Inclusions of unital C*-algebras and the Rokhlin property

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Motivation

Let $P \subset A$ be an inclusion of unital C*-algebras.

It is a natural question whether there is a relation in several permanent properties between P and A.

For example, let A be a unital C*-algebra and α an (amenable) action from a discrete group G on A, and $A \rtimes_\alpha G$ its crossed product algebra. Then several properties of A can be transmitted to $A \rtimes_\alpha G$.

$$A \subset A \rtimes_\alpha G$$

Conditions for A	G	α	$A \rtimes_\alpha G$
(1) Simplicity	any	outer	$\bigcirc$
(2) Property (SP) + Simplicity	any	outer	$\bigcirc$
(3) Stable rank one	$\mathbf{Z}$	any	≤ 2
(2) + (3)	finite	any	≤ 2
(4) Real rank zero	?	?	?
(5) Extremal richness	finite	Rokhlin	$\bigcirc$
(6)Cancellation + (2) + (3)	finite $\mathbf{Z}$	any $\alpha_* = \mathrm{id}_0$	$\bigcirc$ $\bigcirc$
(7) $\mathcal{Z}$-stability	finite $\mathbf{Z}$	Rokhlin Rokhlin	$\bigcirc$ $\bigcirc$
(8) The order on projections is determined by traces	finite	Rokhlin	$\bigcirc$

To generalize the above results for an inclusion of unital C*-algebras $P \subset A$ we will give an attention to a canonical conditional expectation $E\colon A \rtimes_\alpha G \to A$ by $E(\sum_g a_g u_g) = a_0$, where $u\colon G \to A \rtimes_\alpha G$ is a unitary representation such that $u_g a u_g^* = \alpha_g(a)$ for any $a \in A$ and $g \in G$.

In this talk we assume that there is a faithful conditional expectation $E\colon A \to P$.

The following is the contents of this talk:

1. Simplicity
2. Property (SP)
3. Low ranks
4. Cancellation
5. Rokhlin property
6. Applications

Simplicity

Definition 1 (Osaka:98)**.** Let $1 \in A \subset B$ be a pair of C*-algebras. Then we say that a conditional expectation $E\colon B \to A$ is outer if for any element $x \in B$ with $E(x) = 0$ and any nonzero hereditary C*-subalgebra C of A (i,e, if $c \in C$ and $a \in A$ satisfy $0 \leq a \leq c$, then $a \in C$)

$$\inf\{\|cxc\|\colon c \in C^+, \|c\| = 1\} = 0.$$

We note that Kishimoto showed in [Kishimoto:81] that if A is simple unital C*-algebra and α is a representation of a discrete group G onto the set $\mathrm{Aut}(A)$ of automorphisms on A, is outer, then the canonical conditional expectation from the reduced crossed product algebra $A \rtimes_{\alpha r} G$ to A is outer.

Theorem 2 (O-Teruya:10)**.** Let $1 \in A \subset B$ be a pair of C*-algebras and E be a faithful conditional expectation from B to A. Suppose that A is simple and E is outer. Then B is simple.

For further topics we recall Watatani C*-index theory.

Definition 3 (Watatani:90)**.** Let $P \subset A$ be an inclusion of unital C*-algebras with a conditional expectation E from A onto P.

1. A *quasi-basis* for E is a finite set $\{(u_i, v_i)\}_{i=1}^n \subset A \times A$ such that for every $a \in A$,

$$a = \sum_{i=1}^n u_i E(v_i a) = \sum_{i=1}^n E(a u_i) v_i.$$

2. When $\{(u_i, v_i)\}_{i=1}^n$ is a quasi-basis for E, we define $\mathrm{Index}E$ by

$$\mathrm{Index}E = \sum_{i=1}^n u_i v_i.$$

When there is no quasi-basis, we write $\mathrm{Index}E = \infty$. $\mathrm{Index}E$ is called the Watatani index of E.

Remark 4. We give several remarks about the above definitions.

1. $\mathrm{Index}E$ does not depend on the choice of the quasi-basis in the above formula, and it is a central element of A.

