Hopf Algebras, Independences and Dual Groups

Steffi Lachs

University of Greifswald

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Outline

1. Preliminaries
   - Universal products
   - $\mathbb{N}_0$-graded dual groups
2. Connections between dual groups and Hopf algebras
3. Main theorem
4. Applications: CLT's for sums of i.i.d. q.r.v.'s

Free product of $*$-algebras

- Notation:

$$\mathbb{A}_2 = \left\{ \varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \mid m \in \mathbb{N}, \varepsilon_k \in \{1,2\}, \varepsilon_k \neq \varepsilon_{k+1}, k \in \{1, \ldots, m\} \right\}$$

Let $\mathcal{A}_1, \mathcal{A}_2$ be $*$-algebras.

- free product of the vector spaces:

$$\mathcal{A}_1 \sqcup \mathcal{A}_2 := \bigoplus_{\varepsilon \in \mathbb{A}_2} \mathcal{A}_{\varepsilon_1} \otimes \mathcal{A}_{\varepsilon_2} \otimes \cdots \otimes \mathcal{A}_{\varepsilon_m}$$

- with multiplication

$$\underbrace{(a_1 \otimes \cdots \otimes a_m)}_{\in \mathcal{A}_{(\varepsilon_1, \ldots, \varepsilon_m)}} \underbrace{(b_1 \otimes \cdots \otimes b_n)}_{\in \mathcal{A}_{(\varrho_1, \ldots, \varrho_n)}} = \begin{cases} a_1 \otimes \cdots \otimes a_m \otimes b_1 \otimes \cdots \otimes b_n, & \varepsilon_m \neq \varrho_1 \\ a_1 \otimes \cdots \otimes (a_m b_1) \otimes \cdots \otimes b_n, & \varepsilon_m = \varrho_1 \end{cases}$$

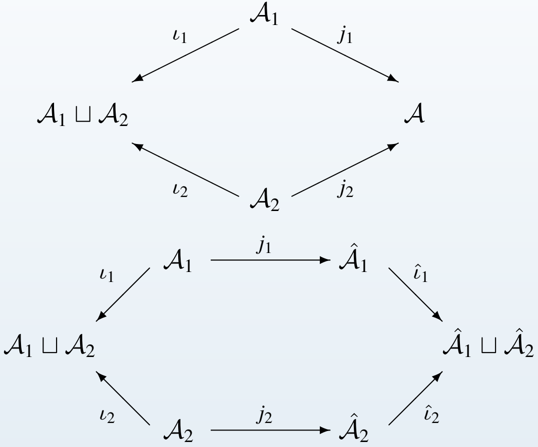
- with involution

$$(a_1 \otimes \cdots \otimes a_m)^* := a_m^* \otimes \cdots \otimes a_1^*$$

$\Rightarrow$ the free product $\mathcal{A}_1 \sqcup \mathcal{A}_2$ becomes a $*$-algebra

Universal property of the free product

$\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{A}_2$, $\hat{\mathcal{A}}_1$, $\hat{\mathcal{A}}_2$ algebras, $\iota_1, \iota_2, \hat{\iota}_1, \hat{\iota}_2$ canonical embeddings, j_1, j_2 algebra homomorphisms



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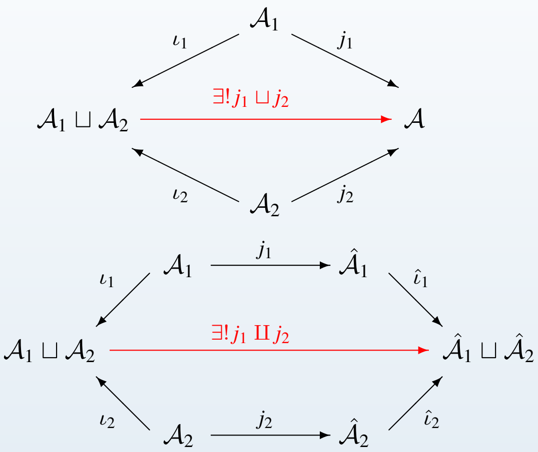

