A Discussion on Multiparameter Quantum Stochastic Processes

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Abstract

Quantum stochastic analogues $(\mathcal{H}, \mathcal{A}, \{\mathcal{A}_z\}_{z\in\mathcal{R}_+}, m, \mathcal{R}_+)$, of a classical stochastic base may be formed whereby a classical sample space Ω is replaced by a Hilbert Space $\mathcal{H}$, σ - field $\mathcal{F}$ is replaced by a von Neumann algebra $\mathcal{A}$, the filtration $\{\mathcal{F}_i\}_{i\in I}$ by a filtration $\{\mathcal{A}_z\}_z$ of von Neumann subalgebras of the von Neumann algebra $\mathcal{A}$ and the probability measure $\mathcal{P}$ with gage m [2].

In this presentation we consider quantum analogues for multiparameter stochastic processes, extending quantum results in [4, 5, 6, 7, 8].

Outline

1. Background
   - Previous Work
   - Construction
   - POSETs
2. Stochastic Integrals
   - Adapted Processes
   - Integrals
   - Properties
   - Representation

Previous Work

- Classical
  - Ito (1 Parameter Space)
  - Cairoli, Walsh, Wong & Zakai, (2 Parameter Space)
  - Imkeller (3 Parameter Space)
- Quantum
  - BSW, Hudson, Lindsay, Parthasarathy, ... (1 Parameter Space)
  - Spring and Wilde (2 Parameter Space)
  - Spring (3, . . . , n Parameter Space)

Stochastic Base

We work with a quantum stochastic base

$$(\mathcal{F}(\mathcal{H}), \mathcal{A}, \{\mathcal{A}_z\}_{z\in\mathcal{R}_+}, m, \mathcal{R}_+)$$

Definition

$\mathcal{F}(\mathcal{H}) = \bigoplus_{r=1}^{\infty} \mathcal{H}_r$ denotes the Fermi-Fock antisymmetric space over $\mathcal{H} = L^2(\mathbb{R}^n_+)$ the Fermion "one particle Hilbert space"

Stochastic Base

$$(\mathcal{F}(\mathcal{H}), \mathcal{A}, \{\mathcal{A}_z\}_{z\in\mathcal{R}_+}, m, \mathcal{R}_+)$$

Definition

$\mathcal{A}$ a non commutative von Neumann Algebra generated by the fermion field $\Psi(u) = a^*(u) + a(u)$, $u \in L^2(\mathbb{R}^n_+)$ real valued, (bounded) operators acting on $\mathcal{F}(\mathcal{H})$, satisfying the CAR relations $\{\Psi(u), \Psi(v)\} = 2(u, v)\mathbb{I}$, a^* and a the creation and annihilation operators acting on $\mathcal{F}(\mathcal{H}), v \in L^2(\mathbb{R}^n_+)$

Stochastic Base

$$(\mathcal{F}(\mathcal{H}), \mathcal{A}, \{\mathcal{A}_z\}_{z \in \mathcal{R}_+}, m, \mathcal{R}_+)$$

Definition

$\{\mathcal{A}_z\}_{z \in \mathcal{R}_+}$ is a filtration [2, 6] of sub von Neumann Algebras in $\mathcal{A}$ with index set $\mathcal{R}_+$. A filtration of von Neumann algebras $(\mathcal{A}_z)$ may be defined in $\mathcal{A}$ by restricting the fermion fields ψ to functions $f \in L^2_{loc}(\mathbb{R}^n_+)$ of the form $\chi_{R_z} g \in L^2(R_z) \subseteq L^2(\mathbb{R}^n_+)$, R_z denoting the n-dimensional cube with inf $R_z = 0$, the origin.

