Liftings of Covariant Representations of $W^*$-correspondences

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Outline

1 Introduction
   • Representations
   • Preliminaries

2 Dilations and Liftings
   • Dilations
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3 References
For a Hilbert $C^*$-module $\mathcal{G}$, let $\mathcal{L}(\mathcal{G})$ the set of all adjointable operators on $\mathcal{G}$.

A $\mathcal{G}$ over a von Neumann algebra $\mathcal{M}$ can be equipped with the $\sigma$-topology induced by $f(.) = \sum_{n=1}^{\infty} \omega_n(\langle \xi_n, . \rangle)$ where $\sum ||\omega_n||||\xi_n|| < \infty$.

$\mathcal{G}$ is called self-dual if $\forall \phi : \mathcal{G} \rightarrow \mathcal{M} \quad \exists \xi_{\phi} \in \mathcal{G}$ so that $\phi(\xi) = \langle \xi_{\phi}, \xi \rangle$, $\xi \in \mathcal{G}$.

For self-dual $\mathcal{G}$, $\mathcal{L}(\mathcal{G})$ is a von Neumann algebra.

A $W^*$-correspondence $\mathcal{E}$ is a self-dual Hilbert $C^*$-bimodule over $\mathcal{M}$, where the left action $\varphi : \mathcal{M} \rightarrow \mathcal{L}(\mathcal{E})$ is normal.

$\varphi(a)\eta = a\eta \quad \forall a \in \mathcal{M}, \eta \in \mathcal{E}$.
For a Hilbert $C^*$-module $G$, let $L(G)$ the set of all adjointable operators on $G$.

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$G$ is called self-dual if $\forall \phi : G \rightarrow M \ \exists \xi_\phi \in G$ so that $\phi(\xi) = \langle \xi_\phi, \xi \rangle$, $\xi \in G$.

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A $W^*$-correspondence $E$ is a self-dual Hilbert $C^*$-bimodule over $M$, where the left action $\varphi : M \rightarrow L(E)$ is normal.

$\varphi(a)\eta = a\eta$ $\forall a \in M, \eta \in E$. 
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Covariant representations

Definition

A pair $(T, \sigma)$ is called a covariant representation of $\mathcal{E}$ over $\mathcal{M}$ on a Hilbert space $\mathcal{H}$, if

(i) $T : \mathcal{E} \to B(\mathcal{H})$ is a linear map that is continuous (w.r.t. $\sigma$ and ultra weak topology)

(ii) $\sigma : \mathcal{M} \to B(\mathcal{H})$ is a normal homomorphism

(iii) $T(a\xi) = \sigma(a)T(\xi)$, $T(\xi a) = T(\xi)\sigma(a)$ \hspace{1cm} $\xi \in \mathcal{E}, a \in \mathcal{M}$.

Moreover if $(T, \sigma)$ satisfies

$$T(\xi)^* T(\eta) = \sigma(\langle \xi, \eta \rangle), \hspace{1cm} \xi, \eta \in \mathcal{E}$$

it is called isometric.
Covariant representations

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A pair \((T, \sigma)\) is called a covariant representation of \(E\) over \(M\) on a Hilbert space \(\mathcal{H}\), if

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σ-dual

For $\mathcal{E}$ over $\mathcal{M}$ and a normal $\sigma : \mathcal{M} \to B(\mathcal{H})$ the induced tensor product $\mathcal{E} \otimes_\sigma \mathcal{H}$ is the unique Hilbert space such that:

$$\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2 \rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2 \rangle)h_2 \rangle, \quad \xi_1, \xi_2 \in \mathcal{E}; h_1, h_2 \in \mathcal{H}.$$  

Define $\sigma$-dual of $\mathcal{E}$ as

$$\mathcal{E}^\sigma := \{ \mu \in B(\mathcal{H}, \mathcal{E} \otimes_\sigma \mathcal{H}) : \mu \sigma(a) = (\varphi(a) \otimes 1)\mu \quad \forall a \in \mathcal{M} \}.$$  

Let $(T, \sigma)$ of $\mathcal{E}$ on $\mathcal{H}$ be such that $T$ is bounded. Then $\tilde{T} : \mathcal{E} \otimes \mathcal{H} \to \mathcal{H}$ can be associated with

$$\tilde{T}(\eta \otimes h) := T(\eta)h, \quad \eta \in \mathcal{E}, h \in \mathcal{H}.$$  

\(\sigma\)-dual

- For \(\mathcal{E}\) over \(\mathcal{M}\) and a normal \(\sigma : \mathcal{M} \to B(\mathcal{H})\) the induced tensor product \(\mathcal{E} \otimes_\sigma \mathcal{H}\) is the unique Hilbert space such that:

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$$\tilde{T}(\eta \otimes h) := T(\eta)h, \quad \eta \in \mathcal{E}, h \in \mathcal{H}.$$
Lemma

let \((T, \sigma)\) be a covariant representation of \(\mathcal{E}\)

(i) \(T\) is completely contractive \(\iff \|\tilde{T}\| \leq 1\)

(ii) \((T, \sigma)\) is isometric if and only if \(\tilde{T}\) is an isometry.

