An Axiomatic Approach to Quantum Lévy Processes on Dual Groups

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Outline

1. Introduction
   - Motivation
   - Details

2. Previous Work
   - Construction of Quantum Lévy Processes

3. Main Result
   - κ-Approximation
   - Basic Idea for Proof
   - Application

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- positive universal products
- Muraki five: (T), (B), (F), (M) and (AM)
- Schoenberg correspondence holds for the five PUP.
- quantum Lévy processes on dual groups (D. Voiculescu)
- formula for quantum Lévy processes in the additive case

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Without the concrete formulas of the five PUP

- assume Schoenberg correspondence
- construct Lévy processes by generators (inductive limit)
- consider minimal version of the quantum Lévy process
- perform κ-approximation
- realize quantum Lévy process on the Fock spaces

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Positive Universal Products

Definition (PUP)

The positive universal product $\bullet$ maps linear functionals (φ_1, φ_2) to $\varphi_1 \bullet \varphi_2 : A_1 \sqcup A_2 \to \mathbb{C}$ such that:

- $(\varphi_1 \bullet \varphi_2) \bullet \varphi_3 = \varphi_1 \bullet (\varphi_2 \bullet \varphi_3)$ $(UP1)$
- $(\varphi_1 \bullet \varphi_2) \circ j_{A_1} = \varphi_1$ and $(\varphi_1 \bullet \varphi_2) \circ j_{A_2} = \varphi_2$ $(UP2)$
- $(\varphi_1 \circ j_1) \bullet (\varphi_2 \circ j_2) = (\varphi_1 \bullet \varphi_2) \circ (j_1 \sqcup' j_2)$ $(UP3)$
- φ_1, φ_2 states $\Rightarrow \varphi_1 \bullet \varphi_2$ state (Pos)

$j_1 : D_1 \to A_1$, $j_2 : D_2 \to A_2$ are algebraic homomorphisms.

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Dual Semi Group

(D. Voiculescu)

(B, Δ, δ)

- B unital *-algebra
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Definition (Generator)

A conditionally positive hermitian linear functional Ψ with $\Psi(\mathbf{1}) = 0$ is called a generator.

Theorem

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Quantum Lévy Process

Definition

$(j_{s,t})_{0 \le s \le t} : (B, \Delta, \delta) \to (A, \Phi)$ algebraic homomorphisms

- evolution: $j_{r,s} \star j_{s,t} = j_{r,t}$
- independence of $j_{t1,t2}, j_{t2,t3}, \ldots, j_{tn,tn+1}$ to one PUP
- stationarity of distribution
- weakly continuous

$f \star g := (f \sqcup g) \circ \Delta$
$0 \le t_1 < t_2 < \cdots < t_{n+1}$
$\Phi \circ j_{s,t} = \Phi \circ j_{0,t-s}$

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Construction of Quantum Lévy Processes (2)

- inductive limit (A, f_σ, Φ)
- quantum Lévy process: $\forall b \in B$, $0 \leq s < t$ put

$$J_{s,t}(b) := f_{\{s,t\}}(b)$$
$$J_{s,s}(b) := \delta(b)\mathbf{1}$$

- let D be a pre hilbertian space with unit vector Ω, $L_a(D)$- linear, adjoint-able functions
- GNS construction for $\Phi \Rightarrow$ exists $*$-representation $\pi : A \to L_a(D)$ with

$$\Phi(b) = < \Omega, \pi(b)\Omega > \quad \forall b \in A$$

- define QLP $j_{s,t} : B \to (L_a(D), < \Omega, (\cdot)\Omega >)$ by

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Minimal Version, Notation

Definition (Minimal Version)

- $F := span\{j_{s_1,s_2}(b_1)\cdots j_{s_n,t_n}(b_n)\Omega\,,\ n\in\mathbb{N}\,,\ b_i\in B\,,\ s_i\leq t_i\,\forall i\}$
- $j_{s,t}: B\to L_a(F)$

Definition (Notation)

For $\alpha=\{t_1<t_2<\cdots<t_{n+1}\}$ define

$$\mathop{\star}_{\alpha}(j_\alpha\circ\kappa):=(j_{t_1,t_2}\circ\kappa)\star\cdots\star(j_{t_n,t_{n+1}}\circ\kappa).$$