2. Once we know that there exists a quasi-basis, we can choose one of the form $\{(w_i, w_i^*)\}_{i=1}^m$, which shows that $\mathrm{Index}E$ is a positive element.

3. By the above statements, if A is a simple C^*-algebra, then $\mathrm{Index}E$ is a positive scalar.

4. If $\mathrm{Index}E < \infty$, then E is faithful, that is, $E(x^*x) = 0$ implies $x = 0$ for $x \in A$.

5. If $\mathrm{Index}E < \infty$, then there is a basic construction $C^*\langle A, e_p\rangle$ such that

$$C^*\langle A, e_p\rangle = \{\sum_{i=1}^n x_i e_P y_i : x_i, y_i \in A, n \in \mathbf{N}\}$$

   and

$$P \subset A \subset C^*\langle A, e_p\rangle,$$

where e_p is called the Jones projection which satisfies $e_p a e_p = E(a) e_p$ for $a \in A$ and $e_p x = x e_p$ for $x \in P$.

6. If $\mathrm{Index} E$ is finite, then $\mathrm{Index} E$ is a central invertible element of A and there is the dual conditional expectation $\hat{E}$ from $C^*\langle A, e_P\rangle$ onto A such that

$$\hat{E}(x e_P y) = (\mathrm{Index} E)^{-1} xy \quad \text{for } x, y \in A$$

by Proposition 2.3.2 of [Watatani:90]. Moreover, $\hat{E}$ has a finite index and faithfulness.

The following is a model for an inclusion of unital C*-algebras:

Let A be a unital C*-algebra and α an action of a finite group G on A. Suppose that α is outer. Then

$$A^G \subset A \subset A \rtimes_\alpha G$$

is a basic construction.

Theorem 5 (Izumi:02)**.** Let $P \subset A$ be an inclusion of unital C*-algebras with index finite type.

1. Suppose that P is simple, then A can be realized as finite direct sums of simple C*-algebras.

2. Suppose that A is simple, then P can be realized as finite direct sums of simple C*-algebras.

Theorem 6 (O-Teruya:10)**.** Let a conditional expectation $E\colon A \to P$ be of index finite type and let $P \subset A \subset B$ a basic construction, that is, $B = \mathrm{span}\{a_1 e_p a_2 \colon a_1, a_2 \in A\}$ and e_p is the Jones projection correspondent to E.

Suppose that there is a projection $e \in A' \cap A^\infty$ such that $e e_p e = (\mathrm{Index} E)^{-1} e$. Then the dual conditional expectation $\hat{E}\colon B \to A$ is outer. In this case if A is simple, then P is simple.

For a C^*-algebra A, we set

$$
\begin{aligned}
c_0(A) &= \{(a_n) \in l^\infty(\mathbf{N}, A) : \lim_{n\to\infty} \|a_n\| = 0\} \\
A^\infty &= l^\infty(\mathbf{N}, A)/c_0(A).
\end{aligned}
$$

Property (SP)

Definition 7. A C*-algebra A is said to have Property (SP) if any nonzero hereditary C*-subalgebra of A has nonzero projection.

Theorem 8 (Osaka:01)**.** Let $P \subset A$ be an inclusion of unital C*-algebras and $E\colon A \to P$ be a conditional expectation of index finite type.

1. Suppose that A is simple and has Property (SP). Then P has Property (SP).

2. Suppose that P is simple and has Property (SP). Then A has Property (SP).

Low ranks

Here low ranks for C*-algebras mean *stable rank one, real rank zero, and extremal richness*.

Definition 9 (Rieffel:86)**.** For a unital C*-algebra A the *topological stable rank* $\mathrm{tsr}(A)$ of A is defined to be the least integer n such that the set $\mathrm{Lg}_n(A)$ of all n-tuples $(a_1, a_2, \ldots, a_n) \in A^n$ which generate A as a left ideal is dense in A^n.

The topological stable rank of a nonunital C*-algebra is defined to be that of its smallest unitization.

Note that

1. $\mathrm{tsr}(A) = 1$ is equivalent to the density of the set of invertible elements in A.

