Generators for Arithmetic Groups

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The Example of $SL_2(\mathbb{R})$

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If $N \subset \Gamma$ is of finite index m, then N is also a free group, on h generators, say, and a comparison of the Euler Characteristics of Γ and N shows that

$$h - 1 = m(g - 1).$$

Consequently, the smallest number of generators of N (namely h) *increases* with respect to the index m.

Co-compact lattices in $SL_2(\mathbb{R})$

Similarly, if $\Gamma \subset SL_2(\mathbb{R})$ is a torsion-free co-compact discrete subgroup, then Γ is generated by $2g$ generators, call them $a_1, a_2, \cdots, a_g$ and $b_1, b_2, \cdots b_g$, with one relation:

$$\prod [a_i, b_i] = 1.$$

In this relation, $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ is the commutator of a_i and b_i.

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If $N \subset \Gamma$ is of finite index m and has $2g'$ generators, then a comparison of Euler characteristics of Γ and N shows that

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If $N \subset \Gamma$ is of finite index m and has $2g'$ generators, then a comparison of Euler characteristics of Γ and N shows that

$$m(2 - 2g) = 2 - 2g'.$$

Consequence: The number of generators for N grows linearly with the index m.

What happens with $SL_3(\mathbb{Z})$?

We replace $SL_2(\mathbb{Z})$ by $SL_3(\mathbb{Z})$. Recall that $SL_3(\mathbb{Z})$ is generated by upper and lower triangular unipotent matrices of the form $1 + E_{ij}$ where 1 is the 3×3 identity matrix and for $1 \leq i \neq j \leq 3$, E_{ij} is the matrix whose ij-th entry is 1 and all other entries are zero.

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A result of Bass, Milnor and Serre tells us that if $m \geq 1$ is any integer, then the subgroup Γ_m generated by the 6 elements $1 + mE_{ij}$ $1 \leq i, j \leq 3$ generates a subgroup of finite index in $SL_3(\mathbb{Z})$.

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Consequence: if $N \subset SL_3(\mathbb{Z})$ is a normal subgroup of finite index of index m, then N contains Γ_m which is also of finite index and has at most 6 generators.

the case of $SL_n(\mathbb{Z})$

Suppose $N \geq 3$, 1 is the $N \times N$ identity matrix, and E_{ij} denotes the $N \times N$ matrix all of whose entries are zero except the ij-th entry, which is 1. Then, for $i \neq j$, the matrix $1 + tE_{ij}$ is unipotent and lies in $SL_n(\mathbb{Z})$ if $t \in \mathbb{Z}$.

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Clearly, Γ_m is generated by $N(N-1)$ elements.
If Δ is a normal subgroup of index m in $SL_n(\mathbb{Z})$, then $\Delta \supset \Gamma_m$.
Consequently, every finite index subgroup Δ of $SL_n(\mathbb{Z})$ contains a finite index subgroup generated by $C = N(N-1)$ elements. Thus the number of elements needed to generate *some* finite index subgroup of Δ is independent of the finite index subgroup Δ.

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Given a finitely generated group Γ, does there exist a number C depending on Γ such that every finite index subgroup Δ of Γ contains a smaller finite index subgroup generated by C elements?

- The answer is **no** for $SL_2(\mathbb{Z})$, as we have seen.
- The answer is **yes** for the group $SL_N(\mathbb{Z})$, for $N \geq 3$. The constant C can be taken to be $N(N-1)$.

Other Groups

The group $SL_N(\mathbb{Z})$ is a discrete subgroup of $SL_N(\mathbb{R})$ with finite covolume with respect to the invariant Haar measure on $SL_N(\mathbb{R})$. Furthermore, the group $SL_N(\mathbb{R})$ is a *simple Lie group*. That is, it does not have connected normal subgroups other than the trivial group and itself.

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We will now consider a simple Lie group G and a discrete subgroup $\Gamma \subset G$ of finite covolume (Γ is then called a **lattice** in G).

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We will now consider a simple Lie group G and a discrete subgroup $\Gamma \subset G$ of finite covolume (Γ is then called a **lattice** in G).

We now ask the question: what are the lattices in simple Lie groups for which the foregoing question has a positive answer? That is, is there a number $C = C(G)$ depending only on G such that if Γ is a lattice in G, then Γ contains a subgroup of finite index which is generated by C elements?

Non-uniform Arithmetic Groups

Suppose G is an algebraic simple group defined over $\mathbb{Q}$. That is , $G \subset SL_N$ is a subgroup which is the set of zeroes of polynomials in the matrix entries x_{ij} $(1 \leq i, j \leq N)$; moreover, these polynomials have coefficients in $\mathbb{Q}$.

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Set $G = G(\mathbb{R})$. That is, $G(\mathbb{R}) = SL_N(\mathbb{R}) \cap G$. Let $\Gamma = G(\mathbb{Z}))$. Then it is a well known result of Borel and Harish-Chandra that Γ is a lattice in $G(\mathbb{R})$.

Suppose $G \subset SL_N$ is an algebraic subgroup defined over $\mathbb{Q}$. let D_N be the subgroup of diagonal matrices in SL_N. Define the $\mathbb{Q}$-rank of G to be the supremum of the dimensions $dim(gGg^{-1} \cap D_N)$ as g runs through elements of $SL_N(\mathbb{Q})$.

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It is a result of Borel and Harish Chandra that G/Γ is non-compact if and only if $\mathbb{Q} - rank(G)$ is at least one.

It also follows from the theory of algebraic groups that there then exist proper parabolic subgroups of G defined over $\mathbb{Q}$.