\(\tilde{T}^*\), when bounded is an element of \(\mathcal{E}^\sigma\) and converse.
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Definition

Let \((C, \sigma_C)\) be a contractive covariant representation of \(\mathcal{E}\) on \(\mathcal{H}_C\). Then a contractive covariant representation \((E, \sigma_E)\) of \(\mathcal{E}\) on a \(\mathcal{H}_E \supset \mathcal{H}_C\) is called a contractive lifting of \((C, \sigma_C)\) if

(i) \(\sigma_E(a)|_{\mathcal{H}_C} = P_{\mathcal{H}_C}\sigma_E(a)|_{\mathcal{H}_C} = \sigma_C(a)\) \(a \in \mathcal{M}\)

(ii) \(\mathcal{H}_C^{\perp}\) is invariant w.r.t. \(E(\xi)\) for all \(\xi \in \mathcal{E}\)

(iii) \(P_{\mathcal{H}_C}E(\xi)|_{\mathcal{H}_C} = C(\xi)\) for all \(\xi \in \mathcal{E}\)

Set \(\mathcal{H}_A := \mathcal{H}_C^{\perp}\), \(A(\xi) := E(\xi)|_{\mathcal{H}_A}\) and \(\sigma_A(a) := \sigma_E(a)|_{\mathcal{H}_A}\) for all \(\xi \in \mathcal{E}, a \in \mathcal{M}\).
Liftings

Definition

Let $(C, \sigma_C)$ be a contractive covariant representation of $\mathcal{E}$ on $\mathcal{H}_C$. Then a contractive covariant representation $(E, \sigma_E)$ of $\mathcal{E}$ on a $\mathcal{H}_E \supset \mathcal{H}_C$ is called a contractive lifting of $(C, \sigma_C)$ if

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(ii) $\mathcal{H}_\perp_C$ is invariant w.r.t. $E(\xi)$ for all $\xi \in \mathcal{E}$

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Set $\mathcal{H}_A := \mathcal{H}_\perp_C$, $A(\xi) := E(\xi)|_{\mathcal{H}_A}$ and $\sigma_A(a) := \sigma_E(a)|_{\mathcal{H}_A}$ for all $\xi \in \mathcal{E}$, $a \in \mathcal{M}$. 
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1. $\sigma_E(a)|_{\mathcal{H}_C} = P_{\mathcal{H}_C}\sigma_E(a)|_{\mathcal{H}_C} = \sigma_C(a)$ for all $a \in \mathcal{M}$
2. $\mathcal{H}_C$ is invariant w.r.t. $E(\xi)$ for all $\xi \in \mathcal{E}$
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Definition

Let \((T, \sigma)\) be a completely contractive covariant (c.c.c. for short) representation of \(\mathcal{E}\) on \(\mathcal{H}\).

An isometric dilation \((V, \pi)\) of \((T, \sigma)\) is an isometric covariant representation of \(\mathcal{E}\) on \(\tilde{\mathcal{H}} \supset \mathcal{H}\) such that \((V, \pi)\) is a lifting of \((T, \sigma)\).

A minimal isometric dilation (\(\text{mid}\)) of \((T, \sigma)\) is an isometric dilation \((V, \pi)\) on \(\hat{\mathcal{H}}\) for which

\[
\hat{\mathcal{H}} = \overline{\text{span}}\{V(\xi_1) \ldots V(\xi_n)h : h \in \mathcal{H}, \xi_i \in \mathcal{E} \text{ for } i = 1, \ldots n\}.
\]

- \(\text{mid}\) is unique up to unitary equivalence.
Definition

Let \((T, \sigma)\) be a completely contractive covariant (c.c.c. for short) representation of \(E\) on \(\mathcal{H}\).