2. If X is a locally compact Hausdorff space, $\mathrm{tsr}(C_0(X)) = [\frac{1}{2}\dim(X \cup \{\infty\})] + 1$.

3. Let A be an AF algebra. Then $\mathrm{tsr}(A) = 1$.

4. [Rieffel:86] Let A be a C*-algebra and α be an automorphism on A. Then $\mathrm{tsr}(A \rtimes_\alpha \mathbf{Z}) \leq \mathrm{tsr}(A) + 1$.

5. [Putnam:90] Irrational rotation algebras have topologigal stable rank one.

6. [Rieffel:86] Let A be a unital C*-algebra with $\mathrm{tsr}(A) = 1$. Then

$$U(A)/U_0(A) \cong K_1(A).$$

Theorem 10 (O-Teruya:07)**.** Let $P \subset A$ be a unital C*-algebras of index finite and depth 2. Suppose that P is simple with $\mathrm{tsr}(P) = 1$ and Property (SP). Then

$$\mathrm{tsr}(A) \leq 2.$$

The inclusion $1 \in A \subset B$ of unital C*-algebras of index-finite type is said to have *finite depth* k if the derived tower obtained by iterating the basic construction

$$A' \cap A \subset A' \cap B \subset A' \cap B_2 \subset A' \cap B_3 \subset \cdots$$

satisfies $(A' \cap B_k)e_k(A' \cap B_k) = A' \cap B_{k+1}$, where $\{e_k\}_{k\geq 1}$ are projections derived obtained by iterating the basic construction such that $B_{k+1} = C^*(B_k, e_k)$ $(k \geq 1)$ $\quad (B_1 = B, e_1 = e_A)$. Let $E_k : B_{k+1} \to B_k$ be a faithful conditional expectation correspondent to e_k for $k \geq 1$.

Corollary 11. Let A be a simple C*-algebra with $\mathrm{tsr}(A) = 1$ and Property (SP), and α an action of a finite group G on A. Then

$$\mathrm{tsr}(A \rtimes_\alpha G) \leq 2.$$

The following is still a open question.

Problem 12 (Blackadar:90)**.** Let A be a AF C*-algebra and α be an action of a finite group on A. Then $\mathrm{tsr}(A \rtimes_\alpha G) = 1$?

In particular, $\mathrm{tsr}(CAR \rtimes_\alpha \mathbf{Z}/2\mathbf{Z}) = 1$?

The following is a general estimate for stable rank for an inclusion of unital C*-algebras.

Theorem 13 (Kodaka-O-Jeong-Phillips:09)**.** Let $P \subset A$ be an inclusion of unital C*-algebras and $\{(v_k, v_k^*)\}_{k=1}^n$ be a quasi-basis for $E\colon A \to P$. Then

$$\mathrm{tsr}(A) \leq \mathrm{tsr}(P) + n - 1.$$

Corollary 14. Let A be a unital C*-algebra and α be an action from a finite group G on A. Then

$$\mathrm{tsr}(A \rtimes_\alpha G) \leq \mathrm{tsr}(A) + |G| - 1.$$

Definition 15 (Brown-Pedersen:91)**.** For a unital C*-algebra A the *Real rank* $\mathrm{RR}(A)$ of A is defined to be the least integer n such that the set $\mathrm{Lg}_{n+1}(A_{sa})$ of all $n+1$-tuples $(a_0, a_1, , a_2 \ldots, a_n) \in A_{sa}^{n+1}$ which generate A_{sa} as a left ideal is dense in A_{sa}^{n+1}.

The real rank of a nonunital C*-algebra is defined to be that of its smallest unitization.

Note that

1. $\mathrm{RR}(A) = 0$ is equivalent to the density of the set of invertible self-adjoint elements in A_{sa}.

2. If X is a locally compact Hausdorff space, $\mathrm{tsr}(C_0(X)) = \dim(X \cup \{\infty\})$.

3. The only general relation between tsr and RR is $\mathrm{RR}(A) \leq 2\mathrm{tsr}(A) - 1$.

4. [Brown-Pedersen:91] If A is a σ-unital C*-algebra with $\mathrm{RR}(A) = 0$, then it has an approximate identity consisting of projections.

