A problem on $c.F_4$-geometries

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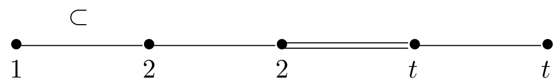
1 Introduction

The following diagram, which we call $c.F_4(2,t)$, is the subject of this paper.

$(c.F_4(2,t))$



The numbers $1, 2, 2, t, t$ are orders and $t < \infty$. Hence $t \in \{1, 2, 4\}$. The label $\subset$ on the first stroke of the diagram stands for the class of complete graphs, regarded as point-line geometries with the vertices as points and the edges as lines.

Several flag-transitive examples are known for this diagram. We will describe some of them later. We only say here that the Fischer group Fi_{22} and the Baby Monster group B occur as automorphism groups of two of them, with $t = 1$ and $t = 4$ respectively.

A number of mathematicians (Ivanov and Wiedorn [14], Ivanov, Pasechnik and Sphectorov [15], Wiedorn [27]) have been busy to prove that, even if we do not know all flag-transitive $c.F_4(2,t)$-geometries, nevertheless we know all of those that satisfy certain hypotheses, which I will discuss in Section 2. They have succeeded to reach this goal when $t = 1$ and $t = 4$, thus obtaining a characterization of Fi_{22}, its non-split central extension $3^{\cdot}Fi_{22}$ and the Baby Monster group B as automorphism groups of $c.F_4(2,t)$-geometries satisfying those hypotheses (Ivanov and Wiedorn [14] for Fi_{22} and $3^{\cdot}Fi_{22}$, where $t = 1$, and Ivanov, Pasechnik and Sphectorov [15] for B, where $t = 4$). However, the case of $t = 2$ still stands against all efforts. A thorough analysis of the possible amalgams in that case has been accomplished by Corinna Wiedorn [27], but that analysis fails to end up with a complete classification. Sorrowfully, Corinna is no more among us. More than one year ago, A. Ivanov and myself agreed to take over that job starting from the point that she had reached. However, so far, we have not been able to make any important progress.

In this paper I will report on the work by A. A. Ivanov, D. V. Pasechnik, S. V. Shpectorov and C. Wiedorn on this topic, with a particular emphasis on the case of $t = 2$, which is still open. I will also give a new construction for one of the examples that arise for $t = 2$.

Added at the last moment. Gernot Stroth has lately informed me that, perhaps, he can prove that we also have a classification for the case of $t = 2$, namely no examples exist besides those that we already know. However, since so far I could have no access to the details of his proof, I cannot yet take this good news into account in this paper.

1.1 Organization of the paper

In Sections 2, 3 and 4 we shall report on the work by Ivanov, Pasechnik, Sphectorov and Wiedorn on flag-transitive $c.F_4(s,t)$-geometries. We shall firstly discuss the additional hypotheses that they assume (Section 2). Next we describe the known examples that satisfy those hypotheses, sticking to the description that Ivanov, Pasechnik, Sphectorov and Wiedorn have chosen for them (Section 3). Finally, we state the classification theorems of Ivanov and Wiedorn [14] and Ivanov, Pasechnik and Shpectorov [15] for the cases of $t = 1$ and $t = 4$ and the quasi-classification theorem by Wiedorn [27] for $t = 2$. We shall also give shortened expositions of the proofs of those theorems, focusing on a few crucial ideas, as the use of shrinkings and geometries at infinity for instance, but avoiding details.

In Section 5 we construct an infinite family of geometries associated to Chevalley groups of type E_6 and belonging to the diagram $Af.F_4$ (which includes $c.F_4(2,t)$ as a special case). The smallest member of that family is one of the known flag-transitive $c.F_4(2,2)$-geometries. Many $Af.F_4$-geometries can be obtained by a different constructions, as affine extensions of F_4-buildings. We discuss them too in Section 5.

The last five sections of this paper are appendices containing notions, constructions and results essential to fully understand Sections 2-5. Section 6 is mainly devoted to embeddings and affine extensions. Section 7 deals with affine polar spaces and their quotients. Section 8 is a concise introduction to shrinkings and geometries at infinity. A few facts and notions on F_4-geometries and F_4-buildings are recalled in Sections 9 and 10.

We could have referred the reader to the literature for the material of Sections 6-8, but fusing the various sources that one can find in the literature in one single picture is not so easy. We have preferred to save the reader that labor.

1.2 Notation and conventions

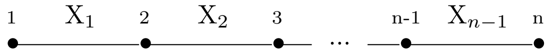
We use the notation of [4] for finite groups. We follow [19] for basic notions of diagram geometry, except that we call $(n-1)$-coverings and $(n-1)$-quotients of geometries of rank n just *coverings* and *quotients* for short and we say that a geometry of rank n is *simply connected* if it is $(n-1)$-simply connected, namely it is its own universal $(n-1)$-cover.

Geometries are defined in [19] in such a way that all geometries are residually connected, by definition. We keep that convention in this paper.

Given a geometry $\mathcal{G}$ and an element x of $\mathcal{G}$, we denote the residue of x in $\mathcal{G}$ by $\mathrm{Res}_{\mathcal{G}}(x)$, also $\mathrm{Res}(x)$ for short when no ambiguity arises. We will often write $x \in \mathcal{G}$ as a shortening of the phrase "x is an element of $\mathcal{G}$".

As in [19], we denote the Intersection Property by the symbol (IP).

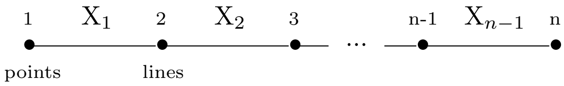
Conventions for geometries with string-shaped diagrams. Let $\mathcal{G}$ be a geometry of rank n belonging to a diagram as follows, where X_1, X_2,...,X_{n-1} denote classes of geometries of rank 2 other than generalized digons and the integers $1, 2, ..., n$ are the types:



We may regard $\mathcal{G}$ as a poset by setting $x < y$ for two elements $x, y \in \mathcal{G}$ if and only if x and y are incident in $\mathcal{G}$ and $\tau(x) < \tau(y)$, where τ is the type-function of $\mathcal{G}$.

The residue of an element $x \in \mathcal{G}$ of type $\tau(x) = k$ with $1 < k < n$ is a direct sum $\mathrm{Res}(x) = \mathrm{Res}^-(x) \oplus \mathrm{Res}^+(x)$ where $\mathrm{Res}^-(x)$ is the residue of any $\{k, k+1, ..., n\}$-flag of $\mathcal{G}$ containing x and $\mathrm{Res}^+(x)$ is the residue of any $\{1, ..., k-1, k\}$-flag containing x. We call $\mathrm{Res}^-(x)$ and $\mathrm{Res}^+(x)$ the *lower* and *upper* residue of x, respectively. We can extend this notation to 1- and n-elements: if $\tau(x) = n$ then $\mathrm{Res}^-(x) := \mathrm{Res}(x)$, and similarly for 1-elements.

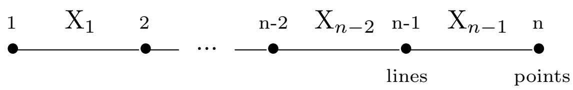
The elements of $\mathcal{G}$ of type 1 and 2 are called *points* and *lines*:



We denote the set of points and the set of lines of $\mathcal{G}$ by $P(\mathcal{G})$ and $L(\mathcal{G})$ respectively. The rank 2 geometry $\mathcal{G}_{|1,2} := (P(\mathcal{G}), L(\mathcal{G}))$ is called the *point-line system* of $\mathcal{G}$. The *collinearity graph* of $\mathcal{G}$ is the collinearity graph of $\mathcal{G}_{|1,2}$.

Suppose that $\mathcal{G}$ satisfies (IP). Then, denoted by $P(x)$ the set of points incident with an element $x \in \mathcal{G}$, if $x \neq y$ then $P(x) \neq P(y)$, we have $x < y$ in the poset $\mathcal{G}$ if and only if $P(x) \subset P(y)$ and, if $P(x) \cap P(y) \neq \emptyset$ then there exists a unique element $z \in \mathcal{G}$ such that $P(z) = P(x) \cap P(y)$. In short, the poset $(\mathcal{G}, <)$ is isomorphic to $(\{P(x)\}_{x \in \mathcal{G}}, \subset)$ and the latter, enriched with $\emptyset$ as the minimal element, is a lower semilattice with respect to $\cap$.

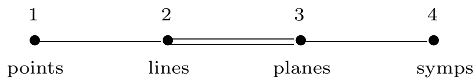
We denote by $\mathcal{G}^*$ the dual poset of $(\mathcal{G}, <)$ and we say that $\mathcal{G}^*$ is the geometry *dual* of $\mathcal{G}$. Clearly, the points and the lines of $\mathcal{G}^*$ are the elements of $\mathcal{G}$ of type n and $n-1$.



Still on points and lines. Points and lines can be defined in a more general setting. Let $\mathcal{G}$ be a geometry of rank > 1 belonging to a connected diagram. Let Δ be the underlying graph of the diagram of $\mathcal{G}$. Chosen a type, say 0, let $\Delta(0)$ be the neighborhood of 0 in Δ. The 0-elements of $\mathcal{G}$ are taken as points while the flags of type $\Delta(0)$ are the lines. In other words, we take as points and lines the points and the lines of the 0-grassmann geometry of $\mathcal{G}$, which indeed belongs to a string-shaped diagram (see [19, Chapter 5]).

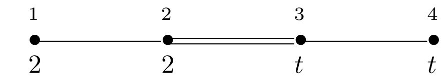
Conventions for F_4-geometries. We choose the integers $1, 2, 3$ and 4 as types for the diagram F_4, calling the elements of type 3 and 4 *planes* and *symps* respectively. Elements of type 1 and 2 are called points and lines, according to the convention stated for geometries with string-shaped diagrams.

(F_4)



We denote by $F_4(2,t)$ the F_4-diagram with orders $2, 2, t, t$:

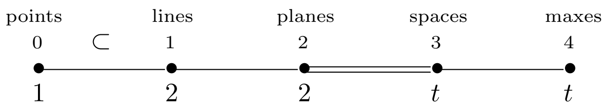
$(F_4(2,t))$



An F_4-geometry (in particular, an F_4-building) with orders $2, 2, t, t$ will be called an $F_4(2,t)$-*geometry* ($F_4(2,t)$-*building*).

Conventions for $c.F_4(2,t)$-geometries. We choose $0, 1, 2, 3$ and 4 as types for the diagram $c.F_4(2,t)$, as follows:

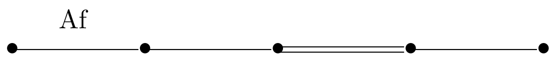
$(c.F_4(2,t))$



The elements of type 0 and 1 are called points and lines. The elements of type 2, 3 and 4 will be called *planes*, *spaces* and *maxes* respectively, as indicated in the above picture.

Note that, as a complete graph on 4 vertices is the same as an affine plane of order 2, the diagram $c.F_4(2,t)$ can also be regarded as a special case of the following, where Af denotes the class of affine planes.