Stochastic Base

$$(\mathcal{F}(\mathcal{H}), \mathcal{A}, \{\mathcal{A}_z\}_{z \in \mathcal{R}_+}, m, \mathcal{R}_+)$$

Definition

m a gage defined upon $\mathcal{A}$ by $m(.) = (\Omega, .\Omega)$; Ω denotes $1 \in \mathbb{C} = \mathcal{H}_0 \subset \mathcal{F}(\mathcal{H})$

A gage m is:

- faithful if $A \in \mathcal{A}, A \geq 0, m(A) = 0 \implies A = 0$
- central if $\forall A, B \in \mathcal{A}$ $m(AB) = m(BA)$
- normal if given a family $\{P_\alpha\}$ of mutually orthogonal projections in $\mathcal{A}$, $m(\sum_\alpha P_\alpha) = \sum_\alpha m(P_\alpha)$

Stochastic Base

- Noncommutative $L^p(\mathcal{A})$ spaces, [3], for $1 \leq p < \infty$ may be formed via the relation $m(|a|) = (\Omega, |a|^p \Omega)^{1/p}$, with $L^\infty(\mathcal{A})$ taking the usual operator norm
- Conditional expectations $\mathbb{E}(\circ|\mathcal{B}) : \mathcal{A} \longrightarrow \mathcal{B}$ may be extended to $L^p(\mathcal{A}) \longrightarrow L^P(\mathcal{B})$ [1, 2]
- Processes $(\psi(\chi_{R_z} g))$ form centred martingales [1, 6]

POSET's

We work with partially ordered sets of the following form:

- $z \prec z'$ if all coordinates in z are less than or equal to those in z'. We say that z' is forward of z in all coordinates.

For the case $n = 2$ we additionally use:

- $z {}^{\wedge}_{\wedge} z'$ if z' is forward of z in all but one coordinate

For $n = 3$ we also use:

- $z {}^{\wedge}_{\wedge} z' {}^{\wedge}_{\wedge} z''$ if z is forward of z' and z'' in just one coordinate (say x), and z' is forward of z and z'' in another coordinate (say y) and z'' is forward of z and z' in the remaining coordinate (say z)

Adapted Processes

Definition

A map $h : \mathbb{R}_+^n \times \cdots \times \mathbb{R}_+^n \longrightarrow L^2(\mathcal{A})$ is said to be a $\mathcal{A}$ valued elementary r adapted process if $\forall z_i \in \mathbb{R}_+^n$ with $1 \leq i \leq n$,

$$\exists\, \Delta_1, \ldots \Delta_r \text{ with } \Delta_1 \overset{\wedge}{\wedge} \ldots \overset{\wedge}{\wedge} \Delta_r$$

and h of the form

$$h(z_1, \ldots, z_r) = a \underset{i=1}{\overset{r}{\pi}} \chi_{\Delta_i}(z_i) \in L^2(\mathcal{A}) \text{ with } a \in \mathcal{A}_{\inf \Delta_1 \wedge \cdots \wedge \inf \Delta_r}$$

An $L^2(\mathcal{A})$ - valued adapted process is said to be *simple* if it is a finite linear combination of elementary $L^2(\mathcal{A})$ - valued adapted processes

Adapted Processes - Examples

Example

- $a\chi_{\Delta}$ with $a \in \mathcal{A}_{inf\,\Delta}$
  - A type 1 adapted process
- $a\chi_{\Delta_1}\chi_{\Delta_2}$ with $a \in \mathcal{A}_{inf\,\Delta_1 \wedge inf\,\Delta_2}$ $\Delta_1 \overset{\wedge}{\wedge} \Delta_2$.
  - A type 2 adapted process
- $a\chi_{\Delta_1}\chi_{\Delta_2}\chi_{\Delta_3}$ with $a \in \mathcal{A}_{inf\,\Delta_1 \wedge inf\,\Delta_2 \wedge inf\,\Delta_3}$. $\Delta_1 \overset{\wedge}{\wedge} \Delta_2 \overset{\wedge}{\wedge} \Delta_3$.
  - A type 3 adapted process

Such processes may be presented in various forms however in the completion of the space of quantum stochastic integrals they are seen to lead to equivalent forms.

