On the geometry of global function fields,

the Riemann–Roch theorem

and

finiteness properties of S-arithmetic groups

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I

What are finiteness properties and how can one determine these?

Finiteness properties

Let G be a group.

A classifying space for G is a pointed CW complex (X, x) such that
- $\pi_1(X, x)$ isomorphic to G and
- $\tilde{X}$ contractible.

A group G is of type F_n, if there exists a classifying space for G with finite n-skeleton.

The finiteness length of a group G is

$$\sup\{n \in \mathbb{N}_0 \mid G \text{ is of type } F_n\}.$$

Fact:
Finite groups are of type F_n for each $n \in \mathbb{N}_0$, i.e. they have finiteness length ∞.

Question: What finiteness properties does $\mathrm{SL}_n(\mathbb{F}_q[t])$ have?

Cayley graphs

Let G be a group with set of generators Z and set of relations R, so that

$$G = \langle Z \mid R \rangle$$

is a presentation of G.

Let Cay(G, Z) be the Cayley graph of G with
- vertex set G and
- edge set $\{(g, gz) \mid g \in G, z \in Z\}$.

Let $X^1 := G \backslash \text{Cay}(G, Z)$.

For each $r \in R$ let Δ_r be a 2-disk whose boundary is divided in $l(r)$ segments which are labelled by the word r (over Z).

Definine X^2 by glueing $(\Delta_r)_{r \in R}$ into X^1 according to the labels.

Observation: $\pi_1(X^2) \cong G$.

Construct a classifying space X for G be glueing higher-dimensional cells into X^2 as necessary.

Finitely generated and finitely presented groups

We conclude:

- Each group G is of type F_0.
- If G finitely generated, then it is of type F_1.
- If G finitely presented, then it is of type F_2.

Conversely, let

- X be a classifying space for G,
- $x \in X$ a 0-cell and
- T a maximal subtree of the graph X^1.

Then $G \cong \pi_1(X, x) \cong \pi_1(X/T, \overline{x})$.

As $\widetilde{X/T}$ contractible, also X/T is a classifying space for G.

Conclusion:
For a group G the following assertions are equivalent:

$$\begin{aligned} \text{finitely generated} &\iff \text{type } F_1, \\ \text{finitely presented} &\iff \text{type } F_2. \end{aligned}$$

A universal tool: Brown's criterion

Theorem 1 (Brown)
Let $n \in \mathbb{N}$ and X a CW complex with $\pi_k(X) = 0$ for $0 \leq k \leq n-1$.

Let Γ be a group that acts cellularly and rigidly on X, such that
- *there exists a Γ-cocompact Γ-filtration*

$$X = \bigcup_{i \in \mathbb{N}} X_i$$

and
- *the stabilizer of each i-cell is of type F_{n-i}.*

Then the group Γ is of type F_n if and only if for each $k \leq n-1$ the directed system

$$\pi_k(X_j) \xrightarrow{j<j'} \pi_k(X_{j'})$$

is essentially trivial.

Goal: Construct a filtration for $\mathrm{SL}_n(\mathbb{F}_q[t])$ on the Bruhat–Tits building of $\mathrm{SL}_n(\mathbb{F}_q((\frac{1}{t})))$.

II

The theorem of Riemann–Roch

and

the geometry of numbers

Non-singular projective curves over $\mathbb{F}_q$

Let k be a perfect field (i.e. irreducible polynomials have zeros of multiplicity 1).

A projective variety over k of dimension 1 is called a projective curve over k.

A projective curve Y is non-singular in $P \in Y$, if the ring $\mathcal{O}_P$ of functions that are regular in P is a discrete valuation ring.

A curve is non-singular, if it is non-singular in each of its points.

Example: The projective line

$$\mathbb{P}_1(\mathbb{C}) \cong \mathbb{C} \cup \{\infty\}$$

is a non-singular projective curve.

For $P \in \mathbb{C}$ one has

$$\begin{aligned} \mathcal{O}_P &= \left\{ \frac{a}{b} \in \mathbb{C}(t) \mid b(P) \neq 0 \right\}, \\ \mathfrak{m} &= \left\{ \frac{a}{b} \in \mathcal{O}_P \mid a(P) = 0 \right\}. \\ \nu_P(x) &:= \sup \left\{ i \in \mathbb{N} \cup \{0\} \mid x \in \mathfrak{m}^i \right\}. \end{aligned}$$

$\nu_P(x)$ counts the multiplicity of the zero P.

Curves and global function fields

Galois descent to a real closed subfield via the action of an involution on $\mathbb{P}_1(\mathbb{C})$:

Closed points $\leftrightarrow$ orbits on $\mathbb{P}_1(\mathbb{C})$

Fixed points $\leftrightarrow$ irr. linear polynomials
Non-fixed points $\leftrightarrow$ irr. quadratic polynomials

Theorem 2
Let $Y/\mathbb{F}_q$ be a non-singular projective curve.