$(Af.F_4)$



Part I

Flag-transitive $c.F_4$-geometries

2 Looking for the right hypotheses

In this section we shall discuss the hypotheses assumed on flag-transitive $c.F_4(2,t)$-geometries in [14], [15] and [27]. However, before to come to those hypotheses, we must see what flag-transitivity implies.

Throughout this section $\mathcal{G}$ is a given flag-transitive $c.F_4(2,t)$-geometry. Lemma 9.1 of Section 9 immediately implies the following:

Lemma 2.1 *The residues of the points of* $\mathcal{G}$ *are buildings of type* F_4.

Lemma 2.2 *No two distinct lines of* $\mathcal{G}$ *are incident with the same pair of points.*

Proof. Given a point a of $\mathcal{G}$, let $L(a)$ be the set of lines of $\mathcal{G}$ incident with a and let Θ_a be the equivalence relation defined on $L(a)$ by declaring that two lines $l, m \in L(a)$ correspond in Θ_a if $P(l) = P(m)$. By Lemma 2.1, $\mathrm{Res}(a)$ is a building. Hence the group induced by G_a on $\mathrm{Res}(a)$ contains a group isomorphic to $O_8^+(2) : S_3$, $F_4(2)$ or ${}^2E_6(2)$, according to whether t is 1, 2 or 4. In any case, $G_{a,l}$ is maximal in G_a. In other words, G_a acts primitively on $L(a)$. On the other hand, G_a preserves Θ_a. Therefore Θ_a is either the identity relation or the trivial relation. In the first case no two lines through a have the same points, as we wanted to prove. In the latter case, there is a unique point b such that $P(l) = \{a, b\}$ for every $l \in L(a)$, but this is clearly impossible. $\square$

Lemma 2.3 *Property* (IP) *holds in the residue of every element of* $\mathcal{G}$.

Proof. This immediately follows from [19, Lemma 7.25], Lemmas 2.1 and 2.2 and the fact that (IP) holds in every building. $\square$

Lemma 2.3 and Proposition 7.3 of Section 7 imply the following:

Lemma 2.4 *The residues of the maxes of* $\mathcal{G}$ *are standard quotients of affine polar spaces.*

(See Section 7 for the definition of affine polar spaces and their standard quotients.)

It is now time to cope with a difficulty that might sit in the shape of the diagram $c.F_4(2,t)$. The underlying minimal circuit diagram of $c.F_4(2,t)$ is the affine diagram $\widetilde{F}_4$. So, it would be no wonder if $\mathcal{G}$ or some of its covers were infinite. However we would like to keep infinite geometries out of the door. Indeed we would like to end up with a classification, but the usual tools we can use to that goal would be ineffective in the infinite case. So, we need residues of maxes to be sufficiently tight to force finiteness. To this aim, we assume the following:

(TR) (**Tight Residues**) The collinearity graph of the residue of any max of $\mathcal{G}$ has diameter equal to 1.

This property drastically reduces the range of possibilities for residues of maxes allowed by Lemma 2.4. Explicitly,

Lemma 2.5 *Let* (TR) *hold. Then the residues of the maxes of* $\mathcal{G}$ *are minimal standard quotients of an affine polar space* $\mathcal{P} \setminus H$ *where* $\mathcal{P}$ *and* H *are as follows, according to whether* t *is* 1, 2 *or* 4.

(1) *Let* $t = 1$. *Then* $\mathcal{P}$ *is the* $O_8^+(2)$*-polar space and* H *is isomorphic to the* $O_7(2)$*-polar space. The minimal standard quotient of* $\mathcal{P}\setminus H$ *is a tangent geometry of the* $O_6^+(2)$*-polar space.*

(2) *Let* $t = 2$. *Then* $\mathcal{P}$ *is the* $S_8(2)$*-polar space and* H *is singular. The minimal standard quotient of* $\mathcal{P} \setminus H$ *is the affine extension of the* $S_6(2)$*-polar space, naturally embedded in* $V(6, 2)$.

(3) *Let* $t = 4$. *Then* $\mathcal{P}$ *is the* $O_{10}^-(2)$*-polar space and* H *is isomorphic to the* $O_9(2)$*-polar space. The minimal standard quotient of* $\mathcal{P}\setminus H$ *is a tangent geometry of the* $O_6^-(2)$*-polar space.*

(See Subsection 7.2 for the minimal standard quotient of an affine polar space, Subsection 7.3 for tangent geometries and Subsection 6.3 for affine extensions.)

Property (TR) has an additional nice effect.

Lemma 2.6 *If* $\mathcal{G}$ *satisfies* (TR) *then* (IP) *also holds in* $\mathcal{G}$.

Proof. Given two elements x and y of $\mathcal{G}$ of type at least 1, suppose that $P(x) \cap P(y) \neq \emptyset$. Pick a point $p \in P(x) \cap P(y)$ and let x' and y' be maxes incident with x and y respectively, possibly, $x' = x$ or $y' = y$. By (TR), $P(x)$ and $P(y)$ are cliques in the collinearity graphs of $\mathrm{Res}(x')$ and $\mathrm{Res}(y')$. For every point $q \in P(x) \cap P(y)$ different from p, let pq_x and pq_y be the lines through p and q in $\mathrm{Res}(x')$ and $\mathrm{Res}(y')$ respectively. Then pq_x is incident with x and pq_y is incident with y. By Lemma 2.2, $pq_x = pq_y$. The line $pq := pq_x = pq_y$ is incident with either of x and y (possibly, $pq = x$ or $pq = y$). Hence we can replace the set of points $P(x) \cap P(y)$ with the set of lines through p incident with both x and y, thus turning to $\mathrm{Res}(p)$. The latter is a building (Lemma 2.1), hence (IP) holds in it. Therefore there exists an element $z \in \mathrm{Res}(p)$ incident with x and y and such that a 2-element of $\mathrm{Res}(p)$ is incident with z if and only if it is incident with both x and y. By the above, $P(z) = P(x) \cap P(y)$. Property (IP) is proved. □

Another hypothesis can be considered besides (TR). When trying to classify a class of geometries, it can be useful to focus on their collinearity graphs, turning the original problem into a problem of local recognition of graphs. However, for we can safely go from point-residues to neighborhoods in the collinearity graph and back, without missing any piece of information along the way, we need a hypothesis that says that, given a point p, every edge that exists in the neighborhood of p in the collinearity graph is somehow recognizable in $\mathrm{Res}(p)$. In our case, this can be phrased as follows, where $\Gamma(\mathcal{G})$ stands for the collinearity graph of $\mathcal{G}$:

(LR) (**Local Recognizability**) Every 3-clique of $\Gamma(\mathcal{G})$ is contained in at least one max of $\mathcal{G}$.

According to (LR), only two kinds of 3-cliques exist in $\Gamma(\mathcal{G})$: a 3-clique can be either contained in a plane or not. In the first case that 3-clique belongs to exactly seven maxes while in the second case it belongs to exactly one max (at least one by (LR) and at most one by (IP)).

Properties (TR) and (LR) are indeed the hypotheses assumed in [14], [15], [11], [12] and [27].

Clearly, each of (TR) and (LR) is preserved when taking universal covers (but (TR) is not preserved when taking 2-covers).

Property (TR) forces finiteness. More explicitly:

Proposition 2.7 *If $\mathcal{G}$ satisfies* (TR) *then $\Gamma(\mathcal{G})$ has diameter at most* 4.

Proof. Assume firstly that $\mathcal{G}$ satisfies both (TR) and (LR). Then the previous statement is the main result of Ivanov and Pasini [12] when $t = 2$ and it is a by-product of the classification theorems of Ivanov and Wiedorn [14] and Ivanov, Pasechnik and Shpectorov [15] when $t = 1$ and $t = 4$ (see Theorems 4.1 and 4.2 of Section 4 and the information given on $\mathcal{G}(3^{\cdot}Fi_{22})$ and $\mathcal{G}(B)$ in Section 3).

Suppose now that $\mathcal{G}$ satisfies (TR) but not (LR). Let p be a point of $\mathcal{G}$ and let $\Gamma(p)$ be its neighborhood in $\Gamma(\mathcal{G})$. By (TR), every collinear pair and every symplectic pair of points of the F_4-building $\mathrm{Res}(p)$ contributes an edge of $\Gamma(p)$. (See Subsection 10 for the definition of symplectic and special pairs of points of an F_4-building.) However, since now (LR) fails to hold in $\mathcal{G}$, additional edges of $\Gamma(p)$ are contributed by special pairs of points of $\mathrm{Res}(p)$ or pairs of points of $\mathrm{Res}(p)$ at distance 3 in $\mathrm{Res}(p)$. It is not difficult to prove that this forces $\mathrm{diam}(\Gamma(\mathcal{G})) \leq 2$. We leave the details of this proof for the reader. □

Problem 2.8 The situation where $\mathcal{G}$ satisfies (TR) but not (LR) has not been investigated in the literature, but it is worth of consideration. What can we say on $\mathcal{G}$ in that case? Can we prove the following conjecture?

Conjecture 2.9 *If $\mathcal{G}$ satisfies* (TR) *and is simply connected then it also satisfies* (LR).

Problem 2.10 Let $\mathcal{G}$ be a simply connected flag-transitive $c.F_4(2,t)$-geometry satisfying (TR) and (LR) and let $\widetilde{\mathcal{G}}$ be its universal 2-cover. Is $\widetilde{\mathcal{G}}$ infinite?

Most likely, (TR) never holds in $\widetilde{\mathcal{G}}$. (In fact, as we shall see in Subsection 5.3.1, property (TR) fails to hold in $\widetilde{\mathcal{G}}$ when $\mathcal{G}$ is the geometry $\mathcal{G}(2^{26}F_4(2))$, to be defined in the next section.)

Is there any $c.F_4(2,t)$-geometry which is its own universal 2-cover, does not satisfy (TR) but, nevertheless, is finite? As we will see in Subsection 5.3, a few finite flag-transitive $c.F_4(2,t)$-geometries exist where (TR) fails to hold, but we do not know if they are 2-simply connected.

3 The examples

We shall now describe the known flag-transitive $c.F_4(2,t)$-geometries that satisfy (TR) and (LR), sticking to the constructions offered in [14], [15] and [11]. Throughout this section $\mathcal{F}$ is an $F_4(2,t)$-building and $\Phi(\mathcal{F})$ is the graph with the points of $\mathcal{F}$ as vertices, two points being adjacent in $\Phi(\mathcal{F})$ when they are either collinear or form a symplectic pair (see Subsection 10).

