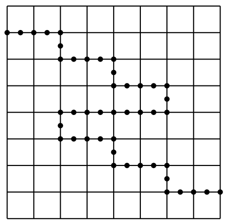
Solutions to July-August Problems



The solution to the carpet problem (a) is as follows. After cutting the 8×8 piece in the manner indicated above, one slides the top piece two squares to the right and brings it down by a square. The 1×6 piece fits exactly in the gap in the middle. This solves (a). The part (b) is exactly similar.

The problem on the distribution of land documents has two parts, the first of which is the standard so-called pirates problem.
The solution for part (a) is as follows. Let $P_1, \cdots, P_{20}$ be the MLA's in increasing order of clout; so P_1 gets to vote first etc. Then, the distribution is : $0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 991$ documents, respectively, for $P_1, \cdots, P_{20}$. It is easy to see that this vote is passed on by the people $P_2, P_4, P_6, \cdots, P_{20}$.

The variant posed as part (b) is due to Stephen Omohundro and appeared in Ian Stewart's column in Sceintific American of May 1999. Let us see how far the previous argument carries over when there are more people than 20. The earlier analysis shows that the same pattern persists for a while - until 200. The most powerful person P_{200} proposes no documents for the odd-numbered people $P_1, P_3, \cdots, P_{199}$ and one document each for the even $P_2, P_4, \cdots, P_{198}$ and himself. Let us examine the next few cases carefully and that will give us the general pattern.
If there are 201 people, then what does P_{201} propose? He needs to survive and is, therefore, forced to propose one document each for the odd-numbered people $P_1, P_3, \cdots, P_{199}$ and nothing for the even-numbered people and nothing for himself !
Now, if there are 202 people, then P_{202} is also forced to propose nothing for himself and he must propose to bribe 100 people, who must be among the people who would get nothing under P_{201}'s proposal. As there are 101 such people, there are 101 ways for P_{202} to propose distibution of one document each among 100 of them. Note that under any of P_{202}'s 101-possible proposals, the odd-numbered people $P_1, P_3, \cdots, P_{199}$ definitely get nothing. Among $P_2, P_4, \cdots, P_{200}, P_{201}$ one person gets nothing and the others get a document each.

If there are 203 people, the most powerful P_{203} has to get 102 favourable votes for his proposal and since he has only 100 documents for bribing, he will be thrown out no matter what he proposes !
If there are 204 people, then P_{204} knows that he can count on the vote of P_{203} who would want to survive; so, he has his own vote and that of P_{203} apart from 100 others who he can bribe. That would give the 102 favourable votes for him to survive (without getting any document though) !
Continuing in this manner, we see in the cases when there are $205, 206$ or 207 people, the top person gets thrown out. When there are 208 people, he can survive by bribing 100 people other than $P_{205}, P_{206}, P_{207}$ whose votes he is assured of as they would want to survive. He thus gets 104 votes. From this analysis, we note :
The people who are lucky enough to be able to survive are P_{200+2^k} for some k.
Therefore, for solving (b), look at the largest number (viz, $200 + 2^8 = 456$) not more than 500. The persons $P_{457}, P_{458}, \cdots, P_{500}$ get thrown out no matter what they propose and then P_{456} makes the following proposal so as to be sure that it is accepted. He proposes one document each for $P_1, P_3, \cdots, P_{199}$ and nothing to the others.

In the number problem, note that n must be square-free. One easily sees that $2, 3, 6, 7, 43$ divide n. Moreover, any such n must be divisible by these numbers (and perhaps others). Because of the hypothesis that a prime $p|n$ iff $(p-1)|n$, if a new prime factor of n arises, it must be one more than a product of smaller prime factors of n. However, the above numbers cannot give a new prime because the numbers

$$2.43 + 1, 2.3.43 + 1, 2.7.43 + 1, 2.3.7.43 + 1$$

are all composite. Therefore, the unique answer to the problem is the number $2.3.7.43 = 1806$.