Then there exists a bijection between
- *the set Y° of $\mathbb{F}_q$-closed points of Y and*
- *the set of places of the field $\mathbb{F}_q(Y)$ of $\mathbb{F}_q$-rational functions of Y.*

The degree of an $\mathbb{F}_q$-closed point equals the degree of the corresponding place.

$\mathbb{F}_q(Y)$ is called a global function field.

Weil divisors

Let
- $Y/\mathbb{F}_q$ a non-singular projective curve,
- $K := \mathbb{F}_q(Y)$.

The Weil divisor group $\mathrm{Div}(Y)$ is the free abelian group over Y°.

An element

$$D = \sum_{P \in Y^\circ} n_P \, P \in \mathrm{Div}(Y)$$

is effective ($D \geq 0$), if $n_P \geq 0$ for all $P \in Y^\circ$.

The degree of D is

$$\deg(D) := \sum_{P \in Y^\circ} n_P \deg P.$$

Define $\nu_P(D) := n_P$.

For $0 \neq x \in K$ define the divisor of x as

$$\mathrm{div}(x) := \sum_{P \in Y^\circ} \nu_P(x) P.$$

Riemann–Roch spaces and adèles

The Riemann–Roch space of a divisor D of $Y/\mathbb{F}_q$ is defined as

$$L(D) := \{0 \neq x \in K \mid \mathrm{div}(x) + D \geq 0\} \cup \{0\}.$$

Example: $L(0) = \mathbb{F}_q$.

Define the **ring of adèles**

$$\begin{aligned} \mathbb{A}_K \;\; := \;\; \{ \, (x_P)_{P \in Y^\circ} \in \prod_{P \in Y^\circ} K_P \mid \\ x_P \in \widehat{\mathcal{O}}_P \text{ for almost all } P \in Y^\circ \;\, \} \end{aligned}$$

where

$$\widehat{\mathcal{O}}_P := \varprojlim \mathcal{O}_P / \mathfrak{m}^i \qquad \text{and} \qquad K_P = Q(\widehat{\mathcal{O}}_P).$$

For a divisor D define

$$\begin{aligned} \mathbb{A}_K(D) \;\; := \;\; \{ \, x \in \mathbb{A}_K \mid \\ \nu_P(x) + \nu_P(D) \geq 0 \text{ for all } P \in Y^\circ \, \} \, . \end{aligned}$$

One has $K \cap \mathbb{A}_K(D) = L(D)$.

The Riemann–Roch theorem

Let $Y/\mathbb{F}_q$ be a non-singular projective curve. Its genus is

$$g := \dim_{\mathbb{F}_q}(\mathbb{A}_K/\mathbb{A}_K(0) + K).$$

Theorem 3 (Riemann, Roch)
For each Weil divisor D one has

$$\begin{aligned} & \dim_{\mathbb{F}_q}(L(D)) - \dim_{\mathbb{F}_q}(\mathbb{A}_K/\mathbb{A}_K(D) + K) \\ = \ & \deg(D) + 1 - g. \end{aligned}$$

Example $D = 0$:

$$\begin{aligned} \deg(0) &= 0 \\ \dim_{\mathbb{F}_q}(L(0)) = \dim_{\mathbb{F}_q}(\mathbb{F}_q) &= 1 \\ \dim_{\mathbb{F}_q}(\mathbb{A}_K/\mathbb{A}_K(0) + K) &\overset{\text{def}}{=} g \end{aligned}$$

The ring of adèles as locally compact topological space

The subring

$$\begin{aligned} \widehat{\mathcal{O}}_K &:= \{ (x_P)_{P \in Y^\circ} \in \prod_{P \in Y^\circ} K_P \mid \\ &\qquad x_P \in \widehat{\mathcal{O}}_P \text{ for each } P \in Y^\circ \ \} \\ &= \mathbb{A}_K(0) \end{aligned}$$

is a compact neighbourhood of 0.

Define

$$\begin{aligned} |\cdot| : \mathbb{A}_K &\to \mathbb{R} \\ x &\mapsto \prod_{P \in Y^\circ} |x|_P \\ &= \prod_{P \in Y^\circ} \left(q^{\deg(P)} \right)^{-\nu_P(x)} . \end{aligned}$$

Let ω be a one-dimensional volume form defined over K.

Serre's formula

Observation 4 (Serre)

Let

- G a unimodular lokally compact group,
- Γ a discrete subgroup,
- H a compacte open subgroup and
- μ a Haar measure.

Assume that $H\backslash G/\Gamma$ is countable.