5. Inductive limits of C*-algebras with real rank zero (resp. stable rank one) have real rank zero (resp. stable rank one). In particular, $\mathrm{RR}(AF) = 0$.

Stable rank and real rank play important roles in the classification of simple unital, separable, nuclear, C*-algebras.

Cancellation

Definition 16. For two projections p, q in a C*-algebra A, we write $p \sim q$ if they are Murray-von Neumann equivalent, that is, there is a partial isometry $s \in A$ such that $s^*s = p$ and $ss^* = q$. A C*-algebra A is said to have *cancellation of projections* if whenever $p, q, r \in A$ are projections with $p \perp r$, $q \perp r$, and $p + r \sim q + r$, then $p \sim q$. If the matrix algebra $M_n(A)$ over A has cancellation of projections for each $n \in \mathbf{N}$, we simply say that A has *cancellation*.

Note that

1. Every C*-algebra A with cancellation is stably finite, that is, for $n \in \mathbf{N}$ if $x \in M_n(A)$ satisfies $x^*x = 1$, then $xx^* = 1$.

2. Every unital C*-algebra of stable rank one has cancellation.

3. [Blackadar-Handelman:82] + [Brown-Pedersen:91]

   Let A be a unital C*-algebra with cancellation and $\mathrm{RR}(A) = 0$. Then $\mathrm{tsr}(A) = 1$.

Theorem 17 (Jeong-O-Phillips-Teruya:09)**.** Let $1 \in A \subset B$ be an inclusion of unital C*-algebras of index-finite type and with finite depth. Suppose that A is simple, $\mathrm{tsr}(A) = 1$, and A has Property (SP). Then B has cancellation.

Corollary 18. Let $1 \in A \subset B$ be a pair of unital C*-algebras of index-finite type and with finite depth. Suppose that A is simple with $\mathrm{tsr}(A) = 1$ and Property (SP), and that B has real rank zero. Then $\mathrm{tsr}(B) = 1$.

Corollary 19. Let A be an infinite dimensional simple unital C*-algebra, let G be a finite group, and let α be an action of G on A. Suppose that $\mathrm{tsr}(A) = 1$ and A has Property (SP). Then $A \rtimes_\alpha G$ has cancellation. Moreover, if $A \rtimes_\alpha G$ has real rank zero, then $\mathrm{tsr}(A \rtimes_\alpha G) = 1$.

Let $\alpha \in \mathrm{Aut}(A)$ be an automorphism of a C*-algebra A. There is no conditional expectation of index-finite type from $A \rtimes_\alpha \mathbf{Z}$ onto A. Nevertheless, we have the following result.

Theorem 20 (Jeong-O-Phillips-Teruya:09)**.** Let A be a simple unital C*-algebra with $\mathrm{tsr}(A) = 1$ and Property (SP). Let $\alpha \in Aut(A)$ generate an outer action of $\mathbf{Z}$ on A such that $\alpha_* = \mathrm{id}$ on $K_0(A)$. Then $A \rtimes_\alpha \mathbf{Z}$ has cancellation.

Rokhlin property

Definition 21 (Izumi:04). Let α be an action of a finite group G on a unital C^*-algebra A. α is said to have the *Rokhlin property* if there exists a partition of unity $\{e_g\}_{g\in G} \subset A' \cap A^\infty$ consisting of projections satisfying

$$(\alpha_g)_\infty(e_h) = e_{gh} \quad \text{for } g, h \in G.$$

We call $\{e_g\}_{g\in G}$ Rokhlin projections.

Motivated by Definition 21 Kodaka, Osaka, and Teruya introduced the Rokhlin property for a inclusion of unital C*-algebras with a finite index.

Definition 22. A conditional expectation E of a unital C^*-algebra A with a finite index is said to have the *Rokhlin property* if there exists a projection $e \in A' \cap A^\infty$ satisfying

$$E^\infty(e) = (\mathrm{Index}E)^{-1} \cdot 1$$

and a map $A \ni x \mapsto xe$ is injective. We call e a Rokhlin projection.

