Hilbert Modules—Square Roots of Positive Maps

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We reflect on notions of positivity and square roots.
More precisely:

- In a good notions of positivity, it should be a theorem that every positive thing has a square root!
- The square root must allow to recover the positive thing in an easy way, making also manifest in that way that the positive thing is positive. ($\rightsquigarrow$ facilitate proofs of positivity.)
- We prefer unique square roots.
- We wish to compose two positive things to get new ones.

To achieve this:

- We will allow for quite general square roots.
- It turns out that it is good to view positive things as maps.

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such that $p(x) := \overline{\varphi}(x)\varphi(x)$ becomes a probability density over $\mathbb{R}^3$.
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Note: Suppose $z' \in \mathbb{C}$ such that $\overline{z'}z' = \lambda > 0$.
Then $u := \frac{z'}{z} = e^{i\alpha} \in \mathbb{S}^1$.
In fact, $u\colon \lambda \mapsto u\lambda$ is a unitary in $\mathcal{B}(\mathbb{C})$ that maps z to z'.
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Note: Positive numbers $\lambda, \mu \geq 0$ can be multiplied.
In fact, if $z, w \in \mathbb{C}$ are square roots of λ, μ, respectively,
then $\overline{(zw)}(zw) = (\bar{z}z)(\overline{w}w) = \lambda\mu$,
so that $\lambda\mu \geq 0$.

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This square root depends on the choice (at least of γ)
and it is noncommutative.
Note: In order to compose in that way a fixed c with any b, we need to know the whole map $\gamma^* \bullet \gamma$! ($\rightsquigarrow$ Hilbert bimodules!)

Example. A kernel $\mathfrak{k}\colon S \times S \to \mathbb{C}$ over a set S is positive definite ④

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Theorem. (Kolmogorov decomposition.) If $\mathfrak{k}$ is $\mathbb{C}$–valued PD-kernel over S, then there exist a Hilbert space H and a map $i\colon S \to H$ such that

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Proof. On $S_{\mathbb{C}} := \bigoplus_{\sigma\in S} \mathbb{C} = \left\{ (z_\sigma)_{\sigma\in S} \;\middle|\; \#\{\sigma\colon z_\sigma \neq 0\} < \infty \right\}$ define the sesquilinear form

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  Try to do the same with the collection of numbers $\sqrt{\sum_{i,j=1}^{n} \overline{z}_i \mathfrak{k}^{\sigma_i,\sigma_j} z_j}$

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  In fact, if also (K, j) fulfills $\overline{\text{span}}\, j(S) = K$, then v becomes a unitary.

  Also, compare positive numbers: $S = \{\omega\}$, $\mathfrak{k}^{\omega,\omega} := \lambda \geq 0$.

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- Composition of PD-kernels is reflected by tensor products.

  $(\mathfrak{l}\mathfrak{k})^{\sigma,\sigma'} := \mathfrak{l}^{\sigma,\sigma'} \mathfrak{k}^{\sigma,\sigma'}$. (Schur prod.) $\mathfrak{k} \rightsquigarrow i\colon S \to H, \quad \mathfrak{l} \rightsquigarrow j\colon S \to K$

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  $(\mathfrak{l}\mathfrak{k}) \rightsquigarrow (i \otimes j)(\sigma) := i(\sigma) \otimes j(\sigma) \in H \otimes K$. Note: $\overline{\text{span}}(i \otimes j)(S) \subsetneq H \otimes K$!

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However:

- It does NOT help composing PD-kernels.

  There is no reasonable tensor product of right Hilbert $\mathcal{B}$–modules that recovers what we did for the one-point set $S = \{\omega\}$.

  In fact, how could it?
  Our composed square root $\beta\gamma$ depends on the choice of γ!

Example. A kernel $\mathfrak{K}\colon S \times S \to \mathbb{B}(\mathcal{A}, \mathcal{B})$ over a set S is completely positive definite (CPD) if

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- Possibly Speicher [Spe98] (Habilitation thesis 1994)?