Then

$$\begin{aligned}
\int_{G/\Gamma} d\mu &= \sum_{x\in(H\backslash G)/\Gamma} \left(\int_{G_x/\Gamma_x} d\mu\right) \\
&= \sum_{x\in H\backslash G/\Gamma} \frac{\int_{G_x} d\mu}{|\Gamma_x|} \\
&= \sum_{x\in H\backslash G/\Gamma} \frac{\int_{H} d\mu}{|\Gamma_x|} \\
&= \int_H d\mu \quad \sum_{x\in H\backslash G/\Gamma} \frac{1}{|\Gamma_x|}.
\end{aligned}$$

The geometry of numbers I

Proposition 5 (Weil)
One has

$$\int_{\mathbb{A}_K/K} \omega_{\mathbb{A}_K} = q^{g-1}.$$

Proof. For the compact open subgroup $\mathbb{A}_K(0)$ of $\mathbb{A}_K$ Serre's formula implies

$$\begin{aligned}
&\int_{\mathbb{A}_K/K} \omega_{\mathbb{A}_K} \\
\overset{4}{=}\ & \int_{\mathbb{A}_K(0)} \omega_{\mathbb{A}_K} \sum_{\mathbb{A}_K(0)\backslash\mathbb{A}_K/K} \frac{1}{|K \cap \mathbb{A}_K(0)|} \\
=\ & \frac{|\mathbb{A}_K/\mathbb{A}_K(0)+K|}{|L(0)|} \int_{\mathbb{A}_K(0)} \omega_{\mathbb{A}_K}.
\end{aligned}$$

Since

$$\int_{\mathbb{A}_K(0)} \omega_{\mathbb{A}_K} = \int_{\widehat{\mathcal{O}}_K} \omega_{\mathbb{A}_K} = 1$$

by Riemann–Roch (or rather its underlying definitions)

$$\int_{\mathbb{A}_K/K} \omega_{\mathbb{A}_K} = q^{g-1}.$$

$\square$

The geometry of numbers II

Proposition 6 (Weil)
One has

$$\int_{\mathbb{A}_K(D)} \omega_{\mathbb{A}_K} = q^{\mathrm{deg}(D)}.$$

Proof. One computes

$$\begin{aligned} q^{g-1} &\stackrel{5}{=} \int_{\mathbb{A}_K/K} \omega_{\mathbb{A}_K} \\ &\stackrel{4}{=} \frac{|\mathbb{A}_K(D)\backslash\mathbb{A}_K/K|}{|K \cap \mathbb{A}_K(D)|} \int_{\mathbb{A}_K(D)} \omega_{\mathbb{A}_K} \\ &= \frac{|\mathbb{A}_K/\mathbb{A}_K(D) + K|}{|L(D)|} \int_{\mathbb{A}_K(D)} \omega_{\mathbb{A}_K}. \end{aligned}$$

Riemann–Roch implies

$$\int_{\mathbb{A}_K(D)} \omega_{\mathbb{A}_K} = q^{\mathrm{deg}(D)}.$$

□

III

Harder's reduction theory

Filtrations of unipotent radicals

Proposition 7 (Demazure, Grothendieck) *Let*

- $Y/\mathbb{F}_q$ *a non-singular projective curve,*
- G/Y *a reductive group* Y*-scheme and*
- P/Y *a parabolic subgroup of* G/Y*.*

Then there exists a filtration

$$R_u(P) = U_0 \supset U_1 \supset \cdots \supset U_k = \{e\}$$

with U_i/U_{i+1} *vector bundles over* Y*.*

More precisely:

$$\begin{aligned} U_i &= \prod_{\alpha\in\Delta_P^+, l(\alpha)>i} P_\alpha \\ U_i/U_{i+1} &\cong \prod_{\alpha\in\Delta_P^+, l(\alpha)=i+1} P_\alpha \end{aligned}$$

where P_α *is the vector bundle over* Y *corresponding to the root space* $\mathfrak{g}^\alpha$*.*

Harder's numerical invariants I

Let
- B/Y a minimal parabolic subgroup of G and
- $(P_i/Y)_i$ the maximal parabolic subgroups of G containing B.

Define

$$p_i(B) := p(P_i) \;=\; \sum_{\alpha \in \Delta^+_{P_i}} \mathrm{deg}(\mathcal{L}(P_\alpha))$$

where $\mathcal{L}(P_\alpha)$ is the divisor/locally free $\mathcal{O}_Y$-module corresponding to P_α.

