O (a disc of radius zero). Suppose P is not O. Let Q be any point on the circle C_0 with centre O passing through P. If the angle $\angle QOP$ is θ, consider the sequence of points Q_n on the line segment OQ defined by $OQ_n = OP(\cos\frac{\theta}{n})^n$. Note that $OQ_n = OP\cos\frac{\theta}{n}(\cos\frac{\theta}{n})^{n-1}$. Now, $OP\cos\frac{\theta}{n}$ is the base of a right triangle with hypotenuse OP. By hypothesis, the point $Q_{n,1} \in C$ where $OQ_{n,1} = OP\cos\frac{\theta}{n}$ and $\angle Q_{n,1}OP = \frac{\theta}{n}$. In this manner, for each $k \leq n$, the point $Q_{n,k} \in C$ where $OQ_{n,k} = OQ_{n,k-1}\cos\frac{\theta}{n}$ and $\angle Q_{n,k}OQ_{n,k-1} = \frac{\theta}{n}$. Thus, $Q_n \in C$. Evidently, $Q_n \to Q$. As C is closed, $Q \in C$. Therefore, each point of the circle C_0 belongs to C. By maximality of OP, it is clear that no point outside C_0 can be in C. Finally, it is clear that each point inside C_0 is in C as the corresponding radial segment intersects C.
Also solved by Ashay Burungale.

Solution 8.
The generating function $1+\sum_{n\geq 1} p_2(n)t^n = \prod_{k\geq 1}\frac{1}{(1-t^k)^k}$ (see P.A.Macmahon's 1916 book 'Combinatorial analysis' for a proof). The interesting thing is that Macmahon's conjectured values

$$1 + \sum_{n\geq 1} p_d(n)t^n = \prod_{k\geq 1}\frac{1}{(1-t^k)^{\binom{k+d-2}{d-1}}}$$

are known to be false for each $d \geq 3$ (!) Nevertheless, there is still hope that the exponent $\binom{k+d-2}{d-1}$ could be replaced by one of the same degree $d-1$ to get a correct generating function. For instance, when $d = 3$, a paper by V.Mustonen & R.Rajesh (see arXiv:cond-mat/0303607v1) shows that when $d = 3$, if there is an exponent of degree 2 in k, it must be of the form $(0.5 \pm 0.012)k^2$. As yet, the question is quite open.