Integral

Definition

Let $h(z_1, \ldots, z_r) = a \overset{r}{\underset{i=1}{\pi}} \chi_{\Delta_i}(z_i)$, denote elementary r adapted processes with $a \in \mathcal{A}_{inf\,\Delta_1 \wedge \cdots \wedge inf\,\Delta_r}$ and each $z_i \in \mathbb{R}^n_+$. We define the type r integral $\mathcal{S}_r$ of h over R_z to be

$$\mathcal{S}_r(h, z, f_1, \ldots, f_r) = \int_{R_z} \ldots \int_{R_z} h(z_1, \ldots, z_r) d\psi_{z_1}(f_1) \ldots d\psi_{z_r}(f_r)$$
$$= a \overset{r}{\underset{i=1}{\pi}} \psi(\chi_{\Delta_i \cap R_z} f_i)$$

We extend to simple adapted processes on $\mathbb{R}^n_+ \times \cdots \times \mathbb{R}^n_+$ and their respective integrals via linearity.

Integrals - Examples

Example

For the case of $\mathbb{R}_+$ the increment forward of $z \in \mathbb{R}_+$ was of the form Δ_z an interval of the form $[z, z')$. This led to the Ito -Clifford integral as discussed in [2].

Integrals - Examples

Example

In $\mathbb{R}^2_+$ the increment Δ_z took the form of a square in which all points were forward (in both parameters) of the point $z \in \mathbb{R}^2_+$ and led to the 2- parameter version of the Ito-Cifford integral. In addition to the square forward of z in 2 parameters it was also possible to form the Wong-Zakai Clifford integral, a quantum analogue of the classical Wong-Zakai integral in which the increments contained points in the parameter space that were forward of the point $z \in \mathbb{R}^2_+$ in just one parameter [7, 8].

Integrals - Examples

Example

The case of $\mathbb{R}^3_+$ leads one to consider increments forward of a point $z \in \mathbb{R}^3_+$ in three, two and one parameters, leading to further new integrals.

Properties

Theorem

Each of the above integrals are

- *orthogonal* $m(\mathcal{S}_i^* \mathcal{S}_j) = 0$ *for* $i \neq j$
- *isometric*

$$\| \mathcal{S}_r(h, z, f_1, \ldots, f_r) \|_2^2 = \int\limits_{R_z} \ldots \int\limits_{R_z} \| h \|_2^2 \prod_{j=1}^{r} |f_j|^2 dz_1 dz_2 \ldots dz_r$$

- *centred martingales*

Completion

- The integrals described above extend via the isometry property to completions $L^2_{\psi}, L^2_{\psi\psi}$ and $L^2_{\psi\psi\psi}$ of the simple type 1, 2 and 3 adapted processes in $L^2(\mathcal{A})$ where they continue to satisfy the same isometry and martingale properties
- Processes belonging to the respective completions of type r simple processes are themselves found to be orthogonal, isometric, centred martingales

Representation

Representation Theorems have been established for Clifford and Quasi-Free settings for the case $n = 2$ and $n = 3$:

Theorem

Let $(X_z)_{z\in\mathbb{R}^3_+}$ denote an $L^2(\mathcal{A})$ valued martingale adapted to the family $(\mathcal{A}_z)_{z\in\mathbb{R}^3_+}$ of von Neumann subalgebras of $\mathcal{A}$. Then $\exists$ unique $f \in L^2_{\psi}$, $g \in L^2_{\psi\psi}$, and $h \in L^2_{\psi\psi\psi}$ s.t.

$$X_z = X_0 + \mathcal{S}_1(f,z) + \mathcal{S}_2(g,z) + \mathcal{S}_3(h,z)$$

Further work has been developed for the general case in both Clifford and quasi-free settings. The results will appear elsewhere.

Summary

- Increments
- Stochastic integrals
- Properties
- Representation Theorem

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