Theorem 23 (Kodaka-O-Teruya:09)**.** Let a conditional expectation $E\colon A \to P$ be of index finite type and have the Rokhlin property. Then if A is simple, then P is simple.

Theorem 24 (Kodaka-O-Teruya:09)**.** Let a conditional expectation $E\colon A \to P$ be of index finite type and have the Rokhlin property.

1. If $\mathrm{tsr}(A) = 1$, then $\mathrm{tsr}(P) = 1$.

2. If $\mathrm{RR}(A) = 0$, then $\mathrm{RR}(P) = 0$.

Definition 25. Let A be a unital C*-algebra. We denote by $T(A)$ the set of all tracial states on A, equipped with the weak* topology. For any element of $T(A)$, we use the same letter for its standard extension to $M_n(A)$ for arbitrary n, and to $M_\infty(A) = \cup_{n=1}^{\infty} M_n(A)$.

We say that *the order on projections over a unital C*-algebra A is determined by traces* if whenever $p, q \in M_\infty(A)$ are projections such that $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \preceq q$.

Proposition 26. [O-Teruya:10] Let $E\colon A \to P$ be of index finite type and has the Rokhlin property. Then the restriction map defines a bijection from the set $T(A)$ to the set $T(P)$.

Theorem 27. [O-Teruya:10] Let A be a unital C*-algebra such that the order on projections over A is determined by traces. Let $E\colon A \to P$ be of index finite type. Suppose that E has the Rokhlin property. Then the order on projections over P is determined by traces.

Applications

In this section we show several results related to the classification problem of simple unital, separable, nuclear, stably finite C*-algebras.

For a unital C*-algebra the Elliott invariant of A, $\mathrm{Ell}(A)$ is the 4-tuple,

$$((K_0(A), K_0(A)^+, [1_A]_0), K_1(A), T(A), r_A),$$

where $S(K_0(A))$ is the state space of $K_0(A)$, that is, the set of all homomorphisms $f\colon K_0(A) \to \mathbf{R}$ such that $f(K_0(A)^+) \subset \mathbf{R}^+$ and $f([1_A]_0) = 1$, and $r_A\colon T(A) \to S(K_0(A))$ defined by

$$r(\tau)([p]_0 - [q]_0) = \tau(p) - \tau(q)$$

for $\tau \in T(A)$ and $[p]_0 - [q]_0 \in K_0(A)$.

The "original" statement of the Elliott conjecture for simple unital, separable, nuclear, C*-algebras as follows:

Conjecture 28 (EC)**.** Let A and B be simple unital, separable, nuclear, C*-algebras, and suppose that there exists an isomorphism $\phi\colon \mathrm{Ell}(A) \to \mathrm{Ell}(B)$. Then there is a *-isomorphism $\Phi\colon A \to B$ which induces ϕ.

We stress that the following references will be useful for reader to desire a fuller introduction to the classification program:

1. M.Rørdam, *A classification of nuclear C*-algebras*, Encyclopaedia of Mathematical Sciences 126, Springer-Verlag, Berlin, Heidelberg, 2002.

2. G. A. Elliott and A. S. Toms, *Regularity properties in the classification program for separable amenable C*-algebras*, Bull. Amer. Math. Soc. , 45(2008), no.2, 229 - 245.

Theorem 29 (Elliott-Gong:96, Dădărlat:95,Gong:97). (EC) holds among simple unital AH algebras with slow dimension growth and real rank zero.

Here an AH algebra A is the limit of an inductive sequences $(A_{,}\phi_i)$, where each A_i is semi-homogeneous

$$A_i = \oplus_{j=1}^{n_i} p_{i,j}(C(X_{i,j}) \otimes \mathbb{K})p_{i,j}$$

for some $n_i \in \mathbf{N}$, compact metric spaces $X_{i,j}$, and projections $p_{i,j} \in C(X_{i,j}) \otimes \mathbb{K}$.

We say an AH algebra has slow dimension growth if it has a decomposition (A_i, ϕ_i) satisfying

$$\lim\sup_{i \to \infty} \max_{1 \leq j \leq n_i} \{\dim X_{i,j}/\mathrm{rank}(p_{i,j})\} = 0.$$

In particular, if each $X_{i,j}$ is point (or the interval $[0, 1]$,or S^1), then we call A AF algebra (AI algebra, or AT algebra).