3.1 Preliminaries

All examples considered in [14], [15] and [11] are constructed as follows. A graph Γ is considered that is locally $\Phi(\mathcal{F})$. The graph Γ is related to a group G which acts transitively on the set of directed edges of Γ and such that the stabilizer G_a of a in G of a vertex a of Γ acts as a flag-transitive subgroup F of $\mathrm{Aut}(\mathcal{F})$ on the neighborhood $\Gamma(a)$ of a. The vertices and the edges of Γ are taken as points and lines respectively of the geometry $\mathcal{G}$ that we are going to define. For every vertex a of Γ, the subgraphs of $\Gamma(a)$ corresponding to lines, planes and symps of $\mathcal{F}$ in the isomorphism $(G_a, \Gamma(a)) \cong (F, \Phi(\mathcal{F}))$, joined with $\{a\}$, are taken as planes, spaces and maxes of $\mathcal{G}$ on a. This definition is consistent. Indeed, in each of the cases to consider the pair (G, Γ) satisfies the following:

(C) (**Consistency Condition**) For every edge $\{a, b\}$ of Γ, for every class $\mathcal{X}$ of subgraphs of $\Gamma(a)$ corresponding to the set of lines, planes or symps of $\mathcal{F}$, and every $X \in \mathcal{X}$ containing b, there is an element $g \in G$ that maps a to b and X to $(X \setminus \{b\}) \cup \{a\}$.

Planes, spaces and maxes of $\mathcal{G}$ can also be recovered from Γ as follows. The maxes of $\mathcal{G}$ are maximal cliques of Γ, but not all maximal cliques of Γ are maxes of $\mathcal{G}$. Indeed Γ admits maximal cliques of different size. The largest ones have size $4 \cdot (7t+2)$. These are the maxes of $\mathcal{G}$. Having recognized maxes of $\mathcal{G}$ in this way, we can recover spaces and planes as intersections of maxes. Indeed, given two maxes X and Y of $\mathcal{G}$, either $|X \cap Y| \leq 2$ or $|X \cap Y| = 8$. If $|X \cap Y| = 8$ then $X \cap Y$ is a space. Finally, given three maxes X, Y and Z, either $|X \cap Y \cap Z| = 8$ (whence $X \cap Y \cap Z = X \cap Y$ is a space) or $|X \cap Y \cap Z| = 4$ or $|X \cap Y \cap Z| \leq 2$. If $|X \cap Y \cap Z| = 4$ then $X \cap Y \cap Z$ is a plane.

By construction, $\mathcal{G}$ belongs to $c.F_4(2,t)$ and G acts flag-transitively on it. Clearly, Γ is the collinearity graph of $\mathcal{G}$. Moreover, $\mathcal{G}$ satisfies the following:

($*$) Three distinct points of $\mathcal{G}$ form a clique in Γ if and only if there is max of $\mathcal{G}$ that contains them all.

This property is just the conjuction of properties (TR) and (SR) considered in Section 2.

We shall now go into details, explaining which group G and which graph Γ are to be chosen. We shall firstly consider the cases of $t = 1$ and $t = 4$, keeping the case of $t = 2$ for last. Our exposition is based on Ivanov and Wiedorn [14] for $t = 1$ and $t = 4$, Ivanov, Pasechnik and Sphectorov [15] for all cases and Ivanov and Pasechnik [11] for $t = 2$.

3.2 $\mathcal{G}(Fi_{22})$ and $\mathcal{G}(3^{\cdot}Fi_{22})$ $(t = 1)$

Let $\Gamma(Fi_{22})$ be the graph with the $2D$-involutions of $\mathrm{Aut}(Fi_{22}) = Fi_{22}{:}2$ as vertices, two such involutions being adjacent if and only if they commute. (Note that these involutions are outer automorphisms of Fi_{22}.) The graph $\Gamma(Fi_{22})$ is locally $\Phi(\mathcal{F})$, where $\mathcal{F}$ is the $F_4(2,1)$-building. Moreover, Fi_{22} is transitive on the set of directed edges of $\Gamma(Fi_{22})$ and the pair $(Fi_{22}, \Gamma(Fi_{22}))$ satisfies condition (C). Thus, a geometry $\mathcal{G}(Fi_{22})$ exists, belonging to $c.F_4(2,1)$, satisfying (TR) and (SR) and admitting Fi_{22} as a flag-transitive automorphism group. Actually, $\mathrm{Aut}(\mathcal{G}(Fi_{22})) = \mathrm{Aut}(Fi_{22}) = Fi_{22}{:}2$.

This geometry is not simply connected. Its universal cover is a 3-fold cover $\mathcal{G}(3^{\cdot}Fi_{22})$ with $\mathrm{Aut}(\mathcal{G}(3^{\cdot}Fi_{22})) = 3^{\cdot}Fi_{22}{:}2$. Accordingly, $\Gamma(Fi_{22})$ admits a 3-fold cover $\Gamma(3^{\cdot}Fi_{22})$ which lives in $3^{\cdot}Fi_{22}{:}2$. The geometry $\mathcal{G}(3^{\cdot}Fi_{22})$ arises from it.

The geometry $\mathcal{G}(3^{\cdot}Fi_{22})$ is simply connected. This is a by-product of Theorem 4.1 of Section 4, but it also follows from Proposition 2.7 and the fact that $\Gamma(3^{\cdot}Fi_{22})$ has diameter equal to 4.

3.3 $\mathcal{G}(B)$ $(t = 4)$

Let $\Gamma(B)$ be the Baby Monster graph, namely the graph on the set of $\{3,4\}$-transpositions in the Baby Monster group B in which two such transposition are adjacent if their product is a central involution in B. The graph $\Gamma(B)$ is locally $\Phi(\mathcal{F})$, where $\mathcal{F}$ is the $F_4(2,4)$-building. Moreover, B acts transitively on the set of directed edges of $\Gamma(B)$ and the pair $(B, \Gamma(B))$ satisfies condition (C). Thus, a geometry $\mathcal{G}(B)$ exists, belonging to $c.F_4(2,4)$, satisfying (TR) and (SR) and admitting B as a flag-transitive automorphism group. Indeed $\mathrm{Aut}(\mathcal{G}(B)) = B$.

The geometry $\mathcal{G}(B)$ is simply connected. This is a by-product of Theorem 4.2 of Section 4. It also follows from Proposition 2.7, recalling that $\Gamma(B)$ has diameter equal to 3.

3.4 $\mathcal{G}(2^{26}F_4(2))$, $\mathcal{G}(E_6(2))$, $\mathcal{G}({}^2E_6(2))$ and $\mathcal{G}(3^{\cdot 2}E_6(2))$ $(t = 2)$

Let G be any of the groups $2^{26}{:}F_4(2)$, $E_6(2)$, ${}^2E_6(2)$ or $3^{\cdot 2}E_6(2)$, where $2^{26}{:}F_4(2)$ is the extension of the Chevalley group $F_4(2)$ by its 26-dimensional GF(2)-module. In any case G admits a maximal subgroup $F \cong F_4(2)$. Let V be the set of cosets of F (left cosets, to fix ideas). The transitive action of G on V by left translation admits several subdegrees but, in any case, one of them is equal to 69615, which is the number of points of the $F_4(2)$-building. Let $\overline{E}$ be the orbital of G on V corresponding to that subdegree, E the set of unordered pairs corresponding to members of $\overline{E}$ and Γ the graph with V as the set of vertices and E as the set of edges. Then Γ is locally $\Phi(\mathcal{F})$, where $\mathcal{F}$ is the $F_4(2)$-building. Moreover, (G, Γ) satisfies condition (C). Thus, we obtain geometries $\mathcal{G}(2^{26}F_4(2))$, $\mathcal{G}(E_6(2))$, $\mathcal{G}({}^2E_6(2))$ and $\mathcal{G}(3^{\cdot 2}E_6(2))$ belonging to $c.F_4(2,2)$, satisfying (TR) and (SR) and admitting $2^{26}{:}F_4(2)$, $E_6(2)$, ${}^2E_6(2)$ and $3^{\cdot 2}E_6(2)$ respectively as flag-transitive automorphism groups. Actually $\mathrm{Aut}(\mathcal{G}(2^{26}F_4(2))) = 2^{26}{:}F_4(2)$

while $\mathrm{Aut}(\mathcal{G}(E_6(2))) = E_6(2){:}2$, $\mathrm{Aut}(\mathcal{G}({}^2E_6(2))) = {}^2E_6(2){:}2$ and $\mathrm{Aut}(\mathcal{G}(3^{\cdot 2}E_6(2))) = 3^{\cdot 2}E_6(2){:}2$.

The graph Γ has diameter 4, 3, 2 or 4 according to whether G is $2^{26}F_4(2)$, $E_6(2)$, ${}^2E_6(2)$ or $3^{\cdot 2}E_6(2)$. Hence $\mathcal{G}(2^{26}F_4(2))$, $\mathcal{G}(E_6(2))$ and $\mathcal{G}(3^{\cdot 2}E_6(2))$ are simply connected, by Proposition 2.7. The geometry $\mathcal{G}({}^2E_6(2))$ is a quotient of $\mathcal{G}(3^{\cdot 2}E_6(2))$.

The geometries $\mathcal{G}(2^{26}F_4(2))$ and $\mathcal{G}(E_6(2))$ can also be produced by constructions of more geometrical nature. Indeed, as proved in [12], $\mathcal{G}(2^{26}F_4(2))$ is nothing but the affine extension of the 26-dimensional projective embedding of the $F_4(2)$-building $\mathcal{F}$ (see Subsection 6.3 for the definition of affine extensions). In Section 5 we will obtain $\mathcal{G}(E_6(2))$ as a special case of a construction which works for any building of type E_6. On the other hand, the graph-theoretic constructions given at the beginning of this subsection and in Remark 3.2 (see below) are so far the only ones available for $\mathcal{G}({}^2E_6(2))$.

Problem 3.1 Find a different, more geometric construction for $\mathcal{G}({}^2E_6(2))$. (Compare Problem 5.9.)

Remark 3.2 Both $\mathcal{G}(Fi_{22})$ and $\mathcal{G}({}^2E_6(2))$ are subgeometries of $\mathcal{G}(B)$. To see this, we must recall a few properties of $\Gamma(B)$. There are two types of pairs of vertices $\{a, b\}$ at distance 2 in $\Gamma(B)$, according to whether ab has order 3 or 4. Let $\{a, b\}$ be a pair of vertices of $\Gamma(B)$ at distance 2 with $(ab)^3 = 1$ and let $\Gamma_3(a, b)$ be the subgraph induced by $\Gamma(B)$ on the set of vertices at distance 3 from either of a and b. Then $\Gamma_3(a, b) \cong \Gamma(Fi_{22})$. We can now recover $\mathcal{G}(Fi_{22})$ inside $\mathcal{G}(B)$. The points and the lines of $\mathcal{G}(Fi_{22})$ are the vertices and the edges of $\Gamma_3(a, b)$. The 4-elements of $\mathcal{G}(Fi_{22})$ are provided by the 4-elements x of $\mathcal{G}(B)$ with $P(x) \cap \Gamma_3(a, b)$ of maximal size, while the 2- and 3-elements of $\Gamma(Fi_{22})$ are the 2- and 3-elements y of $\mathcal{G}(B)$ incident to a 4-element as above and such that $P(y) \subset \Gamma_3(a, b)$.