Universal products and notions of independence

Definition

A **universal product** is a prescription that assigns to every pair of algebras $(\mathcal{A}_1, \mathcal{A}_2)$ and every pair of linear functionals (φ_1, φ_2) a linear functional $\varphi_1 \bullet \varphi_2 : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \mathbb{C}$ in such a way that the following axioms yield:

(A1) $(\varphi_1 \bullet \varphi_2) \bullet \varphi_3 = \varphi_1 \bullet (\varphi_2 \bullet \varphi_3)$
with linear functional $\varphi_3 : \mathcal{A}_3 \to \mathbb{C}$

(A2) $(\varphi_1 \bullet \varphi_2) \circ \iota_1 = \varphi_1$ and $(\varphi_1 \bullet \varphi_2) \circ \iota_2 = \varphi_2$

(A3) $(\varphi_1 \circ j_1) \bullet (\varphi_2 \circ j_2) = (\varphi_1 \bullet \varphi_2) \circ (j_1 \amalg j_2)$
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Definition

Let $\mathcal{A}$ be a unital $*$-algebra and Φ be a state on $\mathcal{A}$. Furthermore let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be $*$-algebras and $j_k : \mathcal{A}_k \to \mathcal{A}$ ($k \in \{1, \ldots, n\}$) $*$-algebra homomorphisms. Then $j_1, \ldots, j_n$ are called **$\bullet$-independent**, if

$$\Phi \circ (j_1 \sqcup \cdots \sqcup j_n) = (\Phi \circ j_1) \bullet \cdots \bullet (\Phi \circ j_n).$$

Some universal products

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$$\big(\varphi_1 \bigstar \varphi_2\big)(a_1 \otimes \cdots \otimes a_m) =$$

$$\sum_{I \subsetneq \{1,\ldots,m\}} (-1)^{m-\#I+1} (\varphi_1 \bigstar \varphi_2) \left(\prod_{k\in I}^{\rightarrow} a_k\right) \left(\prod_{l \notin I} \varphi_{\varepsilon_l}(a_l)\right) \tag{F}$$

(recursively with $\varphi_1 \bigstar \varphi_2 \left(\prod_{k\in\emptyset}^{\rightarrow} a_k\right) := 1$)

for $a_1 \otimes \cdots \otimes a_m \in \mathcal{A}_{(\varepsilon_1,\ldots,\varepsilon_m)}$

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$\mathbb{N}_0$-graded dual groups

Definition

A $\mathbb{N}_0$**-graded dual semigroup** $(\mathcal{D}, \Lambda)$ is a graded $*$-algebra $\mathcal{D}$ with a homogeneous $*$-algebra homomorphism $\Lambda : \mathcal{D} \to \mathcal{D} \sqcup \mathcal{D}$, which fulfills

$$(\Lambda \amalg \mathrm{id}_{\mathcal{D}}) \circ \Lambda = (\mathrm{id}_{\mathcal{D}} \amalg \Lambda) \circ \Lambda$$
$$(\mathbf{0} \sqcup \mathrm{id}_{\mathcal{D}}) \circ \Lambda = \mathrm{id}_{\mathcal{D}} = (\mathrm{id}_{\mathcal{D}} \sqcup \mathbf{0}) \circ \Lambda.$$

If there exists in addition an homogeneous antipode $\kappa : \mathcal{D} \to \mathcal{D}$ s. t.

$$(\kappa \sqcup \mathrm{id}_{\mathcal{D}}) \circ \Lambda = \mathbf{0} = (\mathrm{id}_{\mathcal{D}} \sqcup \kappa) \circ \Lambda,$$

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$(\mathcal{D}, \Lambda, \kappa)$ becomes a **graded dual group**.