Theorem 30 (Kodaka-O-Teruya:09)**.** Let $P \subset A$ be an inclusion of separable unital C*-algebras and E a conditional expectation from A onto P with a finite index. Suppose that E has the Rokhlin property.

1. If A is an AF algebra, then P is also an AF algebra.

2. If A is a unital AI algebra, then P is a unital AI algebra.

3. If A is a unital AT algebra, then P is a unital AT algebra.

There is Lin's classification of simple unital, nuclear, separable, C*-algebras.

Definition 31. A simple unital C*-algebra has tracial topological rank zero if for any finite set $\mathcal{F}$, any $\varepsilon > 0$, and positive element $a \in A$ there exists a unital finite dimensional C*-algebra B of A with unit $1_B = p$ such that

1. $\|px - xp\| < \varepsilon$ and $pxp \in_\varepsilon B$ for $x \in \mathcal{F}$.

2. $1_A - p$ is equivalent to a projection in $\overline{aAa}$.

Theorem 32 (Lin:00)**.** The following conditions are equivalent for any simple, unital, separable C*-algebra A of real rank zero.

1. A is nuclear in the UCT class $\mathcal{N}$, has tracial topological rank zero.

2. A is an AH algebra of slow dimension growth.

Theorem 33 (Kodaka-O-Teruya:09)**.** Let $P \subset A$ be an inclusion of separable unital C*-algebras and E a conditional expectation from A onto P with a finite index. Suppose that A is simple and E has the Rokhlin property. If A has tracial topological rank zero, then P has the tracial topological rank zero.

Recently, there is a counterexample for $\mathrm{Ell}(A)$.

Theorem 34 (Toms:05)**.** There are simple unital, separable, nuclear, and stably finite C*-algebras which agree on the Elliott invariant but are not isomorphic.

But there is possibility if we add some extra condition.

Theorem 35 (Lin:08)**.** Let A and B be two simple unital inductive limits of generalized dimension drop algebras. Then $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

Here the C*-algebra $\mathcal{Z}$ in the above theorem is called the Jiang-Su algebra which is inductive limit of the dimension drop algebras

$$I[p_n, d_n, q_n] = \{f \in C[0,1] \otimes M_{d_n} \colon f(0) \in M_{p_n} \otimes id_{d_n/p_n}, \\ f(1) \in id_{d_n/q_n} \otimes M_{q_n}\},$$

where p_n, d_n, q_n are positive integers with $d_n = p_n q_n$, and p_n and q_n are relatively prime.

[Winter:08] showed that if A is a simple unital ASH-algebra with the no dimension growth, then $A \otimes \mathcal{Z} \cong A$.

Theorem 36 (O-Teruya:09)**.** Let $P \subset A$ be an inclusion of separable unital C*-algebras and E a conditional expectation from A onto P with a finite index. If A is $\mathcal{D}$-absorbing, then P is $\mathcal{D}$-absorbing, where D is a separable unital strongly self-absorbing C*-algebra, that is, there is an isomorphism $\phi\colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ such that ϕ is approximate unitarily equivalent to the embedding $d \mapsto d \otimes 1_{\mathcal{D}}$.

Note that

1. [Jiang-Su:99]

   The Jiang-Su algebra $\mathcal{Z}$ is strongly self-absorbing.

2. In the case that G is a compact second countable group, or, $\mathbf{Z}$ [Hirshberg and Winter:07] showed that if A is $\mathcal{D}$-absorbing, then the crossed product $A \rtimes_\alpha G$ is $\mathcal{D}$-absorbing assuming that the action of G on A has the Rokhlin property in the sense of Kishimoto.

3. [Winter:09] Every strongly self-absorbing C*-algerba is $\mathcal{Z}$ stable.

There are several other important properties in the classification theory such as *the strict comparison of positive elements* in the Cuntz semigroups and *the finite decomposition property* etc. Those properties also hold under the assumption that a conditional expectation $E\colon A \to P$ has the Rokhlin property.

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