Turning to $\mathcal{G}({}^2E_6(2))$, its collinearity graph is isomorphic to the graph $\Gamma_3(a)$ induced by $\Gamma(B)$ on the set of vertices at distance 3 from a given vertex a. The geometry $\mathcal{G}({}^2E_6(2))$ can be recovered from $\Gamma_3(a)$ like $\mathcal{G}(Fi_{22})$ from $\Gamma_3(a, b)$.

4 A quasi-classification

4.1 The case of $t = 1$ and $t = 4$

The next two theorems give a complete classification of flag-transitive $c.F_4(2, t)$-geometries satisfying (TR) and (LR) when $t = 1$ and $t = 4$.

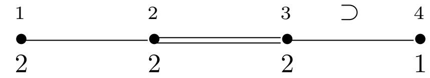
Theorem 4.1 (Ivanov and Wiedorn [14]) *$\mathcal{G}(Fi_{22})$ and its cover $\mathcal{G}(3^{\cdot}Fi_{22})$ are the only flag-transitive $c.F_4(2, 1)$-geometries that satisfy both properties* (TR) *and* (LR)*.*

Theorem 4.2 (Ivanov, Pasechnik and Shpectorov [15]) *$\mathcal{G}(B)$ is the unique flag-transitive $c.F_4(2, 4)$-geometry satisfying* (TR) *and* (LR)*.*

4.1.1 A sketch of the proof of Theorem 4.1

Let $\mathcal{G}$ be a flag-transitive $c.F_4(2,1)$-geometry satisfying (TR) and (LR). Ivanov and Wiedorn firstly investigate the graphs defined on the set of 1-elements and the set of 2-elements of $\mathcal{G}$ by taking 'being parallel inside the same $\{0,1,2\}$-residue' as the adjacency relations. The connected components of these two graphs are the classes of the equivalence relations called Λ_1 and Λ_2 in Subsection 8.2. Condition (RS) of 8.1 and its type 2 analogue hold. So, we can consider the geometry at infinity $\mathcal{G}^{\infty,2}$, which is also flag-transitive. Ivanov and Wiedorn prove that $\mathcal{G}^{\infty,2}$ belongs to the following diagram:

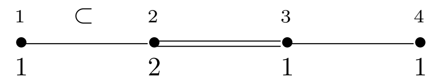
$(C_3.c^*)$



If we find out which group $\mathrm{Aut}(\mathcal{G}^{\infty,2})$ is, then we are done. Indeed $\mathrm{Aut}(\mathcal{G}^{\infty,2}) = \mathrm{Aut}(\mathcal{G})$ by Proposition 8.4.

Thus, we turn from $\mathcal{G}$ to $\mathcal{G}^{\infty,2}$. One can prove that (IP) holds in all residues of $\mathcal{G}^{\infty,2}$ of rank 3. Hence the $\{1,2,3\}$-residues of $\mathcal{G}^{\infty,2}$ are polar spaces (isomorphic to the $S_6(2)$-polar space). So, $\mathcal{G}^{\infty,2}$ is the dual of a c-extended dual polar space. Flag-transitive c-extended dual polar spaces have been classified by Ivanov [9], [10]. We can exploit that classification to finish (but I warn that this is not the way that Ivanov and Wiedorn choose in [14]). In view of that, we must determine the isomorphism type of the $\{2,3,4\}$-residues of $\mathcal{G}^{\infty,2}$. As (IP) holds in these residues, they are dually isomorphic to standard quotients of affine polar spaces (Proposition 7.3). Having stated this, a range of possibilities still remains for those residues. In order to select the right one from it, we ask shrinkings for help. The classes of Λ_1, which provide the 1-elements of $\mathcal{G}^{\infty,2}$, are the point-sets of the shrinkings of $\mathcal{G}$. Hence the stabilizer in $\mathrm{Aut}(\mathcal{G})$ of a shrinking of $\mathcal{G}$ is the same as the stabilizer in $\mathrm{Aut}(\mathcal{G}^{\infty,2})$ of a 1-element of $\mathcal{G}^{\infty,2}$. Let $\mathrm{Shr}(\mathcal{G})$ be a shrinking of $\mathcal{G}$. Then $\mathrm{Shr}(\mathcal{G})$ is a flag-transitive with diagram and orders as follows:

$(c.C_3^*)$



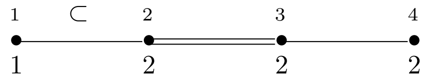
The $\{2,3,4\}$-residues of $\mathrm{Shr}(\mathcal{G})$ are isomorphic to the $\{2,3,4\}$-residues of $\mathcal{G}$. The latter are C_3-buildings. It is well known that every C_3-building with orders $1,1,2$ is isomorphic to the geometry of vertices, edges and 3-cliques of a complete 3-partite graph with all classes of size 3. Denoted this geometry by $\mathcal{K}$, let $\mathcal{K}^*$ be its dual. Thus, the $\{2,3,4\}$-residues of $\mathrm{Shr}(\mathcal{G})$ are isomorphic to $\mathcal{K}^*$. On the other hand, the $\{0,1,2,3\}$-residues of $\mathcal{G}$ are isomorphic to the minimal standard quotient of the affine polar space obtained from the $O_8^+(2)$-polar space by removing an $O_7(2)$-hyperplane (Lemma 2.5). Hence the $\{1,2,3\}$-residues of $\mathrm{Shr}(\mathcal{G})$ are isomorphic to the minimal standard quotient of the affine polar space obtained from the $O_6^+(2)$-polar space by removing an $O_5(2)$-hyperplane (compare Example 8.2). It follows that the collinearity graph of $\mathrm{Shr}(\mathcal{G})$ is locally the distance 1-or-2 graph of the collinearity graph of $\mathcal{K}^*$. We can now apply a result of Cuypers [5] to recover $\mathrm{Shr}(\mathcal{G})$ as a subgeometry of the $F_4(2,1)$-building with $\mathrm{Aut}(\mathrm{Shr}(\mathcal{G})) \cong U_4(2){:}2$ (see also [15, Propsition 5.2]).

By the above, the stabilizer of a 1-element of $\mathcal{G}^{\infty,2}$ in $\mathrm{Aut}(\mathcal{G}^{\infty,2})$ acts as $U_4(2){:}2 \cong \mathrm{PSO}_6^-(2)$ in the residue of that element. This makes it clear that the $\{2,3,4\}$-residues of $\mathcal{G}^{\infty,2}$ are dually isomorphic to the affine polar space obtained by removing an $O_6^-(2)$-hyperplane from the $O_7(2)$-polar space. By Ivanov [9, Proposition 2.5], $\mathrm{Aut}(\mathcal{G}^{\infty,2})$ is either $Fi_{22}{:}2$ or $3^{\cdot}Fi_{22}{:}2$.

However, as I have said before, the argument used by Ivanov and Wiedorn in [14] is different from the above. It is worth to give a shortened exposition of it, too.

The dual $\mathcal{G}^{\infty,2,*}$ of $\mathcal{G}^{\infty,2}$ belongs to the following diagram, which is the same as that of $\mathcal{G}^{\infty,2}$, but switched. The types 1, 2, 3, 4 of $\mathcal{G}^{\infty,2,*}$ correspond to the types 4, 3, 2, 1 of $\mathcal{G}^{\infty,2}$.

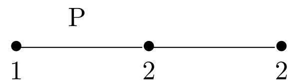
$(c.C_3^*)$



The $\{1,2,3\}$-residues of $\mathcal{G}^{\infty,2,*}$ are the duals of the $\{2,3,4\}$-residues of $\mathcal{G}^{\infty,2}$. According to what we have previously said on the latter, the $\{1,2,3\}$-residues of $\mathcal{G}^{\infty,2,*}$ are isomorphic to the affine polar space obtained by a removing an $O_6^-(2)$-hyperplane from the $O_7(2)$-polar space.

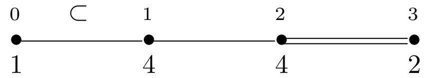
The geometry $\mathcal{G}^{\infty,2,*}$ admits shrinkings. Let $\mathrm{Shr}(\mathcal{G}^{\infty,2,*})$ be one of them. The isomorphism type of the $\{1,2,3\}$-residues of $\mathcal{G}^{\infty,2,*}$ forces $\mathrm{Shr}(\mathcal{G}^{\infty,2,*})$ to belong to the following diagram, where P stands for the Petersen graph (compare Example 8.2):

(P_3)



So, $\mathrm{Shr}(\mathcal{G}^{\infty,2,*})$ is a flag-transitive P_3-geometry. Flag-transitive P_n-geometries are classified (see Ivanov and Shpectorov [13]). Only two flag-transitive P_3-geometries exist. They arise from M_{22} and $3^{\cdot}M_{22}$ respectively. $\mathrm{Shr}(\mathcal{G}^{\infty,2,*})$ is isomorphic to one of these two geometries. In order to finish, Ivanov and Wiedorn must combine the previous information on shrinkings with the investigation of another geometry $\mathcal{B}$ such that $\mathrm{Aut}(\mathcal{B}) = \mathrm{Aut}(\mathcal{G}^{\infty,2,*})$. The construction of $\mathcal{B}$ by Ivanov and Wiedorn is beautiful but its details are too complicated for I are can summarize them here. I only mention that they consider a graph B having the shrinkings of $\mathcal{G}^{\infty,2,*}$ as the vertices, two shrinkings being adjacent in B precisely when they contain 2-elements supported by the same 2-element of $\mathcal{G}^{\infty,2,*}$. The shrinkings of $\mathcal{G}^{\infty,2,*}$ are taken as 3-elements of $\mathcal{B}$, like if we constructed the dual of a geometry at infinity. The 2- and 1-elements of $\mathcal{B}$ are certain 3-cliques and certain subgraphs of B. Since the shrinkings of $\mathcal{G}^{\infty,2,*}$ are P_3-geometries for M_{22} or $3^{\cdot}M_{22}$, the stabilizer in $\mathrm{Aut}(\mathcal{B})$ of a 3-element x of $\mathcal{B}$ induces on $\mathrm{Res}_{\mathcal{B}}(x)$ a group isomorphic to M_{22}, $M_{22}{:}2$, $3^{\cdot}M_{22}$ or $3^{\cdot}M_{22}{:}2$. With this information at hand, Ivanov and Wiedorn can prove that $\mathcal{B}$ is a truncation of a flag-transitive $c.C_3$-geometry $\overline{\mathcal{B}}$ with diagram and orders as below:

$(c.C_3)$

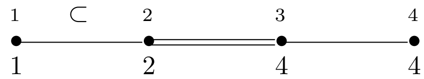


We obtain $\mathcal{B}$ from $\overline{\mathcal{B}}$ by removing all 0-elements of $\overline{\mathcal{B}}$. Flag-transitive $c.C_3$-geometries with orders as above have been characterized since long ago (Buekenhout and Hubaut [3]; see also Del Fra, Ghinelli, Meixner and Pasini [7]). In view of that characterization, $\mathrm{Aut}(\overline{\mathcal{B}})$ is isomorphic to either $Fi_{22}{:}2$ or $3^{\cdot}Fi_{22}{:}2$. Ivanov and Wiedorn also prove that $\mathrm{Aut}(\mathcal{B}) = \mathrm{Aut}(\mathcal{G}^{\infty,2})$. Hence $\mathrm{Aut}(\mathcal{G})$ is either $Fi_{22}{:}2$ or $3^{\cdot}Fi_{22}{:}2$.