- for $\varphi_1, \varphi_2 \in \mathcal{D}'$ the **convolution** is defined by

$$\varphi_1 \star \varphi_2 := (\varphi_1 \bullet \varphi_2) \circ \Lambda$$

(here $\varphi_1 \bullet \varphi_2 \in (\mathcal{D} \sqcup \mathcal{D})'$ stands for one universal product)

Connections between dual groups and Hopf algebras

AIM: Constructing a functor by using the symmetric tensor algebra $\mathcal{S}(V)$ in order to carry out an algebraic reduction of a graded dual group according to one of the universal products to a graded commutative Hopf $*$-algebra.

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$$\varepsilon_{\mathcal{A}_1,\mathcal{A}_2} : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \mathcal{S}(\mathcal{A}_1) \otimes \mathcal{S}(\mathcal{A}_2),$$

  which satisfies for all $\varphi_1 \in \mathcal{A}_1', \varphi_2 \in \mathcal{A}_2'$

$$\varphi_1 \bullet \varphi_2 = \big(\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2)\big) \circ \varepsilon_{\mathcal{A}_1,\mathcal{A}_2}$$

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- these maps fulfill similar axioms as the universal product

Reduction of dual groups to Hopf algebras

- $\mathfrak{gDG}$ - category of graded dual groups
- $\mathfrak{gComH}$ - category of graded commutative Hopf $*$-algebras

Theorem (Functor $\mathcal{S} : \mathfrak{gDG} \to \mathfrak{gComH}$)

- choose one universal product "$\bullet$", then

$$(\mathcal{D}, \Lambda, \kappa) \mapsto \big(\mathcal{S}(\mathcal{D}), \mathcal{S}(\varepsilon_{\mathcal{D},\mathcal{D}} \circ \Lambda), \mathcal{S}(\mathbf{0}), \mathcal{S}(\kappa)\big)$$

$$\mathrm{Mor}(\mathcal{D}_1, \mathcal{D}_2) \ni \quad g \mapsto \mathcal{S}(g) \quad \in \mathrm{Mor}\big(\mathcal{S}(\mathcal{D}_1), \mathcal{S}(\mathcal{D}_2)\big)$$

  describes a functor $\mathcal{S}$ from $\mathfrak{gDG}$ to $\mathfrak{gComH}$

- $\mathcal{S} : \mathcal{D}' \to \mathrm{Hom}(\mathcal{S}(\mathcal{D}), \mathbb{C})$ is a homomorphism between the semigroup $\mathcal{D}'$ with convolution

$$\varphi_1 \star \varphi_2 = (\varphi_1 \bullet \varphi_2) \circ \Lambda \qquad \forall\, \varphi_1, \varphi_2 \in \mathcal{D}'$$

  and the monoid $\mathrm{Hom}\big(\mathcal{S}(\mathcal{D}), \mathbb{C}\big)$ with convolution

$$\mathcal{S}(\varphi_1) \circledast \mathcal{S}(\varphi_2) := \big(\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2)\big) \circ \mathcal{S}(\varepsilon_{\mathcal{D},\mathcal{D}} \circ \Lambda)$$