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\]

- \(\text{mid}\) is unique up to unitary equivalence.
Full Fock module

- \( \mathcal{E} \otimes \mathcal{E} \) w.r.t.

\[
\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \varphi(\langle \xi_1, \xi_2 \rangle)\eta_2 \rangle, \quad \xi_1, \xi_2, \eta_1, \eta_2 \in \mathcal{E}.
\]

- **Full Fock module over** \( \mathcal{M} \) is

\[
\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{E} \otimes^n \text{ where } \mathcal{E} \otimes^0 = \mathcal{M}
\]

- For \( \xi \in \mathcal{E} \) \( L_\xi \eta = \xi \otimes \eta \) \( \forall \eta \in \mathcal{E} \)

- Define \( L \otimes 1_{D_T} : \mathcal{E} \to B(\mathcal{F} \otimes D_T) \) by

\[
(L \otimes 1_{D_T})(\xi) = L_\xi \otimes 1_{D_T}.
\]
**Full Fock module**

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  \]

- **Full Fock module** over $\mathcal{M}$ is

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  \mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{E}^\otimes n \quad \text{where} \quad \mathcal{E}^\otimes 0 = \mathcal{M}
  \]

- For $\xi \in \mathcal{E}$ \quad $L_{\xi} \eta = \xi \otimes \eta \quad \forall \eta \in \mathcal{E}$

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- $\mathcal{E} \otimes \mathcal{E}$ w.r.t.

  \[ \langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \varphi(\langle \xi_1, \xi_2 \rangle) \eta_2 \rangle, \quad \xi_1, \xi_2, \eta_1, \eta_2 \in \mathcal{E}. \]

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- For $\xi \in \mathcal{E}$ \quad $L_\xi \eta = \xi \otimes \eta \quad \forall \eta \in \mathcal{E}$

- Define $L \otimes 1_{\mathcal{D}_T} : \mathcal{E} \to B(\mathcal{F} \otimes \mathcal{D}_T)$ by

  \[ (L \otimes 1_{\mathcal{D}_T})(\xi) = L_\xi \otimes 1_{\mathcal{D}_T}. \]
Presentation of mid

Set $D_{*,T} := (1 - \tilde{T}\tilde{T}^*)^{\frac{1}{2}}$ (in $B(\mathcal{H})$) and $D_T := (1 - \tilde{T}^*\tilde{T})^{\frac{1}{2}}$ (in $B(\mathcal{E} \otimes_{\sigma} \mathcal{H})$).

Let $D_{*,T} := \text{range } D_{*,T}$ and $D_T = \text{range } D_T$.

Every c.c.c. representation $(T, \sigma)$ of $\mathcal{E}$ has a mid $(V, \pi)$, with the representation Hilbert space:

$$\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{F} \otimes_{\sigma_1} D_T$$

$$V(\xi) = \begin{pmatrix}
T(\xi) & 0 & 0 & \ldots \\
D_T(\xi \otimes .) & 0 & 0 & \ldots \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\vdots
\end{pmatrix}$$
Presentation of mid

- Set $D_{*, T} := (1 - \tilde{T} \tilde{T}^*)^{\frac{1}{2}}$ (in $B(\mathcal{H})$) and $D_T := (1 - \tilde{T}^* \tilde{T})^{\frac{1}{2}}$ (in $B(\mathcal{E} \otimes_\sigma \mathcal{H})$).
- Let $\mathcal{D}_{*, T} := \text{range } D_{*, T}$ and $\mathcal{D}_T = \text{range } D_T$.
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\vdots & \vdots & \vdots & \ddots
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Let $\mathcal{D}_{*,T} := \text{range } D_{*,T}$ and $\mathcal{D}_T = \text{range } D_T$.

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Let \((E, \sigma_E)\) be a contractive lifting of \((C, \sigma_C)\). Clearly mid \((V^C, \pi_C)\) is embedded in \((V^E, \pi_E)\). We introduce a c.c.c. representation \((Y, \pi_Y)\) on the orthogonal complement \(\mathcal{K}\) of the space of mid \((V^C, \pi_C)\) to encode this.

Hence we can get a unitary \(W\) such that

\[
W : \mathcal{H}_E \oplus (\mathcal{F} \otimes \mathcal{D}_E) \to \mathcal{H}_C \oplus (\mathcal{F} \otimes \mathcal{D}_C) \oplus \mathcal{K}
\]

\[
\hat{V}^E(\xi) W = WV^E(\xi), \quad (\pi_C \oplus \pi_Y)(a) W = W \pi_E(a),
\]

\[
W|_{\mathcal{H}_C} = 1|_{\mathcal{H}_C}, \text{ with } \hat{V}^E(\xi) = V^C(\xi) \oplus Y(\xi)
\]
Let \((E, \sigma_E)\) be a contractive lifting of \((C, \sigma_C)\). Clearly mid \((V^C, \pi_C)\) is embedded in \((V^E, \pi_E)\). We introduce a c.c.c. representation \((Y, \pi_Y)\) on the orthogonal complement \(K\) of the space of mid \((V^C, \pi_C)\) to encode this.