4.1.2 A sketch of the proof of Theorem 4.2

Let $\mathcal{G}$ be a flag-transitive $c.F_4(2,4)$-geometry satisfying (TR) and (LR). In this case the geometry at infinity $\mathcal{G}^{\infty,2}$ is harder to understand than $\mathcal{G}$ itself. No insight into $\mathcal{G}$ can be got from it. On the other hand, shrinkings can be of some help. Every shrinking $\mathrm{Shr}(\mathcal{G})$ of $\mathcal{G}$ is a flag-transitive c-extended dual polar space with orders as follows:

$(c.C_3)$



Its $\{2,3,4\}$-residues are isomorphic to the dual of the $U_6(2)$-polar space while its $\{1,2,3\}$-residues are isomorphic to the minimal standard quotient of the affine polar space obtained by removing an $O_7(2)$-hyperplane from the $O_8^-(2)$-polar space (see Example 8.2). Hence the collinearity graph of $\mathrm{Shr}(\mathcal{G})$ is locally the distance 1-or-2 graph of the dual of the $U_6(2)$-polar space. One can now exploit the classification of flag-transitive c-extended dual polar spaces by Ivanov [9], [10] or a result by Cuypers [5] to determine the isomorphism type of $\mathrm{Shr}(\mathcal{G})$. It turns out that $\mathrm{Shr}(\mathcal{G})$ is isomorphic to a well known geometry admitting the Conway group Co_2 as its full automorphism group [15, Proposition 5.2].

The isomorphism type of $\mathrm{Shr}(\mathcal{G})$ is thus determined, but a lot of work still remains to do in order to finish the proof. At this stage the authors of [15] turn to a thorough investigation of the collinearity graph Γ of $\mathcal{G}$, focusing on the common neighborhood $\Gamma(x,y)$ of two vertices x and y at distance 2 and the second neighborhood $\Gamma_2(x)$ of a vertex x. In particular, they prove that, for every vertex $y \in \Gamma_2(x)$, there is exactly one vertex $y' \in \Gamma_2(x)$ different from y and such that $\Gamma(x,y) = \Gamma(x,y')$. Let π_x be the permutation of Γ that fixes all vertices of $\Gamma\backslash\Gamma_2(x)$ and switches all pairs $\{y,y'\} \subset \Gamma_2(x)$ with $\Gamma(x,y) = \Gamma(x,y')$. The authors of [15] prove that π_x is an automorphism of Γ (hence it also induces an automorphism of $\mathcal{G}$). Some amount of work is needed to prove this claim. We will not go into the details of that proof. We only mention that the information previously obtained on $\mathrm{Shr}(\mathcal{G})$ is also exploited in it, at a certain stage.

If x and z are adjacent vertices of Γ then $\pi_z(x) = x$, whence π_z induces an automorphism of the neighborhood $\Gamma(x)$ of x. In fact, the family $\{\pi_z\}_{z\in\Gamma(x)}$

generates a copy of ${}^2E_6(2)$. So far, we know that the graph Γ is locally $\Phi(\mathcal{F})$, where $\mathcal{F}$ is the $F_4(2,4)$-building, and it admits an automorphism group G such that, for every vertex x, the stabilizer of G_x of x in G induces on $\Gamma(x)$ an action containing ${}^2E_6(2)$. By a result of Ivanov [8], G is the Baby Monster group B and $\Gamma = \Gamma(B)$ (the Baby Monster graph).

4.2 The case of $t = 2$

Theorem 4.3 (Wiedorn [27]) *Besides* $\mathcal{G}(2^{26}F_4(2))$*,* $\mathcal{G}(E_6(2))$ *and* $\mathcal{G}(3^{\cdot 2}E_6(2))$*, at most four simply connected flag-transitive* $c.F_4(2,2)$*-geometries exist that satisfy* (TR).

4.2.1 A sketch of the proof of Theorem 4.3

Let G be a group acting flag-transitively on a $c.F_4(2,2)$-geometry $\mathcal{G}$ satisfying (TR) and for $i = 0,1,2,3,4$ let G_i be the stabilizer in G of the i-element a_i of a given chamber $c = \{a_i\}_{i=0}^4$ of $\mathcal{G}$. Let $\mathcal{A} = (G_i)_{i=0}^4$ be the amalgam of the subgroups G_0, G_1, G_2, G_3, G_4 with intersections as in G. Then $\mathcal{G}$ is simply connected if and only if G is the universal completion of $\mathcal{A}$ (see [19, Theorem 12.28]). So, we must determine all possibilities for the amalgam $\mathcal{A}$.

Henceforth, for an element $x \in \mathcal{G}$ we denote by G_x the stabilizer of x in G and by K_x the kernel of the action of G_x on $\mathrm{Res}(x)$ and we put $\overline{G}_x := G_x/K_x$. We also put $G_{x,y} := G_x \cap G_y$. If x has type $\neq 0,4$, we denote by K_x^+ and K_x^- the kernels of the actions of G_x on $\mathrm{Res}^+(x)$ and $\mathrm{Res}^-(x)$ respectively. In the previous paragraph we wrote G_i for G_{a_i}. Accordingly, we write K_i for K_{a_i}, $G_{i,j}$ for G_{a_i,a_j} and so on.

Clearly, $\overline{G}_0 \cong F_4(2)$, $\overline{G}_1 \cong 2 \times S_6(2)$ and $\overline{G}_4 \cong 2^6{:}S_6(2)$ (recall that, by Lemma 2.5, $\mathrm{Res}(a_4)$ is the affine extension of the $S_6(2)$-polar space embedded in $V(6,2)$). Moreover $\overline{G}_2$ is isomorphic to either $S_4 \times L_3(2)$ or $A_4 \times L_3(2)$ and $\overline{G}_3$ is isomorphic to either $(2^3{:}L_3(2)) \times S_3$ or $(2^3{:}L_3(2)) \times 3$. However, $\overline{G}_2$ must fit with $\overline{G}_3$. Hence $\overline{G}_2 \cong S_4 \times L_3(2)$ and $\overline{G}_3 \cong (2^3{:}L_3(2)) \times S_3$.

Wiedorn firstly proves that $|K_0| \leq 2$ and that if $|K_0| = 2$ then $G_0 = K_0 \times \overline{G}_0$. Hence $G_{0,1} = K_0 \times 2^{1+6+8}{:}S_6(2)$. We must now describe G_1. Clearly $[G_1 : G_{0,1}] = 2$. Hence $G_1 = G_{0,1}\langle t\rangle$, where t switches the two points of the line a_1 and $t^2 \in G_{0,1}$. Modulo multiplying t be a suitable element of $G_{0,1}$ we can assume that $t \in K_{a_1}^+$. We can do more, choosing $t \in K_{a_1}^+$ in such a way that it moves as little as possible of $G_{0,1}$. Wiedorn proves that if we can choose $t \in C_G(G_{0,1})$ (which can be only if $K_0 = 1$) then $\mathcal{G} = \mathcal{G}(2^{26}F_2(4))$.

Let $\mathcal{G} \neq \mathcal{G}(2^{26}F_2(4))$. Hence $t \notin C_G(G_{0,1})$. Let $G'_{0,1} \cong 2^{1+6+8}{:}S_6(2)$ be the commutator subgroup of $G_{0,1}$. Wiedorn proves that we can choose t in such a way that it centralizes the setwise stabilizer $G'_{0,1,H_t}$ in $G'_{0,1}$ of a suitable set H_t of 4-elements of $\mathrm{Res}^+(a_1)$. More explicitly, recall that the dual $(\mathrm{Res}^+(a_1))^*$ of $\mathrm{Res}^+(a_1)$ is isomorphic to the $S_6(2)$-polar space. The set H_t is an $O_6^+(2)$-hyperplane of the polar space $(\mathrm{Res}^+(a_1))^*$. With the above constraints, once H_t has been chosen, the element t is uniquely determined modulo multiplication by elements of $Z(G_{0,1}) = K_0 \times \langle z\rangle$, where z is the unique involution of $Z(G'_{0,1})$.

Wiedorn also proves that t has order either 2 or 4 and if t has order 4 then $t^2 = z$. It is worth giving t a name. We call it the *switch element* of G_1.

At this stage we have got at most four isomorphism types $\mathcal{A}^{(1)}_{\varepsilon,\eta}$ for the amalgam $\mathcal{A}^{(1)} := (G_0, G_1)$, where $\varepsilon \in \{1,2\}$ is the size of K_0 and $\eta \in \{2,4\}$ is the order $o(t)$ of the switch element t. However, when $\varepsilon = 2$ we can replace t with tk, where k is the unique involution of K_0. If $o(t) = 2$ then $o(tk) = 4$ and if $o(t) = 4$ then $o(tk) = 2$. Thus, three possibilities exist for $\mathcal{A}^{(1)}$, namely $\mathcal{A}^{(1)}_{1,2}, \mathcal{A}^{(1)}_{1,4}$ and $\mathcal{A}^{(1)}_{2,*}$, where $*$ can be read as 2 or 4, as we like.

Turning to the rank 3 amalgam $\mathcal{A}^{(2)} = (G_0, G_1, G_2)$, Wiedorn proves that each of the amalgams $\mathcal{A}^{(1)}_{1,2}$ and $\mathcal{A}^{(1)}_{1,4}$ can be extended in exactly one way to a realization of $\mathcal{A}^{(2)}$, which we will denote by $\mathcal{A}^{(2)}_{1,2,*}$ and $\mathcal{A}^{(2)}_{1,4,*}$ respectively. On the other hand, $\mathcal{A}^{(1)}_{2,*}$ can be extended in at most to ways. We denote these two extensions by the symbols $\mathcal{A}^{(2)}_{2,*,\diamond}$ and $\mathcal{A}^{(2)}_{2,*,\circ}$. We warn that no explicit description of $\mathcal{A}^{(2)}_{2,*,\diamond}$ and $\mathcal{A}^{(2)}_{2,*,\circ}$ is given in [27]. Wiedorn only gives an indirect argument which implies that at most two ways exist to extend $\mathcal{A}^{(1)}_{2,*}$.