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Theorem. **(Kolmogorov decomposition.)** $\mathcal{A} \ni \mathbf{1}$. If $\mathfrak{K}$ is a CPD-kernel over S from $\mathcal{A}$ to $\mathcal{B}$, then there exist an $\mathcal{A}$–$\mathcal{B}$–correspondence E and a map $i\colon S \to E$ such that

$$\langle i(\sigma), a i(\sigma') \rangle \;=\; \mathfrak{K}^{\sigma,\sigma'}(a)$$

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Example. A kernel $\mathfrak{K}\colon S \times S \to \mathfrak{B}(\mathcal{A}, \mathcal{B})$ over a set S is completely positive definite (CPD) if

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Note: $S = \{\omega\} \rightsquigarrow$ CP-maps. (Do NOT use n–positive for all n!)

Kolmogorov $\rightsquigarrow$ Paschke's GNS-construction [Pas73].

1st proof. The $\mathcal{B}$–valued kernel $\mathfrak{k}^{(a,\sigma),(a',\sigma')} := \mathfrak{K}^{\sigma,\sigma'}(a^*a')$ over $\mathcal{A} \times S$ is, clearly, PD. On its Kolmogorov decomposition $(E, \tilde{i})$ check that $a\tilde{i}(a',\sigma) := \tilde{i}(aa',\sigma)$ defines a left action of $\mathcal{A}$ on E. Put $i(\sigma) := \tilde{i}(\mathbf{1},\sigma)$. ■ 9

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The first proof is "classical":
Guess a PD-kernel, do Kolmogorov, show its algebraic properties.

The second proof is "modern": Start with a bimodule, define the only reasonable inner product that emerges from CPD. (The algebraic properties are general theory of correspondences.)

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  Here for ${}_{\mathcal{A}}E_{\mathcal{B}}$ and ${}_{\mathcal{B}}F_{\mathcal{C}}$, the internal tensor product $E \odot F$ is the unique $\mathcal{A}$–$\mathcal{C}$–correspondence that is spanned by elementary tensors $x \odot y$ fulfilling

  $\langle x \odot y, x' \odot y' \rangle = \langle y, \langle x, x' \rangle y' \rangle$ and $a(x \odot y) = (ax) \odot y$.

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- Put $xy^* \colon z \mapsto x\langle y, z \rangle$ and $E^* := \{x^* \colon x \in E\}$.
  Then $\langle x'^*, x^* \rangle := x' x^*$ and $b x^* a := (a^* x b^*)^*$ turns E^* into a $\mathcal{B}$–$\mathcal{B}^a(E)$–correspondence.
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- Define the Hilbert $M_n(\mathcal{B})$–module $E_n := ((E^*)^n)^*$. Check that $\langle X_n, X_n' \rangle = (\langle x_i, x_j' \rangle)_{ij}$ and $(X_n B)_i = \sum_j x_j b_{ji}$.

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- Then $\left\langle \sum_i x_i \otimes y_i \,,\, \sum_i x_i \otimes y_i \right\rangle = \langle X_n \otimes Y^n, X_n \otimes Y^n \rangle \geq 0$.

Note:

- A CPD-kernel $\mathfrak{K}$ from $\mathcal{A}$ to $\mathcal{B}$ and a CPD-kernel $\mathfrak{L}$ from $\mathcal{B}$ to $\mathcal{C}$ can be composed to form a CPD-kernel $\mathfrak{L} \circ \mathfrak{K}$ from $\mathcal{A}$ to $\mathcal{C}$.

- Viewing $w \in \mathbb{C}$ as map $z \mapsto zw$ on $\mathbb{C}$
  $\mathbb{C}$–valued PD-kernels correspond 1-1 with CPD-kernel from $\mathbb{C}$ to $\mathbb{C}$.
  Schur product of PD-kernels=compositions of CPD-kernels.

- Viewing $b \in \mathcal{B}$ as map $z \mapsto zb$ from $\mathbb{C}$ to $\mathcal{B}$
  $\mathcal{B}$–valued PD-kernels correspond 1-1 with CPD-kernel from $\mathbb{C}$ to $\mathcal{B}$.
  Usually, no composition! (Codomain and domain match only in the $\mathbb{C}$–valued case.)

Recall: $\mathfrak{K} \rightsquigarrow (E, i)$, $\mathfrak{L} \rightsquigarrow (F, j)$, then $\mathfrak{L} \circ \mathfrak{K} \rightsquigarrow$

$$\overline{\text{span}}\{ai(\sigma) \odot j(\sigma)c \colon a \in \mathcal{A}, c \in C, \sigma \in S\}$$

with embedding $i \odot j \colon \sigma \mapsto i(\sigma) \odot j(\sigma)$. This is (usually much!) smaller than

$$\begin{aligned} E \odot F &= (\overline{\text{span}}\,\mathcal{A}i(S)\mathcal{B}) \odot (\overline{\text{span}}\,\mathcal{B}j(S)C) \\ &= \overline{\text{span}}\{ai(\sigma) \odot bj(\sigma')c \colon a \in \mathcal{A}; b \in \mathcal{B}; c \in C; \sigma, \sigma' \in S\}. \end{aligned}$$

So, $E \odot F$ does not coincide but at least contains the GNS-correspondence of $\mathfrak{L} \circ \mathfrak{K}$.

> The GNS-correspondences for $\mathfrak{K}$ and $\mathfrak{L}$ allow easily to compute GNS-correspondence for $\mathfrak{L} \circ \mathfrak{K}$.
> Nothing like this is true for Stinespring constructions!

Recall: (For simplicity for CP-maps.)
$T\colon \mathcal{A} \to \mathcal{B} \subset \mathscr{B}(G) \rightsquigarrow H = E \odot G$, $v = \xi \odot \mathrm{id}_G$, $\rho(a) = a \odot \mathrm{id}_G$.
$S\colon \mathcal{B} \to \mathcal{C} \subset \mathscr{B}(K) \rightsquigarrow L = F \odot K$, $w = \zeta \odot \mathrm{id}_K$, $\pi(b) = b \odot \mathrm{id}_K$.

By no means does the Stinespring representation ρ for T help to construct the Stinespring representation for $S \circ T$!
(One needs to “tensor” E with the representation space $L = F \odot G$ of the Stinespring representation π for S, **not** with G!)

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The GNS-correspondences E and F, on the other hand, are **universal**! (For each CP-map they need to be computed only once.)

Doing Stinespring representations for the individual members of a CP-semigroup on $\mathcal{B} \subset \mathcal{B}(G)$, is like considering a 2×2–system of complex linear equations as a real 4×4–system (ignoring all the structure hidden in the fact that certain 2×2–submatrices are very special) and applying the Gauß algorithm to the 4×4–system instead of trivially resolving the 2×2–system by hand.

$\mathfrak{T} = (\mathfrak{T}_t)_{t\geq 0}$ a CPD-semigroup over S on $\mathcal{B} \ni \mathbf{1}$.
Then the GNS-correspondences $\mathcal{E}_t$ of the $\mathfrak{T}_t$ fulfill $\mathcal{E}_s \odot \mathcal{E}_t \supset \mathcal{E}_{s+t}$, so

$$(\mathcal{E}_{s^n_{m_n}} \odot \ldots \odot \mathcal{E}_{s^n_1}) \odot \ldots \odot (\mathcal{E}_{s^1_{m_1}} \odot \ldots \odot \mathcal{E}_{s^1_1}) \supset \mathcal{E}_{s^n_{m_n}+\ldots+s^n_1} \odot \ldots \odot \mathcal{E}_{s^1_{m_1}+\ldots+s^1_1}$$

Fix $t > 0$, $\rightsquigarrow$ inductive limit over $\mathbb{t} = (t_n, \ldots, t_1) \in (0, \infty)^n$ with $t_n + \ldots + t_1 = t$. For $E_t = \liminf \mathrm{ind}_{\mathbb{t}}\, \mathcal{E}_{\mathbb{t}} \supset \mathcal{E}_t$

$$\mathcal{E}_s \odot \mathcal{E}_t \supset \mathcal{E}_{s+t} \quad \text{becomes equality} \quad E_s \odot E_t \;=\; E_{s+t},$$

so $E^{\odot} = (E_t)_{t\in\mathbb{R}_+}$ is a product system. The $\xi^\sigma_t := i_t(\sigma) \in \mathcal{E}_t \subset E_t$ fulfill $\xi^\sigma_s \odot \xi^\sigma_t = \xi^\sigma_{s+t}$ that is, for each $\sigma \in S$ the family $\xi^{\sigma\odot} = (\xi^\sigma_t)_{t\geq 0}$ is a unit, such that $\langle \xi^\sigma_t, \bullet \xi^{\sigma'}_t \rangle = \mathfrak{T}^{\sigma,\sigma'}_t$ for all $\sigma, \sigma' \in S$, and the set $\{\xi^{\sigma\odot} \colon \sigma \in S\}$ of units generates $E^{\odot}$ as a product system. We see:

The square root of a CPD-semigroup (in particular, of a CP-semigroup) is a product system with generating set of units; Bhat and MS [BS00].

- The product system of a PD-semigroup consists of symmetric Fock spaces. Applications:
  Classical Lévy processes (Parthasarathy and Schmidt [PS72].)
  Quantum Lévy processes (Schürmann, MS, and Volkwardt [SSV07].)

- The product system of uniformly continuous normal CPD-semigroups on von Neumann algebras consists of time ordered Fock modules (Barreto, Bhat, Liebscher, and MS [BBLS04]).
  For C^*–algebras this may fail (Bhat, Liebscher, and MS [BLS10])!

- The Markov semigroups that admit dilations by cocycle perturbations of “noises” are precisely the “spatial” Markov semigroups (MS [Ske09a]). Proof: Via “spatial” product systems (MS [Ske06] (preprint 2001))!

CP-semigroups on $\mathcal{B}^a(E)$

Let ϑ be a semigroup of (unital, for simplicity) endomorphisms ϑ_t of $\mathcal{B}$. Then $\mathcal{B}_t := \mathcal{B}$ with $b.x_t := \vartheta_t(b)x_t$ is its GNS-system with unit $(\mathbf{1})_{t\in\mathbb{R}_+}$.

It is not a good idea to tensor with G when $\mathcal{B} \subset \mathcal{B}(G)$. (Unless vN-alg.) This **changes** when $\mathcal{B} = \mathcal{B}(G)$ — or better $\mathcal{B} = \mathcal{B}^a(E)$. But only, if we tensor "from both sides"!

General: $T\colon \mathcal{B}^a(E_{\mathcal{B}}) \to \mathcal{B}^a(F_{\mathcal{C}})$ and $S\colon \mathcal{B}^a(F_{\mathcal{C}}) \to \mathcal{B}^a(G_{\mathcal{D}})$ CP-maps. Their GNS-correspondences $\mathcal{E}$ and $\mathcal{F}$. Require $\overline{\operatorname{span}}\,\mathcal{K}(E)\mathcal{E} = \mathcal{E}$ and $\overline{\operatorname{span}}\,\mathcal{K}(F)\mathcal{F} = \mathcal{F}$ (strictness!). Then

$$(E^* \odot \mathcal{E} \odot F) \odot (F^* \odot \mathcal{F} \odot G) \;=\; E^* \odot \mathcal{E} \odot (F \odot F^*) \odot \mathcal{F} \odot G$$
$$=\; E^* \odot \mathcal{E} \odot \mathcal{K}(F) \odot \mathcal{F} \odot G \;=\; E^* \odot (\mathcal{E} \odot \mathcal{F}) \odot G.$$

So "sandwiching" between the representation modules (or spaces) preserves tensor products! ($\rightsquigarrow$ Morita equivalence.)