Let

- P_0 be a minimal parabolic subgroup of G defined over $\mathbb{Q}$.
- U_0 be its unipotent radical.
- $T \subset P_0$ be a maximal $\mathbb{Q}$-split torus. Then on the Lie algebra of U_0, T acts by positive roots.
- U_0^- the subgroup of G whose Lie algebra is spanned by the negative root spaces for the action of T.

If $\Gamma \subset G(\mathbb{Z})$ is of finite index, then the intersection $\Gamma \cap U_0$ is generated by $dim(U_0)$-elements. This folows by an induction on the degree of nilpotency of the nilpotent group U_0.

Real Rank of a Simple Lie Group

A linear simple Lie group is a subgroup of $SL_n(\mathbb{R})$ for some integer n. Suppose that the real rank of G is at least one: recall that the real rank of G is the supremum of the numbers

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For example, the real rank of $SL_N(\mathbb{R})$ is $N - 1$. The real rank of $SL_2(\mathbb{R})$ is 1 and for $N \geq 3$, the real rank of $SL_N(\mathbb{R})$ is $N - 1 \geq 2$.

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- When $G = SL_N$ with $N \geq 3$,and $G = Sp_{2g}$ with $g \geq 2$ this result is the Theorem of Bass,Milnor and Serre.

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If G is a $\mathbb{Q}$-simple algebraic group defined over $\mathbb{Q}$, and $\mathbb{R}-rank(G) \geq 2$, $\mathbb{Q}-rank(G) \geq 1$. Suppose $\Gamma \subset G(\mathbb{Z})$ is a subgroup of finite index. Then the subgroup Γ' generated by $\Gamma \cap U_0$ and $\Gamma \cap U_0^-$ is of finite index in Γ. In particular, Γ contains a subgroup of finite index generated by $C = 2dim(U_0)$ elements.

- When $G = SL_N$ with $N \geq 3$,and $G = Sp_{2g}$ with $g \geq 2$ this result is the Theorem of Bass,Milnor and Serre.
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- (V.) In the remaining cases.

Minimal number of generators

If a group Γ has a subgroup of finite index with m generators, let us say that Γ is **virtually** m generated. Thus, the foregoing theorem says that every $\Gamma \subset G(\mathbb{Z})$ is virtually $2dimU_0$-generated.

It is thus of interest to know the smallest number m such that every arithmetic $\Gamma \subset G(\mathbb{Z})$ is virtually m-generated.

The Statement of the Theorem

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Suppose Γ is a lattice in a simple Lie group of real rank least two, Suppose that the quotient G/Γ is not compact. Then every subgroup Δ of finite index in Γ contains a smaller subgroup of finite index generated by **three** *elements.*

- We have already seen that if $G = SL_2(\mathbb{R})$ then the result is false. The real rank of $SL_2(\mathbb{R})$ is one.
- More generally, if Γ is an arithmetic subgroup of $SO(n, 1)$ with non-vasnishing first cohomology, (such groups exist), then the analogue of the result is false for Γ. Thus, the assumption on the rank being at least two is necessary.

Proof for $SL_3(\mathbb{Z})$

Consider in $SL_3(\mathbb{Z})$, the set P of elements of the form

$$p = \begin{pmatrix} A & B \\ 0 & d \end{pmatrix}$$

where $A \in GL_2$ and B is a 2×1 matrix (a column vector in $\mathbb{Z}^2$), and 0 is the row vector (a 1×2-matrix) all of whose entries are zero. d is a scalar.

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P is a maximal parabolic subgroup of SL_3. Its unipotent radical U consists of matrixes u

$$u = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$$

where $A = 1$ is the identity 2×2-matrix and $d = 1$.

Denote by M the set of matrices m in P of the form

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If $\theta \in M(\mathbb{Z}) = GL_2(\mathbb{Z})$ is an element of the group of units in a real quadratic extension then θ acts irreducibly on the $\mathbb{Q}$-representation $U = \mathbb{Q}^2$.

If $\Gamma \subset SL_3(\mathbb{Z})$ is of finite index, then, Γ intersects non-trivially with the groups U and U^-. Pick an element $u \in U \cap \Gamma$ with $u \neq 1$. Similarly, pick an element $u^- \in U^- \cap \Gamma$, with $u^- \neq 1$.

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If $\theta \in \Gamma \cap M$ is a unit in a real quadratic extension of $\mathbb{Q}$, then the group generated by the conjugates $\theta u \theta^{-1}$ in $U(\mathbb{Z})$ is of finite index in $U(\mathbb{Z})$. Consequently, the group Δ generated by θ, u, u^- intersects $U(\mathbb{Z})$ and $U^-(\mathbb{Z})$ in finite index subgroups. We now use the Bass-Milnor-Serre theorem to conclude that Δ is of finite index in $SL_3(\mathbb{Z})$.

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- For most lattices $\Gamma \subset G$, there exist elements $u \in U \cap \Gamma$, $u^- \in U^- \cap \Gamma$ and elements $\theta \in M \cap \Gamma$ such that u, u^- and θ generate a subgroup of finite index in Γ.

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  For some groups there does not exist such parabolic subgroup and we check case by case that by different methods, these arithmetic groups can also be shown to have the "‘virtual three generator"’ property.

Open Questions

Question

Suppose that Γ is a co-compact lattice in a simple Lie group of real rank at least two. Is there a number C such that every subgroup Δ of finite index in Γ is virtually C generated.

I don't know a SINGLE example of Γ with this property.

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Recently, I saw on Arxiv, a paper by Alan Ried which says that every subgroup of finite index in $SL_N(\mathbb{Z})$ ($N \geq 3$) is virtually 2 generated.

THANK YOU FOR YOUR ATTENTION