Next Wiedorn proves that, for every choice of $(\varepsilon, \eta, \theta) = (1, 2, *)$, $(1, 4, *)$, $(2, *, \diamond)$ or $(2, *, \circ)$ the amalgam $\mathcal{A}^{(2)}_{\varepsilon,\eta,\theta}$ can be extended in a unique way to a rank 4 amalgam $\mathcal{A}^{(3)}_{\varepsilon,\eta,\theta}$. Finally, again by an indirect argument, she proves that for every choice of $(\varepsilon, \eta, \theta)$ there are at most two ways to extend $\mathcal{A}^{(3)}_{\varepsilon,\eta,\theta}$ to a rank 5 amalgam $\mathcal{A} = (G_0, G_1, G_2, G_3, G_4)$. We denote those two extensions by the symbols $\mathcal{A}^{(4)}_{\varepsilon,\eta,\theta,\diamond}$ and $\mathcal{A}^{(4)}_{\varepsilon,\eta,\theta,\circ}$.

So far, at most eight possibilities exist for $\mathcal{A} = (G_0, G_1, G_2, G_3, G_4)$, corresponding to the eight quadruples $(1, 2, *, \diamond)$, $(1, 2, *, \circ)$, $(1, 4, *, \diamond)$, $(1, 4, *, \circ)$, $(2, *, \diamond, \diamond)$, $(2, *, \diamond, \circ)$, $(2, *, \circ, \diamond)$ and $(2, *, \circ, \circ)$. However, it might happen that two of these possibilities are mutually isomorphic or that one of them does not admit any completion. Moreover, if for some choice of $\theta, \zeta, \eta, \zeta'$ the amalgam $\mathcal{A}^{(4)}_{2,*,\theta,\zeta}$ embodies $\mathcal{A}^{(4)}_{1,\eta,*,\zeta'}$, then these two amalgams define the same geometry.

Each of $\mathcal{G}(E_6(2))$ and $\mathcal{G}(3{\cdot}{}^2E_6(2))$ gives rise to an amalgam $\mathcal{A}^{(4)}_{1,\eta,*,\zeta}$ as well as an amalgam $\mathcal{A}^{(4)}_{2,*,\theta,\zeta}$. So, at most four possibilities remain for $\mathcal{A}^{(4)}_{\varepsilon,\eta,\theta,\zeta}$ that do not correspond to any of the known examples, as stated in Theorem 4.3.

4.2.2 Problems and remarks

A. Ivanov and me have been busy with an attempt to prove that none of the five extra possibilities mentioned in Theorem 4.3 actually exists, but so far we have failed. A number of problems that we have faced but we have not been able to answer are mentioned in the sequel. Some additional information will also be given. For the rest of this section $\mathcal{G}$ is a simply-connected $c.F_2(2,2)$-geometry satisying (TR) but different from $\mathcal{G}(2^{26}F_4(2))$.

Problem 4.4 A detailed description of G_4 is missing in [27]. We know that $\overline{G}_4 \cong 2^6{:}S_6(2)$ and that $K_4 = K_0 \times 2^{1+6+8}$. We also know the action of an

$S_6(2)$-subgroup of $\overline{G}_4$ on K_4, but we don't know enough on the structure of $O_2(G_4)$, even if $K_0 = 1$.

When $K_0 = 1$ it is likely that $O_2(G_4) = 2^{1+20}$ (extraspecial) with $S_6(2)$ acting on $O_2(G_4)/Z(O_2(G_4))$ as on the 20-dimensional module $\wedge^3 V$ where V is the natural 6-dimensional module for $S_6(2)$ (compare Remark [?]). If so, then the switching element t has order $o(t) = 2$. This would kill both amalgams $\mathcal{A}^{(4)}_{1,4,*,\circ}$ and $\mathcal{A}^{(4)}_{1,4,*,\diamond}$. Hence $\mathcal{G}(E_6(2))$ and $\mathcal{G}(3^{\cdot 2}E_6(2))$ would be the only possibilities for $\mathcal{G}$ when $K_0 = 1$.

One might now wonder if we really have $O_2(G_4) = 2^{1+20}$ when G is $E_6(2)$ or $3^{\cdot 2}E_6(2)$. The answer is certainly affirmative when $G = E_6(2)$. Indeed the construction of $\mathcal{G}(E_6(2))$ to be given in the next section shows that in this case $G_4 < X$ for a maximal subgroup $X \cong 2^{1+20}{:}L_6(2)$ of G. Most likely $O_2(G_4) = 2^{1+20}$ when $G = 3^{\cdot 2}E_6(2)$ too. Indeed, among the maximal subgroups of ${}^2E_2(6)$ listed in [4], $2^{1+20}{:}U_6(2)$ is the only one that might host G_4. If really $G_4 < 2^{1+20}{:}U_6(2)$ then $O_2(G_4) = 2^{1+20}$ in this case too.

Problem 4.5 Is it true that the completion of an amalgam $\mathcal{A}^{(4)}_{2,*,\theta,\zeta}$ always contains a completion of an amalgam $\mathcal{A}^{(4)}_{1,\eta,*,\zeta'}$? If this would be true, then we should only prove that $o(t) = 2$ when $K_0 = 1$ (see the previous problem) and we would have finished.

Problem 4.6 It is not difficult to prove that, if an amalgam $\mathcal{A}^{(4)}_{1,\eta,*,\zeta}$ admits a completion, then it also admits a (possibly non-universal) completion G where G is simple. By exploiting Proposition 2.7 and the information on μ-subgraphs of the collinearity graph of $\mathcal{G}$ available in [15], one can compute an upper bound for the index $[G : G_0]$. A lower bound can also be determined. By very rough computations, we have obtained the following:

$$\frac{k^2}{2^7} \; < \; |G : G_0| \; < \; \frac{k^4}{2^{13}}, \quad \text{where } k = 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17.$$

We can now look for a simple group G containing a subgroup $G_0 = K_0 \times F_4(2)$ with $|K_0| \leq 1$ and $|G : G_0|$ between the above two bounds. It turns out that, apart from a few linear groups of rank $n \leq 10$ and alternating groups A_n of degree $27 \leq n \leq 29$, $E_6(2)$, ${}^2E_6(2)$ and $F_4(4)$ are the only simple groups of size compatible with the above. However $F_4(2)$ is neither a subgroup of $L_n(q)$ for $n \leq 10$ (Kleidman [17]) nor of A_{27} (since $|F_4(2)|$ does not divide $27!/2$). Hence A_{28}, A_{29}, $E_6(2)$, ${}^2E_6(2)$ and $F_4(4)$ only survive. On the other hand, $|G_4| = |K_0| \cdot 2^{21} \cdot |S_6(2)| = |K_0| \cdot 2^{30} \cdot 3^4 \cdot 5 \cdot 7$ while 2^{25} is the highest power of 2 dividing 29!. This rules out both A_{29} and A_{28}. Thus $F_4(4)$ only survives besides $E_6(2)$ and ${}^2E_6(2)$. Does it act on a $c.F_4(2,2)$-geometry as we want?

Remark 4.7 By arguments similar to those used in Subsections 4.1.1 and 4.1.2 one can prove that the shrinkings of $\mathcal{G}$ are isomorphic to the affine extension of the dual of the $O_7(2)$-polar space embedded in $V(8,2)$ via the spin embedding (see also Ivanov, Pasechnik and Shpectorov [15, Proposition 5.2]). Perhaps, this information can be exploited in the investigation of $\mathcal{G}$. One can also prove that property (RS) of Section 8 as well as its type 2 analogue hold in $\mathcal{G}$. Hence a geometry at infinity can also be defined, either as in Section 8 or in other ways (see [22]).

Remark 4.8 Wiedorn assumes property (LR) in [27], but she never uses it in her investigation of $\mathcal{A}$. That's why we have not mentioned (LR) in Theorem 4.3. Anyway, we do not see how to translate (LR) in the language of amalgams. Such a translation would obviously be impossible if simple connectedness implies (LR), as suggested in Conjecture 2.9.

5 A geometric construction of $\mathcal{G}(E_6(2))$ and more $Af.F_4$-geometries

The main goal of this section is to produce a family of $Af.F_4$-geometries which contains $\mathcal{G}(E_6(2))$ as its smallest member. We shall firstly recall a construction of buildings of type E_6 and F_4 as geometries embedded in vector spaces of dimension 27 and 26 respectively. After that, we will construct our family of $Af.F_4$-geometries. In the last part of this section we will discuss a few more families of $Af.F_4$-geometries, which can be created as affine extensions from projective embeddings of F_4-buildings.

5.1 Preliminaries on buildings of type E_6 and F_4

The material of this subsection is taken from Chapter 18 of Buekenhout and Cohen [2]. We will omit many details and all proofs. The reader can find them in the above quoted chapter.

Given a field $\mathbb{F}$, let $M = M_3(\mathbb{F})$ be the algebra of (3×3)-matrices over $\mathbb{F}$ and let $V = M \times M \times M$ (a 27-dimensional vector space over $\mathbb{F}$). For a vector $x = (X_1, X_2, X_3)$ of V, put

$$(1) \qquad f(x) = \det(X_1) + \det(X_2) + \det(X_3) - \mathrm{tr}(X_1X_2X_3),$$

$$(2) \qquad x^\sharp = (\mathrm{adj}(X_1) - X_2X_3,\ \mathrm{adj}(X_3) - X_1X_2,\ \mathrm{adj}(X_2) - X_3X_1)$$

where for a matrix $X \in M$ we denote by $\mathrm{adj}(X)$ the so-called adjoint matrix of X, with $(-1)^{i+j}\det(X_{j,i})$ as the (i, j)-entry, $X_{j,i}$ being the (2×2)-submatrix of X obtained by removing the jth row and the ith column. Thus we have defined a cubic form f on V and a mapping $(.)^\sharp : V \to V$. A symmetric bilinear form ϕ can also be defined on V by putting

$$(3) \qquad \phi(x, y) = \mathrm{Tr}(X_1Y_1 + X_2Y_3 + X_3Y_2)$$

for any two vectors $x = (X_1, X_2, X_3)$ and $y = (Y_1, Y_2, Y_3)$ of V. We have

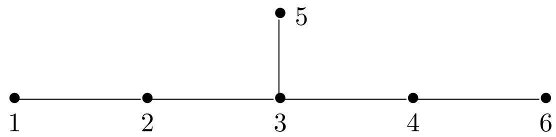
$$(4) \qquad f(x + y) = f(x) + f(y) + \phi(x^\sharp, y) + \phi(x, y^\sharp) \quad (\forall x, y \in V).$$

Also, $x^{\sharp\sharp} = f(x)x$ for every $x \in V$. Hence $x^{\sharp} = 0$ implies $f(x) = 0$ (but the converse is false in general). A commutative (but non-associative) operation $\times$ can also be defined, as follows:

$$(5) \qquad x \times y := (x + y)^{\sharp} - x^{\sharp} - y^{\sharp}.$$

We are now ready to construct the $E_6(\mathbb{F})$-building, denoted by $\mathcal{E}$ throughout the rest of this section. We choose types as follows:

(E_6)



The elements of type 1, 2 and 3 will be called *points*, *lines* and *planes* respectively. The points of $\mathcal{E}$ are the points $\langle x \rangle$ of $\mathrm{PG}(V)$ such that $x^{\sharp} = 0$. Denoted the set of points of $\mathcal{E}$ by $P(\mathcal{E})$, the lines and the planes of $\mathcal{E}$ are the lines and the planes of $\mathrm{PG}(V)$ entirely contained in $P(\mathcal{E})$. Note that two points $\langle x \rangle$ and $\langle y \rangle$ of $\mathcal{E}$ are collinear in $\mathcal{E}$ if and only if $x \times y = 0$.