Hence we can get a unitary \(W\) such that

\[
W : \mathcal{H}_E \oplus (\mathcal{F} \otimes \mathcal{D}_E) \rightarrow \mathcal{H}_C \oplus (\mathcal{F} \otimes \mathcal{D}_C) \oplus K
\]

\[
\tilde{V}^E(\xi)W = WV^E(\xi), \quad (\pi_C \oplus \pi_Y)(a)W = W\pi_E(a),
\]

\[
W|_{\mathcal{H}_C} = 1|_{\mathcal{H}_C}, \text{ with } \quad \tilde{V}^E(\xi) = V^C(\xi) \oplus Y(\xi)
\]
Lemma

\((E, \sigma_E)\) is c.c.c. if and only if \((C, \sigma_C)\) and \((A, \sigma_A)\) are c.c.c. and there exists a contraction \(\gamma : D^*_A \rightarrow D_C\) such that

\[
\tilde{B} = D^*_A \gamma^* D_C.
\]

\((A, \sigma_A)\) is called completely non-coisometric (c.n.c.), if

\[
\mathcal{H}^1_A := \{ h \in \mathcal{H}_A : \| (\tilde{A}^n)^* h \| = \| h \| \text{ for all } n \in \mathbb{N} \} = 0.
\]
Lemma

\((E, \sigma_E)\) is c.c.c. if and only if \((C, \sigma_C)\) and \((A, \sigma_A)\) are c.c.c. and there exists a contraction \(\gamma : D_{*,A} \to D_C\) such that

\[
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Reduced liftings

Definition

A completely contractive lifting \((E, \sigma_E)\) of \((C, \sigma_C)\) by \((A, \sigma_A)\) is called reduced if

1. \(\gamma\) is resolving, i.e., for \(h \in H_A\)

\[
\left( \gamma D_{*,A}(A(\xi))^* h = 0 \text{ for all } \xi \in \mathcal{E} \right) \Rightarrow \left( D_{*,A}(A(\xi))^* h = 0 \text{ for all } \xi \in \mathcal{E} \right), \text{ and}
\]

2. \((A, \sigma_A)\) is c.n.c.

Definition

The characteristic function of reduced lifting \((E, \sigma_E)\) of \((C, \sigma_C)\) is defined as

\[
M_{C,E} := P_{\mathcal{F} \otimes \mathcal{D}_C} W |_{\mathcal{F} \otimes \mathcal{D}_E}.
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The characteristic function of reduced lifting \((E, \sigma_E)\) of \((C, \sigma_C)\) is defined as

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M_{C,E} := P_{\mathcal{F} \otimes D_C} W|_{\mathcal{F} \otimes D_E}.
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Main theorem

\[ M_{C,E}(L_\xi \otimes 1_E) = (L_\xi \otimes 1_C)M_{C,E}, \quad \xi \in \mathcal{E}. \]

\[ WH_A = \left[ (\mathcal{F} \otimes \mathcal{D}_C) \oplus \mathcal{K} \right] \ominus WH(\mathcal{F} \otimes \mathcal{D}_E) \]
\[ = \left[ (\mathcal{F} \otimes \mathcal{D}_C) \oplus \Delta_{C,E}(\mathcal{F} \otimes \mathcal{D}_E) \right] \ominus \{ M_{C,E} x \oplus \Delta_{C,E} x : x \in \mathcal{F} \otimes \mathcal{D}_C \} \]

Theorem

For any c.c.c. representation \((C, \sigma_C)\) of \(\mathcal{E}\), the equivalence classes of characteristic functions are complete invariants for reduced liftings of \((C, \sigma_C)\) up to unitary equivalence.

\(M_{C,E}\) is an element of generalized \(H^\infty(\mathcal{D}_E, \mathcal{D}_C)\).
Main theorem

\[ M_{C,E}(L_\xi \otimes 1_E) = (L_\xi \otimes 1_C) M_{C,E}, \quad \xi \in \mathcal{E}. \]

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Theorem

*For any c.c.c. representation \((C, \sigma_C)\) of \(\mathcal{E}\), the equivalence classes of characteristic functions are complete invariants for reduced liftings of \((C, \sigma_C)\) up to unitary equivalence.*
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\[ \mathcal{WH}_A = \left[ (\mathcal{F} \otimes \mathcal{D}_C) \oplus \mathcal{K} \right] \oplus \mathcal{W}(\mathcal{F} \otimes \mathcal{D}_E) \]

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References: