Noncommutative Poisson Boundaries

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Introduction

Let $H^\infty(D)$ be the set of bounded harmonic functions of the unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$, which is a closed subspace of $L^\infty(D)$.

The classical Poisson integral formula

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} \tilde{f}(e^{i\varphi}) d\varphi,$$

shows that the map $H^\infty(D) \ni f \mapsto \tilde{f} \in L^\infty(\partial D)$ is an isometry.

$H^\infty(D)$ has a hidden algebra structure.

Where does it come from?

Answer

It comes from the heat semigroup $\{P_t\}_{t\geq 0}$ acting on $L^\infty(D)$, where $P_t = e^{t\Delta}$ and Δ is the Laplacian with respect to the Poincare metric.

$H^\infty(D)$ is nothing but the set of fixed points $\{f \in L^\infty(D) \mid P_t(f) = f, \ \forall t > 0\}$.

The von Neumann algebra structure of $L^\infty(\partial D)$ can been seen from the pair $(L^\infty(D), \{P_t\}_{t\geq 0})$.

For $f, g \in H^\infty(D)$,

$$\text{“}\lim_{T\to\infty} \frac{1}{T} \int_0^T P_t(fg)dt\text{”}$$

gives the hidden product of f and g.

The space of harmonic elements

Definition

A Markov operator P on a von Neumann algebra M is a unital normal completely positive map from M to itself.
For a Markov operator P on M, set
$H^\infty(M,P) = \{x \in M \mid P(x) = x\}$.

$H^\infty(M,P)$ is an operator system, i.e., it is a self-adjoint subspace of M containing scalars, though it is not an algebra in general.

Choose $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, and set

$$E_\omega(x) = \mathrm{w-}\lim_{n\to\omega} \frac{1}{N} \sum_{n=0}^{n-1} P^n(x), \quad x \in M.$$

$E_\omega : M \to H^\infty(M,P)$ is a completely positive projection.

Choi-Effros product

Theorem (Choi-Effros 77)

Let M be a von Neumann algebra, and let $X \subset M$ be a weakly closed operator system.
If there exists a completely positive projection $E : M \to X$, X is a von Neumann algebra with respect to the product $x \circ y = E(xy)$.

$H^\infty(M, P)$ is a von Neumann algebra with the Choi-Effros product $x \circ y = E_\omega(xy)$, which does not depend on $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$.

Definition

A concrete realization of the von Neumann algebra structure of $H^\infty(M, P)$ is called the noncommutative Poisson boundary for (M, P).

Example

Let S be the unilateral shift on $\ell^2(\mathbb{Z}_+)$.
Let $P(x) = S^*xS$ for $x \in B(\ell^2(\mathbb{Z}_+))$.
P is a Markov operator acting on $B(\ell^2(\mathbb{Z}_+))$.

Then

$$H^\infty(B(\ell^2(\mathbb{Z}_+)), P) = \{T_f \mid f \in L^\infty(\mathbb{T})\},$$

where T_f is the Toeplitz operator with symbol f
(identify $\ell^2(\mathbb{Z}_+)$ with the Hardy space $H^2(\mathbb{T})$).

The noncommutative Poisson boundary for $(B(\ell^2(\mathbb{Z}+)), P)$ is
$L^\infty(\mathbb{T})$.

Random walk on a group

Let Γ be a discrete group, and let μ be a probability measure on Γ whose support generate Γ as a semigroup.
For $f \in \ell^\infty(\Gamma)$, set $P_\mu(f) = f * \mu$.
Transition probability: $p(g,h) = \mu\{g^{-1}h\}$.

Since $\ell^\infty(\Gamma)$ is commutative, the Choi-Effros product on $H^\infty(\Gamma,\mu) := H^\infty(\ell^\infty(\Gamma), P_\mu)$ is commutative too.
There exists a completely positive isometry θ from $H^\infty(\Gamma,\mu)$ onto an abelian von Neumann algebra A satisfying $\theta(f \circ g) = \theta(f)\theta(g)$.

If $f \in H^\infty(\Gamma,\mu)$ is non-negative and $f(e) = 0$, the mean-value property of f implies $f = 0$.
$A \ni \varphi \mapsto \theta^{-1}(\varphi)(e) \in \mathbb{C}$ is a faithful normal state.

There exists a probability space (Ω, ν) with $A = L^\infty(\Omega, \nu)$ satisfying

$$f(e) = \int_\Omega \theta(f)(\omega) d\nu(\omega), \quad \forall f \in H^\infty(\Gamma, \mu).$$

For $H^\infty(\Gamma, \nu)$, we set $\alpha_\gamma f(\sigma) = f(\gamma^{-1}\sigma)$.
Since α_γ commutes with P_μ, α_γ induces an automorphism $\tilde{\alpha}_\gamma$ of $L^\infty(\Omega, \nu)$, and hence a Γ-action on (Ω, ν) by nonsingular transformations.
$\theta(\alpha_\gamma(f))(\omega) = \theta(f)(\gamma^{-1} \cdot \omega)$.

(Ω, ν) is called the Poisson boundary for (Γ, μ).

Poisson integral formula

For $f \in H^\infty(\Gamma, \mu)$,

$$\begin{aligned} f(\gamma) &= \alpha_{\gamma^{-1}}(f)(e) \\ &= \int_\Omega \theta(\alpha_{\gamma^{-1}}(f))(\omega) d\nu(\omega) \\ &= \int_\Omega \theta(f)(\gamma \cdot \omega) d\nu(\omega) \\ &= \int_\Omega \frac{d\nu(\gamma^{-1} \cdot)}{d\nu}(\omega) \theta(f)(\omega) d\nu(\omega). \end{aligned}$$

$\frac{d\nu(\gamma^{-1}\cdot)}{d\nu}(\omega)$ is an analogue of the Poisson kernel.

Noncommutative extension

Let ρ be the right regular representation of Γ.
For $\ell^\infty(\Gamma) \subset B(\ell^2(\Gamma))$,

$$P_\mu(f) = \sum_{\gamma\in\Gamma} \mu(\gamma)\rho(\gamma) f \rho(\gamma)^{-1}.$$

Set

$$Q_\mu(x) = \sum_{\gamma\in\Gamma} \mu(\gamma)\rho(\gamma) x \rho(\gamma)^{-1}, \quad x \in B(\ell^2(\Gamma)).$$

What is the noncommutative Poisson boundary for $(B(\ell^2(\Gamma)), Q_\mu)$?

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What is the noncommutative Poisson boundary for $(B(\ell^2(\Gamma)), Q_\mu)$?

Answer

The crossed product $L^\infty(\Omega, \nu) \rtimes \Gamma$,
(I. 2004, Jaworski-Neufang 2007).

Random walk on compact group dual

Let $R(G)$ be the group von Neumann algebra of a compact group G, which is generated by the right regular representation $(\rho, L^2(G))$, and

$$R(G) = \bigoplus_{\pi \in \hat{G}} R(G)_\pi, \quad R(G)_\pi \cong M_{n_\pi}(\mathbb{C}),$$

$R(G)$ has a coproduct
$\Delta_G : R(G) \ni \rho(g) \mapsto \rho(g) \otimes \rho(g) \in R(G) \otimes R(G)$.
For a probability measure μ on $\hat{G}$, we set

$$P_\mu = \sum_{\pi \in \hat{G}} \mu(\pi)(\mathrm{id} \otimes \mathrm{tr}_\pi) \circ \Delta_G.$$

$H^\infty(R(G), P_\mu) = \mathbb{C}$.

ITP action

Let (π, H_π) be a finite dimensional unitary representation of a compact group G.
The infinite tensor product (ITP)

$$(M, \tau, \alpha_g) = \bigotimes_{k=1}^{\infty} (B(H_\pi), \mathrm{tr}_\pi, \mathrm{Ad}\, \pi(g))$$

gives an action α on the hyperfinite II_1 factor M.
α is minimal, i.e., $M \cap M^{\alpha\prime} = \mathbb{C}$.

The ITP action makes sense for a compact quantum group $\mathbb{G}$ if tr_π is replace by the so-called quantum trace τ_π.
But $M \cap M^{\alpha\prime}$ is not trivial in general.

Theorem (I.2002)

Let (π, H_π) be a finite dimensional unitary representation of a compact quantum group $\mathbb{G}$ such that every irreducible representation of $\mathbb{G}$ is contained in a tensor power of π.
Let α be the corresponding ITP action of $\mathbb{G}$ on M.
Then $M \cap M^{\alpha\prime}$ is the noncommutative Poisson boundary for $(R(\mathbb{G}), P_\mu)$, where μ is a probability measure determined by (π, H_π).

Corollary

Let the notation be as above.
If $\mathbb{G}$ is not a Kac algebra, the ITP action α is not minimal.

Denote $H^\infty(\hat{\mathbb{G}}, \mu) = H^\infty(R(\mathbb{G}), P_\mu)$.

Outline of proof

Let $M_n = \bigotimes_{k=1}^{n} B(H_\pi)$.

Let $E_n : M \to M_n$ be the $\bigotimes_{k=1}^{\infty} \tau_\pi$-preserving conditional expectation.

π induces a homomorphism $\tilde{\pi} : R(\mathbb{G}) \to B(H_\pi)$,
and hence a homomorphism $\tilde{\pi}^{\otimes n} : R(\mathbb{G}) \to M_n$ satisfying
$E_n \circ \tilde{\pi}^{\otimes(n+1)} = \tilde{\pi}^{\otimes n} \circ P_\mu$.

For $x \in H^\infty(\hat{\mathbb{G}}, \mu)$, $\{\tilde{\pi}^{\otimes n}(x)\}_{n=1}^{\infty}$ is a martingale, and

$$\mathrm{s-}\lim_{n\to\infty} \tilde{\pi}^{\otimes n}(x) \in M \cap M^{\alpha\prime}.$$

Identification problem

I. 2002,
$\mathbb{G} = SU_q(2)$, $H^\infty(\hat{\mathbb{G}}, \mu) \cong L^\infty(SU_q(2)/\mathbb{T})$,

I.-Neshveyev-Tuset 2006,
$\mathbb{G} = SU_q(N)$, $H^\infty(\hat{\mathbb{G}}, \mu) \cong L^\infty(SU_q(N)/\mathbb{T}^{N-1})$.

Tomatsu 2007,
$\mathbb{G} = q$-deformation of a classical group, $H^\infty(\hat{\mathbb{G}}, \mu) \cong L^\infty(\mathbb{G}/T)$.

Vaes-Vander Vennet 2008,
$\mathbb{G} = A_o(F)$.

Vaes-Vander Vennet 2010,
$\mathbb{G} = A_u(F)$.

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