The set $P(\mathcal{E})$ spans $\mathrm{PG}(V)$. Thus, we have realized a projective embedding of the point-line-plane system $\mathcal{E}_{|1,2,3}$ of $\mathcal{E}$ in $\mathrm{PG}(V)$, which sends point, lines and planes of $\mathcal{E}$ to points, lines and planes of $\mathrm{PG}(V)$. It remains to recover the remaining elements of $\mathcal{E}$ as subspaces or subgeometries of $\mathrm{PG}(V)$.

The geometry $\mathcal{E}_{|1,2,3}$ admits two families of maximal singular subspaces (see Subsection 6.4). One of those two families is formed by 4-dimensional projective spaces. These are the 4-elements of $\mathcal{E}$. The other family consists of 5-dimensional projective spaces. These are the 5-elements of $\mathcal{E}$.

We shall now define the elements of type 6 (called *symps*). For every point $\langle x \rangle \in P(\mathcal{E})$, let $\sigma(x)$ be the projective subspace of $\mathrm{PG}(V)$ corresponding to the subspace $\{x \times y\}_{y \in V}$ of V (clearly, the choice of the representative x of $\langle x \rangle$ is irrelevant in this definition). The points and the lines of $\mathcal{E}$ contained in $\sigma(x)$ form the point-line system of a $D_5(\mathbb{F})$-building, say $\mathcal{S}(x)$. The planes of $\mathcal{S}(x)$ are the planes of $\mathcal{E}$ contained in $\sigma(x)$. One of the two families of maximal singular subspaces of $\mathcal{S}(x)$ consists of the 4-elements of $\mathcal{E}$ contained in $\sigma(x)$. The other family is formed by the intersections $\sigma(x) \cap y$ for y a 5-element of $\mathcal{E}$ such that $y \cap \sigma(x)$ has projective dimension 4. (The reader should notice that we are taking the liberty not to distinguish between an element of $\mathcal{E}$ of type 4 or 5 and the subspace of $\mathrm{PG}(V)$ that corresponds to it.)

The incidence relation is defined via inclusion except when an element of type 5 is involved together with an element of type 4 or 6. Let y be an element of type 5. If z is a 4-element then y and z are incident if and only if they intersect in a 3-dimensional subspace of $\mathrm{PG}(V)$. If $z = \mathcal{S}(x)$ is an element of type 6, then y and z are incident if and only if $y \cap \sigma(x)$ is 3-dimensional.

The group $\mathrm{Aut}(f)$ of all invertible linear mappings of V that preserve f acts flag-transitively on $\mathcal{E}$ with kernel Z_0 isomorphic to the group of cubic roots of 1 in $\mathbb{F}$. The quotient $\mathrm{Aut}(f)/Z_0$ is the Chevalley group of adjoint type $E_6(\mathbb{F})$.

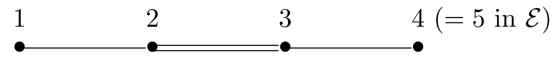
Let Z be the group of scalar transformations of V. The product $Z \cdot \mathrm{Aut}(f)$ is a subgroup of the group of all linear transformations of V that preserve f modulo a scalar. Clearly $Z \cap \mathrm{Aut}(f) = Z_0$ and $(Z \cdot \mathrm{Aut}(f))/Z \cong \mathrm{Aut}(f)/Z_0$. Hence $\mathrm{Aut}(f)$ and $Z \cdot \mathrm{Aut}(f)$ induce the same group on $\mathcal{E}$.

Note that, in general, the elements of $\mathrm{Aut}(f)$ neither preserve ϕ nor commute with the mapping $(.)^\sharp$ or the operation $\times$. However, if $g \in \mathrm{Aut}(f)$ then $x^\sharp = 0$ if and only if $g(x)^\sharp = 0$.

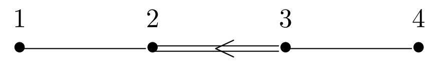
So far, we have constructed $\mathcal{E}$ together with a 27-dimensional projective embedding $\varepsilon : \mathcal{E} \to V$. We shall now turn to the $F_4(\mathbb{F})$-building, denoted by $\mathcal{F}$ in the sequel.

The mapping sending a point x of $\mathcal{E}$ to $\mathcal{S}(x)$ can be extended to a polarity of $\mathcal{E}$, which we will also denote by the symbol σ. Referring the reader to [2, Chapter 18] for details, we only say that this polarity permutes a point x with $\mathcal{S}(x)$ and sends every element y of $\mathcal{E}$ of type 2, 3 or 4 to $\cap_{x \in P(y)} \sigma(x)$, where $P(y)$ is the set of points of y. If y is a 5-element then $\sigma(y) = \cup_{x \in \pi(y)} \sigma(z)$, where $\pi(y)$ stands for the set of planes contained in y. A point x of $\mathcal{E}$ is σ-absolute if and only if $x \in \sigma(x)$, a line x is σ-absolute if and only if $x \subset \sigma(x)$, a 4- or 6-element x is σ-absolute if and only if $\sigma(x)$ is contained in x, a plane or a 5-element x is σ-absolute if and only if $x = \sigma(x)$ (if and only if all points of x are σ-absolute).

The building $\mathcal{F}$ is formed by the σ-absolute elements of $\mathcal{E}$ of type 1, 2, 3 and 5. If we like, we can also regard σ-absolute elements of type 1 or 2 as σ-absolute flags of type $\{1, 6\}$ and $\{2, 4\}$. We keep the types 1, 2 and 3 for σ-absolute elements of $\mathcal{E}$ of type 1, 2 or 3 when we regard them as elements of $\mathcal{F}$ but we take 4 as the type of a σ-absolute 5-element of $\mathcal{E}$ when we regard it as an element of $\mathcal{F}$.

(F_4) 

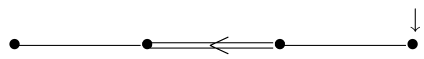
The $\{1, 2, 3\}$-residues of $\mathcal{F}$ are isomorphic to polar spaces of symplectic type while $\{2, 3, 4\}$-residues are dually isomorphic to polar spaces of orthogonal type. In Dynkin notation,

(F_4) 

Another description of $\mathcal{F}$ is possible. Let $\iota := (I, O, O)$, where I is the identity 3×3-matrix while O stands for the null matrix. Then $f(\iota) = 1$. As proved in [2, Chapter 18], a vector $x \in V$ represents a σ-absolute point of $\mathcal{E}$ if and only if $\phi(\iota, x) = 0$. Hence the points (1-elements) of $\mathcal{F}$ are the points of $\mathcal{E}$ contained in the hyperplane H_ι of V represented by the equation $\phi(\iota, x) = 0$. The σ-absolute lines and the σ-absolute planes of $\mathcal{E}$ are contained in H_ι, but not all lines and planes of $\mathcal{E}$ contained in H_ι are σ-absolute. The σ-absolute points, lines and planes incident to a given σ-absolute 5-element x of $\mathcal{E}$ form a polar space of symplectic type, naturally embedded in the 5-dimensional projective space x. The stabilizer $\mathrm{Aut}(f)_\iota$ of ι in $\mathrm{Aut}(f)$ induces on $\mathcal{F}$ a flag-transitive group of automorphisms, isomorphic to the Chevalley group of adjoint type $F_4(\mathbb{F})$.

Note that we have defined $\mathcal{F}$ as a poset together with a 26-dimensional projective embedding $e : \mathcal{F} \to H_\iota$, induced by the 27-dimensional projective embedding $\varepsilon : \mathcal{E} \to V$. If x is a symp of $\mathcal{F}$ (namely a 4-element of $\mathcal{F}$) and e_x is the embedding of $\mathrm{Res}_\mathcal{F}(x)$ induced by e, then x is a 6-dimensional subspace of H_ι and e_x embeds $\mathrm{Res}_\mathcal{F}(x)$ in that subspace as a polar space of symplectic type.

The building $\mathcal{E}$ admits projective embeddings different from ε. Every Weyl module provides one of them. For instance, with the 5- and 3-elements of $\mathcal{E}$ chosen as points and lines, we can embed $\mathcal{E}$ in a 78-dimensional vector space W. Let us denote this 78-dimensional embedding by ε^*. As the 4- and 3-elements of $\mathcal{F}$ arise from 5- and 3-elements of $\mathcal{E}$, the embedding ε^* induces a projective embedding e^* of the dual $\mathcal{F}^*$ of $\mathcal{F}$ in the subspace W' of W spanned by the image $\varepsilon^*(\mathcal{F}^*)$ of $\mathcal{F}^*$ via ε^*. The embedding $e^* : \mathcal{F}^* \to W'$ arises from the 52-dimensional Weyl module associated to the root corresponding to the right hand node of the Dynkin diagram:

(F_4) 

Hence $\dim(W') = 52$. Let x be a symp of $\mathcal{F}^*$ (namely a point of $\mathcal{F}$) and let e^*_x be the embedding induced by e^* on $\mathrm{Res}_{\mathcal{F}^*}(x)$. Then e^*_x embeds $\mathrm{Res}_{\mathcal{F}^*}(x)$ as a polar space of orthogonal type in a 7-dimensional subspace of W'.

When either $\mathrm{char}(\mathbb{F}) \neq 2$ or $\mathrm{char}(\mathbb{F}) = 2$ but $\mathbb{F}$ is non-perfect, the posets $\mathcal{F}$ and $\mathcal{F}^*$ are not isomorphic. On the other hand, let $\mathbb{F}$ be perfect of characteristic 2. Then $\mathcal{F} \cong \mathcal{F}^*$. In this case e is not its own linear hull. Indeed let $\tilde{e}$ be the absolutely universal embedding of $\mathcal{F}$, which exists whatever $\mathbb{F}$ is (Kasikova and Shult [16]). Since now $\mathcal{F} \cong \mathcal{F}^*$, both e and e^* are projective embeddings of $\mathcal{F}$, hence each of them is a morphic image of $\tilde{e}$.

Problem 5.1 Is it true that e^* is its own linear hull? In other words, is e^* absolutely universal?

Suppose that either $\mathrm{char}(\mathbb{F}) \neq 2$ or $\mathrm{char}(\mathbb{F}) = 2$ but $\mathbb{F}$ is non-perfect. In this case, is e absolutely universal?

5.2 A construction for $\mathcal{G}(E_6(2))$

We keep the notation of the previous subsection. For every $k \in \mathbb{F} \setminus \{0\}$, the group $\mathrm{Aut}(f)$ is transitive on the set $\mathcal{V}_k$ of vectors $x \in V$ such that $f(x) = k$ (see Buekenhout and Cohen [2, Chapter 18]).