Applications:

- ϑ a strict E_0–semigroup on $\mathcal{B}^a(E)$ with GNS-systems $(\mathcal{B}^a(E)_t)_{t\in\mathbb{R}_+}$.
$\rightsquigarrow$ $E_t := E^* \odot \mathcal{B}^a(E)_t \odot E = E^* \odot_t E$ is product system via

$$(x^* \odot_s x') \odot (y^* \odot_t y') \longmapsto x^* \odot_{s+t} \vartheta_t(x'y^*)y'.$$

(With "unit vector" MS [Ske02]. General [Ske09b] (preprint 2004).

- Special case: E a Hilbert spaces gives Bhat's construction [Bha96] of the (anti-)Arveson system [Arv89] of ϑ. ("Reverse" difficult!)

- $\mathcal{E}^\odot = (\mathcal{E}_t)_{t\in\mathbb{R}_+}$ the GNS-system of a strict CP-semigroup T on $\mathcal{B}^a(E)$. Then $E_t := E^* \odot \mathcal{E}_t \odot E$ gives a product system $E^\odot = (E_t)_{t\in\mathbb{R}_+}$ of $\mathcal{B}$–correspondences.

- Special case: E a Hilbert spaces gives Bhat's Arveson system of T [Bha96] **without** dilating T first to an endomorphism semigroup.

Only briefly: Positivity in $*$–algebras

- For instance: b in a pre-C^*–algebra is positive when positive in $\overline{\mathcal{B}}$.
  b has a square root $\beta \in \overline{\mathcal{B}}$.

- For instance: $b \in \mathcal{L}^a(G)$ (G a pre-Hilbert space) is positive if $\langle g, bg \rangle \geq 0$ for every $g \in G$.
  By an application of Friedrich's theorem, $b \in \mathcal{B}$ has a square root $\beta \in \mathcal{L}^a(G, G')$ where $G \subset G' \subset \overline{G}$).

- New: Let $\mathcal{B}$ be a unital $*$–algebra and $\mathcal{S}$ a set of positive linear functionals on $\mathcal{B}$.
  $b \in \mathcal{B}$ is $\mathcal{S}$–positive if $\varphi(c^*bc) \geq 0$ for all $\varphi \in \mathcal{S}$ and $c \in \mathcal{B}$.
  $\mathcal{B}$ is $\mathcal{S}$–separated if $\varphi(cbc') = 0 \forall \varphi \in \mathcal{S}; c, c' \in \mathcal{B}$ implies $b = 0$.

Example: Let $\mathcal{B} = \mathbb{C}\langle x \rangle$. Let $Z \subset \mathbb{C}$. Put $\mathcal{S} = \{\varphi_w \colon p \mapsto p(w), w \in Z\}$.

- $Z = \mathbb{R}$ or $Z = \mathbb{S}^1$. Then $p \geq 0 \iff \exists q \in \mathcal{B} \colon \overline{q}q = p$.

- $Z = \mathbb{C}$. Then $p \geq 0 \implies p = 0$. (Liouville.)

- $Z \subset \mathbb{C}$ compact and $Z \backslash \partial Z \neq \emptyset$. Then $\mathcal{B} \subset C(Z) = \overline{\mathcal{B}}$ and $p \geq 0 \iff \exists f \in C(Z) \colon \overline{f}f = p$.
  For instance, $Z = [-1, 0]$, $p = -x$
  $\rightsquigarrow p = \overline{f}f \geq 0$ where $f = \sqrt{-x} \in C[-1, 0]$.

Denote by G the direct sum of the GNS-pre-Hilbert spaces of all $\varphi \in \mathcal{S}$. Identify $\mathcal{B} \subset \mathcal{L}^a(G)$.