Proof of the Theorem

Sketch of proof

1. coassociativity:
$$\big(\mathcal{S}(\varepsilon_{\mathcal{D},\mathcal{D}}\circ\Lambda)\otimes \mathrm{id}_{\mathcal{S}(\mathcal{D})}\big)\circ\mathcal{S}(\varepsilon_{\mathcal{D},\mathcal{D}}\circ\Lambda) = \big(\mathrm{id}_{\mathcal{S}(\mathcal{D})}\otimes\mathcal{S}(\varepsilon_{\mathcal{D},\mathcal{D}}\circ\Lambda)\big)\circ\mathcal{S}(\varepsilon_{\mathcal{D},\mathcal{D}}\circ\Lambda)$$
2. counit property:
$$\big(\mathcal{S}(\mathbf{0})\otimes \mathrm{id}_{\mathcal{S}(\mathcal{D})}\big)\circ\mathcal{S}(\varepsilon_{\mathcal{D},\mathcal{D}}\circ\Lambda) = \iota_{\mathcal{D}} = \big(\mathrm{id}_{\mathcal{S}(\mathcal{D})}\otimes\mathcal{S}(\mathbf{0})\big)\circ\mathcal{S}(\varepsilon_{\mathcal{D},\mathcal{D}}\circ\Lambda)$$
3. antipode property:
$$M_{\mathcal{S}(\mathcal{D})}\circ\big(\mathcal{S}(\kappa)\otimes \mathrm{id}_{\mathcal{S}(\mathcal{D})}\big)\circ\mathcal{S}(\varepsilon_{\mathcal{D},\mathcal{D}}\circ\Lambda) = \mathcal{S}(\mathbf{0}) = M_{\mathcal{S}(\mathcal{D})}\circ\big(\mathrm{id}_{\mathcal{S}(\mathcal{D})}\otimes\mathcal{S}(\kappa)\big)\circ\mathcal{S}(\varepsilon_{\mathcal{D},\mathcal{D}}\circ\Lambda)$$
4. $g\mapsto\mathcal{S}(g)$ maps morphisms g between dual groups to morphisms $\mathcal{S}(g)$ between Hopf $*$-algebras s. t. $\mathcal{S}(g\circ h)=\mathcal{S}(g)\circ\mathcal{S}(h)$
5. $\mathcal{S}(\varphi_1\star\varphi_2)=\mathcal{S}(\varphi_1)\circledast\mathcal{S}(\varphi_2) \qquad \forall\,\varphi_1,\varphi_2\in\mathcal{D}'$
6. $\mathcal{S}$ protects the grading, because $\varepsilon_{\mathcal{D},\mathcal{D}}$ is homogeneous

LT's for graded coalgebras

In which way does the functor gives us new results?

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Theorem (M. Schürmann'93)

- $\mathcal{C} = \bigoplus_{k \in \mathbb{N}_0} \mathcal{C}^{(k)}$ graded coalgebra, $\nu \in \mathbb{N}$
- $\varphi \in \mathcal{C}'$ fulfills
  - (i) $\varphi \restriction \mathcal{C}^{(k)} = 0$ for $0 < k < \nu$
  - (ii) $\varphi \restriction \mathcal{C}^{(0)} = \delta \restriction \mathcal{C}^{(0)}$,

$$\Rightarrow \lim_{N \to \infty} \varphi^{\star N}\left(\frac{c}{N^{-\frac{k}{\nu}}}\right) = (\exp_\star g_\varphi)(c) \qquad \forall\, c \in \mathcal{C}^{(k)}, \tag{1}$$

whereby $g_\varphi \in \mathcal{C}'$ is defined by $g_\varphi(c) = \begin{cases} 0, & \text{if } c \in \mathcal{C}^{(k)}, k \neq \nu \\ \varphi(c), & \text{if } c \in \mathcal{C}^{(\nu)} \end{cases}$

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- $\nu = 1 \Rightarrow$ Law of Large Numbers
- $\nu = 2 \Rightarrow$ CLT for coalgebras

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Main theorem

- $\mathcal{D} = \bigoplus_{k\in\mathbb{N}_0} \mathcal{D}^{(k)}$ graded dual group, $\nu \in \mathbb{N}$
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Sketch of proof

- use functor $\mathcal{S}$ to get the graded Hopf $*$-algebra $\mathcal{S}(\mathcal{D})$ out of $\mathcal{D}$
- LT's for coalgebras $\Rightarrow$ (1) on $(\mathcal{S}(\mathcal{D}), \circledast)$
- restrict (1) to $(\mathcal{D}, \star)$ in order to get (2)

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Let $(\mathcal{Q}, \Phi)$ be a q.p.s.