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κ-Approximation

Assume: unital $*$-Alg.hom. $\kappa : C \to B$ with

$$\delta \circ \kappa = \lambda.$$

dual semi group generator	(B, Δ, δ) Ψ	(C, Λ, λ) $\Psi \circ \kappa$
	$\downarrow$	$\downarrow$
QLP, min. version Q.Prob.Space	$j_{s,t}$ $(L_a(E), \Phi), \Omega$	$k_{s,t}$ $(L_a(F), \Phi'), \Theta$

κ-approximation means, one gets an isometry

$$\bar{W} : \bar{F} \longrightarrow \bar{E}.$$

κ-Approximation

Assume: unital $*$-Alg.hom. $\kappa : C \to B$ with

$$\delta \circ \kappa = \lambda.$$

dual semi group	(B, Δ, δ)	(C, Λ, λ)
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Define map $W : F \to \bar{E}$:
i.e. for disjoint intervals $I_1 = [s_1, t_1], I_2 = [s_2, t_2]$ put

$$W(k_{s_1,t_1}(c_1)k_{s_2,t_2}(c_2)\Theta)$$
$$= \lim_{net} \left(\left(\overset{\wedge}{\underset{\alpha\restriction_{[s_1,t_1]}}{\star}} (j_\alpha \circ \kappa) \right) (c_1) \left(\overset{\wedge}{\underset{\alpha\restriction_{[s_2,t_2]}}{\ast}} (j_\alpha \circ \kappa) \right) (c_2)\Omega \right)_{\alpha \in partition([0,T])}$$

(with $c_1, c_2 \in C$, $T > max\{s_1, t_1, s_2, t_2\}$)
and so on for all $x \in F$

Theorem

The equation

$$< W(x), W(y) >=< x, y > \quad \forall x, y \in F$$

holds, and so there exists an isometry $\bar{W} : \bar{F} \to \bar{E}$.

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1 Introduction
- Motivation
- Details

2 Previous Work
- Construction of Quantum Lévy Processes

3 Main Result
- κ-Approximation
- Basic Idea for Proof
- Application

Basic Idea for Proof

Be $x\Theta, y\Theta \in F$ such that

$$W(x\Theta) = \lim_{net} (F_\alpha \Omega)_{\alpha \in partition([0,T])}$$

$$W(y\Theta) = \lim_{net} (G_\alpha \Omega)_{\alpha \in partition([0,T])} \, .$$

By continuity of the scalar product $< \cdot, \cdot >$,

$$\begin{array}{ccc} < W(x\Theta), W(y\Theta) > & \overset{net}{\longleftarrow} & (< F_\alpha \Omega, G_\alpha \Omega >)_{\alpha \in partition([0,T])} \\ & & \| \\ < x\Theta, y\Theta > & & (< \Omega, H_\alpha \Omega >)_{\alpha \in partition([0,T])} \\ \| & & \| \\ < \Theta, x^* y\Theta > = \Phi'(x^* y) & \overset{net}{\longleftarrow} & (\Phi(H_\alpha))_{\alpha \in partition([0,T])} \end{array}$$

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Realization on Fock spaces

(B, Δ, δ) dual semi group, Ψ generator. Let $B_0 := ker(\delta)$

dual semi group	$(T(B_0), \Delta_{prim}, T(0))$	$(T(B_0), T(\Delta'), T(0))$
generator	Ψ	$\Psi \circ id$
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Q.Prob.Space	$(L_a(E), \Phi), \Omega$	$(L_a(F), \Phi'), \Omega$

- $Fock_i$, $i \in \{(T), (B), (F), (M), (AM)\}$
- $E \subset Fock_i(L^2(\mathbb{R}_+) \otimes D)$, $F \subset Fock_i(L^2(\mathbb{R}_+) \otimes D)$
- for $Fock_{(F)}$ $b_0 \in B_0$ there is

$$j_{s,t}(b_0) = \text{annihilation} + \text{creation} + \text{preservation} + (t-s)\Psi(b_0)id$$

$$W(k_{s,t}\Omega) := \lim_{net} \left(\mathop{\bigstar}_{\alpha}^{T(\Delta')} (j_\alpha \circ id)\Omega \right)_{\alpha \in partition([s,t])}$$

(note: Ω is cyclic)

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Thank you.