1 Theorem. *Let $\mathcal{A}$ be a unital $*$–algebra. Let $\mathfrak{K}\colon S \times S \to \mathcal{L}(\mathcal{A}, \mathcal{B})$ be a kernel over S from $\mathcal{A}$ to $\mathcal{B}$. If $\mathfrak{K}$ is CPD in the sense that*

$$\sum_{i,j} b_i^* \mathfrak{K}^{(\sigma_i,\sigma_j)}(a_i^* a_j) b_j$$

is $\mathcal{S}$–positive for all finite choices, then there exists a pre-Hilbert space H with a left action of $\mathcal{A}$, and a map $i\colon S \to \mathcal{L}^a(G, H)$ such that

$$\mathfrak{K}^{\sigma,\sigma'}(a) \;=\; i(\sigma)^* a i(\sigma')$$

for all $\sigma, \sigma' \in S$ and $a \in \mathcal{A}$.

We refer to $(\operatorname{span} \mathcal{A} i(S) \mathcal{B}, i)$ *as the Kolmogorov decomposition of $\mathfrak{K}$.*

References

[AK01] L. Accardi and S. Kozyrev, *On the structure of Markov flows*, Chaos Solitons Fractals **12** (2001), 2639–2655.

[Arv89] W. Arveson, *Continuous analogues of Fock space*, Mem. Amer. Math. Soc., no. 409, American Mathematical Society, 1989.

[BBLS04] S.D. Barreto, B.V.R. Bhat, V. Liebscher, and M. Skeide, *Type I product systems of Hilbert modules*, J. Funct. Anal. **212** (2004), 121–181, (Preprint, Cottbus 2001).

[Bha96] B.V.R. Bhat, *An index theory for quantum dynamical semigroups*, Trans. Amer. Math. Soc. **348** (1996), 561–583.

[BLS10] B.V.R. Bhat, V. Liebscher, and M. Skeide, *Subsystems of Fock need not be Fock: Spatial CP-semigroups*, Proc. Amer. Math. Soc. **138** (2010), 2443–2456, electronically Feb 2010, (arXiv: 0804.2169v2).

[BRS10] B.V.R. Bhat, G. Ramesh, and K. Sumesh, *Stinespring's theorem for maps on Hilbert C^*–modules*, Preprint, arXiv: 1001.3743v1, 2010, To appear in J. Operator Theory.

[BS00] B.V.R. Bhat and M. Skeide, *Tensor product systems of Hilbert modules and dilations of completely positive semigroups*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **3** (2000), 519–575, (Rome, Volterra-Preprint 1999/0370).

[Heo99] J. Heo, *Completely multi-positive linear maps and* *representations on Hilbert C^*–modules*, J. Operator Theory **41** (1999), 3–22.

[Pas73] W.L. Paschke, *Inner product modules over B^*–algebras*, Trans. Amer. Math. Soc. **182** (1973), 443–468.

[PS72] K.R. Parthasarathy and K. Schmidt, *Positive definite kernels, continuous tensor products, and central limit theorems of probability theory*, Lect. Notes Math., no. 272, Springer, 1972.

[Ske02] M. Skeide, *Dilations, product systems and weak dilations*, Math. Notes **71** (2002), 914–923.

[Ske06] ______, *The index of (white) noises and their product systems*, Infin. Dimens. Anal. Quantum Probab. Relat. Top.

9 (2006), 617–655, (Rome, Volterra-Preprint 2001/0458, arXiv: math.OA/0601228).

[Ske09a] ______, *Classification of E_0–semigroups by product systems*, Preprint, arXiv: 0901.1798v2, 2009.

[Ske09b] ______, *Unit vectors, Morita equivalence and endomorphisms*, Publ. Res. Inst. Math. Sci. **45** (2009), 475–518, (arXiv: math.OA/0412231v5 (Version 5)).

[Spe98] R. Speicher, *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, Mem. Amer. Math. Soc., no. 627, American Mathematical Society, 1998.

[SSV07] M. Schürmann, M. Skeide, and S. Volkwardt, *Transformations of Lévy processes*, Greifswald-Preprint no.13/2007, arXiv: 0712.3504v2, 2007.