Corollary

- $(j_n : \mathbb{C}[x] \to \mathcal{Q})_{n\in\mathbb{N}}$ sequence i.i.d. q.r.v. (same distribution $\varphi = \Phi \circ j_n$)
- $\varphi(x) = 0$

$$\Rightarrow \lim_{N\to\infty} \varphi^{\star N}\left(P\left(\frac{x}{\sqrt{N}}\right)\right) = (\exp_\star g_\varphi)(P(x))$$

for all $P(x) \in \mathbb{C}[x]$, whereby $g_\varphi(x^k) = \begin{cases} 0, & \text{if } k \neq 2 \\ \varphi(x^2), & \text{if } k = 2 \end{cases}$

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In what form do we get $(\exp_\star g_\varphi)$ for respective notions of independence?

Universal products and independences again

Let $\bullet$ be a universal product, which fulfills additionally

(A4) $(\varphi_1 \bullet \varphi_2)(a_1 a_2) = \varphi_1(a_1)\varphi_2(a_2)$ and $(\varphi_2 \bullet \varphi_1)(a_2 a_1) = \varphi_2(a_2)\varphi_1(a_1)$.

Theorem (N. Muraki'02)

There exist exactly five universal products: the tensor (T), the free (F), the Boolean (B), the monotone (M) and the anti-monotone (AM) product.

$\Rightarrow$ There are exactly five independences!

Special CLT

Theorem (CLT - one-dimensional case)

- $(j_n : \mathbb{C}[x] \to \mathcal{Q})_{n\in\mathbb{N}}$ sequence of (tensor, free, Boolean, monotone or anti-monotone) i.i.d. q.r.v.'s with $\varphi = \Phi \circ j_n$ and $j_n(x) = q_n$
- $\varphi(x) = 0, \quad \varphi(x^2) = 1,$

$\Rightarrow$ for all $k = 1, 2, \ldots$

$$\lim_{N\to\infty} \varphi^{\star N}\left(\frac{x^k}{\sqrt{N}}\right) = \lim_{N\to\infty} \Phi\left(\left(\frac{q_1 + \cdots + q_N}{\sqrt{N}}\right)^k\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^k \cdot \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x, \qquad \text{(T)}$$

$$= \frac{1}{2\pi} \int_{-2}^{+2} x^k \cdot \sqrt{4 - x^2}\, \mathrm{d}x, \qquad \text{(F)}$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} x^k \cdot (\delta_{-1} + \delta_{+1})\, \mathrm{d}x, \qquad \text{(B)}$$

$$= \frac{1}{\pi} \int_{-\sqrt{2}}^{+\sqrt{2}} x^k \cdot \frac{1}{\sqrt{2 - x^2}}\, \mathrm{d}x. \qquad \text{(M+AM)}$$

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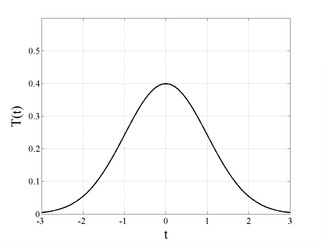
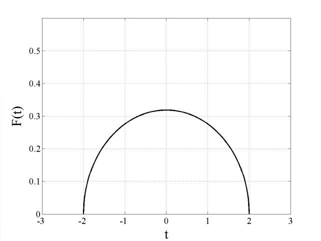
tensor case



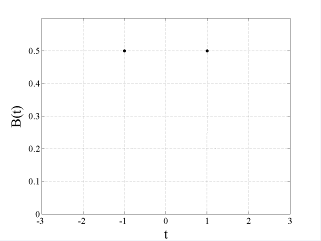
$$\mathfrak{M}_{2k} = \frac{k!}{2^k}\binom{2k}{k}$$

free case



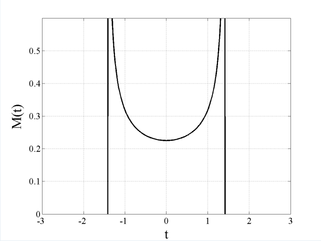
$$\mathfrak{M}_{2k} = \frac{1}{k+1}\binom{2k}{k}$$

Boolean case



$$\mathfrak{M}_{2k} = 1$$

(anti-)monotone case



$$\mathfrak{M}_{2k} = \frac{1}{2^k}\binom{2k}{k}$$

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