Define moreover

$$\chi_{P_i} := \sum_{\alpha \in \Delta^+_{P_i}} \mathrm{dim}(P_\alpha)\alpha.$$

Harder's numerical invariants II

Theorem 8 (Harder)

Let

- G/Y *a reductive group* Y*-scheme,*
- P/Y *a max. parabolic subgroup,*
- $\mathfrak{K} := G(\widehat{\mathcal{O}}_K)$ *and*
- ω *a volume form on* $R_u(P)$ *defined over* K*.*

Then

$$\int_{R_u(P(\mathbb{A}_K))\cap\mathfrak{K}} \omega_{\mathbb{A}_K} = q^{p(P)}.$$

Proof. One computes

$$\begin{aligned}
\int_{R_u(P(\mathbb{A}_K))\cap\mathfrak{K}} \omega_{\mathbb{A}_K} &\overset{7}{=} \prod_{\alpha\in\Delta_P^+} \left(\int_{P_\alpha(\mathbb{A}_K)\cap\widehat{\mathcal{O}}_K} \omega_{\mathbb{A}_K}\right) \\
&\overset{6}{=} \prod_{\alpha\in\Delta_P^+} q^{\deg(\mathcal{L}(P_\alpha))} \\
&\overset{\text{def}}{=} q^{p(P)}.
\end{aligned}$$

$\square$

A transformation formula

Theorem 9 (Harder)
For each $x \in P(\mathbb{A}_K)$

$$\int_{R_u(P(\mathbb{A}_K))\cap\mathfrak{K}} \omega_{\mathbb{A}_K} = |\chi_P(x)| \int_{R_u(P(\mathbb{A}_K))\cap{}^x\mathfrak{K}} \omega_{\mathbb{A}_K}.$$

Proof. The absolute value of the determinant of the derivative of conjugation by x

$$\begin{array}{rcl} |\chi_P(\cdot)| : P(\mathbb{A}_K) & \stackrel{\text{ad}}{\to} & \mathsf{GL}\,(\mathsf{Lie}(R_u(P(\mathbb{A}_K)))) \\ & \stackrel{\det}{\to} & \mathsf{GL}\left(\bigwedge^d \mathsf{Lie}(R_u(P(\mathbb{A}_K)))\right) \\ & \stackrel{|\cdot|}{\to} & \mathbb{R} \\ x & \mapsto & |\chi_P(x)| \end{array}$$

measure the ration of the volumes of

$$R_u(P(\mathbb{A})) \cap \mathfrak{K} \quad \text{and} \quad R_u(P(\mathbb{A})) \cap {}^x\mathfrak{K}.$$

$\square$

A Morse function

Let
- $\emptyset \neq S \subset Y^\circ$ finite,
- X product of the affine buildings of $G(K_P)_{P \in S}$,
- $\mathfrak{K} := G(\widehat{\mathcal{O}}_K)$.

For $g \in \prod_{P \in S} G(K_P)$ and $x = \mathrm{Fix}_X({}^g\mathfrak{K})$ define

$$\begin{aligned} p_i(B,x) &:= \log_q \left(\int_{R_u(P_i(\mathbb{A}_K)) \cap {}^g\mathfrak{K}} \omega_{\mathbb{A}_K} \right) \\ &\overset{8,9}{=} p_i(B) + \sum_{P \in S} \deg(P) \nu_P(g). \end{aligned}$$

This function
- is affine linear on each appartment whose boundary at infinity contains P_i,
- and can therefore be extended to all of X.

Then

$$X^p(c) = \{x \in X_S \mid p_i(B,x) \leq c \text{ for } B \text{ nice}\}$$

yields a filtration which is suitable for studying finiteness properties of the S-arithmetic group $G(\mathcal{O}_S)$.

Study these via
- algebraic geometry,
- CAT(0) theory,
- building theory.

Applications

Theorem 10 (Bux, Wortman 2007)
Let G be an absolutely almost simple $\mathbb{F}_q(t)$-isotropic algebraic $\mathbb{F}_q(t)$-group with $\mathrm{rk}_{\mathbb{F}_q(t)} G = 1$.

The finiteness length of an S-arithmetic subgroup of $G(F_q(t))$ is

$$\left(\sum_{P \in S} \mathrm{rk}_{\mathbb{F}_q(t)_P} G \right) - 1.$$

Theorem 11 (Bux, G., Witzel 2009/10)
Let G be an absolutely almost simple $\mathbb{F}_q$-group of rank $n \geq 1$. Then
- *$G(\mathbb{F}_q[t])$ is of type F_{n-1} but not F_n and*
- *$G(\mathbb{F}_q[t, \frac{1}{t}])$ is of type F_{2n-1} but not F_{2n}.*

Conjecture 12
The finiteness length of an S-arithmetic subgroup of $G(F_q(t))$ is always

$$\left(\sum_{P \in S} \mathrm{rk}_{\mathbb{F}_q(t)_P} G \right) - 1.$$