Put $G := Z \cdot \mathrm{Aut}(f)$ and $\mathbb{F}^3 := \{t^3\}_{t \in \mathbb{F}}$. The orbits of G on $V \setminus \{0\}$ are joins $\overline{\mathcal{V}}_k := \cup(\mathcal{V}_{sk} \mid s \in \mathbb{F}^3 \setminus \{0\})$ of orbits of $\mathrm{Aut}(f)$. Clearly, $\overline{\mathcal{V}}_k = \overline{\mathcal{V}}_h$ if and only if $h^{-1}k \in \mathbb{F}^3 \setminus \{0\}$. Moreover, $\overline{\mathcal{V}}_k \cup \{0\}$ is the union of 1-dimensional linear subspaces of V (points of $\mathrm{PG}(V)$). We denote by $\mathcal{P}_k$ the set of points of $\mathrm{PG}(V)$ contained in $\overline{\mathcal{V}}_k \cup \{0\}$.

Let $\iota = (I, O, O)$, as in Subsection 5.1. Then $\langle \iota \rangle \in \mathcal{P}_1$. As in Subsection 5.1, H_ι is the hyperplane of V orthogonal to ι with respect to ϕ, but we take the liberty to use the symbol H_ι also for the corresponding hyperplane of $\mathrm{PG}(V)$.

Recalling the construction of $\mathcal{F}$ as a subgeometry of H_ι, we define $R(\langle\iota\rangle)$ as the collection of all projective subspaces of $\mathrm{PG}(V)$ that contain $\langle\iota\rangle$ and meet H_ι in an element of $\mathcal{F}$.

Lemma 5.2 *Let $X \in R(\langle\iota\rangle)$. Then:*
(i) $X \setminus (X \cap H_\iota) \subset \mathcal{P}_1$.
(ii) Let $p \in X \cap \mathcal{P}_1$. Then there exists an element $g \in G$ such that $g(X) = X$ and $g(\langle\iota\rangle) = p$. (Compare condition (C) of Section 3.)

Proof. Let x be a non-zero vector of H_ι with $x^\sharp = 0$. Then $f(x) = 0$. Note also that
$iota^\sharp = \iota$. Hence (4) of Subsection 5.1 forces $f(\iota + tx) = f(\iota) = 1$ for all $t \in \mathbb{F}$. Therefore, all projective lines, planes and 3-spaces of $R(\langle\iota\rangle)$ are contained in $\mathcal{P}_1 \cup H_\iota$. Moreover, all points of $\mathrm{PG}(V)$ contained in a 4-element of $\mathcal{F}$ are points of $\mathcal{F}$. Hence the 4-elements of $R(\langle\iota\rangle)$ are joins of lines of $R(\langle\iota\rangle)$. By the above, they are contained in $\mathcal{P}_1 \cup H_\iota$. Claim (i) is proved.

Turning to (ii), let $X \in R(\langle\iota\rangle)$ and $p \in X \cap \mathcal{P}_1$. Let $L = \langle\iota, p\rangle$. Then $L \in R(\langle\iota\rangle)$, by (i). We shall firstly prove that there exists an element $g_L \in G$ such that $g_L(L) = L$ and $g_L(\langle\iota\rangle) = p$. Let $\epsilon := (E_{1,2}, O, O) \in V$, where

$$E_{1,2} := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $\epsilon \in H_\iota$ and $\epsilon^\sharp = 0$, namely ϵ represents a point of $\mathcal{F}$. Without loss, we may assume that $L = \langle\iota, \epsilon\rangle$ and $p = \langle\iota + \epsilon\rangle$. It is easy to see that, for every choice of three non-singular matrices $A, B, C \in M$ with the same determinant, the mapping $g_{A,B,C}$ sending every vector $x = (X_1, X_2, X_3) \in V$ to $(AX_1B^{-1}, BX_2C^{-1}, CX_3A^{-1})$ belongs to $\mathrm{Aut}(f)$. Choose A, B and C as follows: ϵ:

$$A := \begin{bmatrix} a & r & s \\ 0 & a & 0 \\ 0 & t & b \end{bmatrix} \quad \text{for } a, b, , r, s, t \in \mathbb{F},\ a, b \neq 0,$$

$B := A - aE_{1,2}$ and C is any (3×3)-matrix with $\det(C) = a^2b$. Then $g_{A,B,C}$ sends ι to $\iota + \epsilon$ and fixes ϵ. If $X = L$ then (ii) is proved. Let $L \subset X$. As $R(\langle\iota\rangle) \cong \mathcal{F}$ and the stabilizer G_ι of $\langle\iota\rangle$ in G induces a flag-transitive group on $\mathcal{F}$, G_ι also acts flag-transitively on $R(\langle\iota\rangle)$. Hence there exists an element $g_X \in G_\iota$ such that $g_X(L) = L$ and $g_X(X) = g^{-1}_{A,B,C}(X)$. The element $g := g_{A,B,C}g_X$ has the properties required in (ii). □

Let now $p \in \mathcal{P}_1$ and let $g \in G$ map $\langle\iota\rangle$ onto p. Define $R(p) := g(R(\langle\iota\rangle))$. Since the stabilizer of ι in $\mathrm{Aut}(f)$ yields an automorphism group of $\mathcal{F}$, this definition does not depend on the choice of g.

We now define an incidence structure $\mathcal{G}$ having $\mathcal{P}_1 \cup \bigcup_{p \in \mathcal{P}_1} R(p)$ as the set of elements and inclusion as the incidence relation. Types are defined as follows: the points of $\mathcal{P}_1$ are the 0-elements of $\mathcal{G}$. If $Y \in R(p)$ for a point $p \in \mathcal{P}_1$ and

$Y = g(X)$ for an element $g \in G$ sending $\langle \iota \rangle$ to p and an element $X \in R(\langle \iota \rangle)$, then the type of the element $X \cap H_\iota$ of $\mathcal{F}$ is taken as the type $\tau(Y)$ of Y.

Recall that, according to the conventions stated in the introduction of this paper, only residually connected structures deserve to be called geometries.

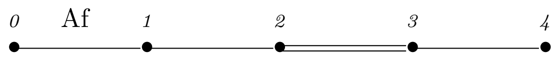
Lemma 5.3 *The structure $\mathcal{G}$ is residually connected (whence it is a geometry). Moreover:*
(1) $\mathrm{Res}_{\mathcal{G}}(p) = R(p) \cong \mathcal{F}$ *for every point* $p \in \mathcal{P}_1$.
(2) If X is an element of $\mathcal{G}$ with type $0 < \tau(X) < 4$ *then* $\mathrm{Res}^-_{\mathcal{G}}(X)$ *is the set of points and subspaces of the affine space* $X \setminus (X \cap H_\iota)$.
(3) If X is a 4-element of $\mathcal{G}$ then $\mathrm{Res}_{\mathcal{G}}(X)$ *is isomorphic to the affine extension of the* $\mathrm{Sp}(6, \mathbb{F})$*-symplectic polar space, embedded in* $V(6, \mathbb{F})$.

Proof. We firstly prove that $\mathcal{G}$ is connected. As the group G acts transitively on $\mathcal{P}_1$, it transitively permutes the connected component of $\mathcal{G}$. Let $\mathcal{X}$ be the component of $\mathcal{G}$ containing $\langle \iota \rangle$ and $G_{\mathcal{X}}$ the stabilizer of $\mathcal{X}$ in G. The stabilizer G_ι of $\langle \iota \rangle$ is contained in $G_{\mathcal{X}}$. However $G_{\mathcal{X}}$ is larger than G_ι, by (ii) of lemma 5.2. On the other hand, G_ι is maximal in G, since the Chevalley group $F_4(\mathbb{F})$ is a maximal subgroup of $E_6(\mathbb{F})$. It follows that $G_{\mathcal{X}} = G$, namely $\mathcal{X} = \mathcal{G}$, since G acts transitively on the set of components of $\mathcal{G}$. The connectedness of $\mathcal{G}$ is proved. The residual connectedness of the residues of the elements of $\mathcal{G}$ follows from claims (1), (2) and (3). So, it only remains to prove these claims.

Let $Y = g_1(X)$ for $X \in R(\langle \iota \rangle)$ and $g_1 \in G$. Let $p \in \mathcal{P}_1 \cap Y$. Then $g_1^{-1}(p) \in X$. By (ii) of Lemma 5.2, an element $g_2 \in G$ exists such that $g_2(X) = X$ and $g_2(\langle \iota \rangle) = g_1^{-1}(p)$. Hence $Y = g_1 g_2(X)$ and $p = g_1 g_2(\langle \iota \rangle)$. Claim (1) is proved. Claim (2) follows from (i) of Lemma 5.2 and claim (1). Finally, (3) follows from (2) and the fact that, if X is a 4-element of $R(\langle \iota \rangle)$, all points of $X \cap H_\iota$ belong to $\mathcal{F}$. □

Theorem 5.4 *The geometry $\mathcal{G}$ belongs to the following diagram:*

$(Af.F_4)$



(Easy, by Lemma 5.3.)

Theorem 5.5 *The group G acts flag-transitively on $\mathcal{G}$ with Z as the kernel of that action.*

Proof. Let c be a given chamber of $\mathcal{G}$ containing $\langle \iota \rangle$ as the 0-element and let d be any other chamber of $\mathcal{G}$. We must prove that $g(d) = c$ for a suitable element $g \in G/Z$. As G/Z is transitive on the point-set $\mathcal{P}_1$ of $\mathcal{G}$, there are elements of G/Z that map the 0-element of d onto $\langle \iota \rangle$. So, we may assume that $\langle \iota \rangle$ is the 0-element of d too. The stabilizer G_ι of ι in G acts flag-transitively on $\mathcal{F}$ and $\mathrm{Res}_{\mathcal{G}}(\langle \iota \rangle) \cong \mathcal{F}$. Hence there exists an element $g \in G_\iota$ such that $g(d) = c$. □

Corollary 5.6 *Let* $\mathbb{F} = \mathrm{GF}(2)$. *Then* $\mathcal{G} = \mathcal{G}(E_6(2))$.

Proof. In this case $Z = 1$. Hence $G = G/Z = E_6(2)$. The group $E_6(2)$ contains only two conjugacy classes of subgroups isomorphic to $F_4(2)$, switched by the polarity of the $E_6(2)$-building. Moreover, the $\{0, 1, 2, 3\}$-residues of $\mathcal{G}$ are just as in $\mathcal{G}(E_6(2))$, by (3) of Lemma 5.3 (see also Lemma 2.5). The conclusion is now obvious. □

Problem 5.7 As noticed in Subsection 3.4, the geometry $\mathcal{G}(E_6(2))$ is simply connected. Is $\mathcal{G}$ simply connected whatever $\mathbb{F}$ is?

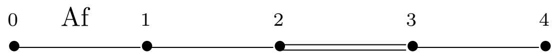
Remark 5.8 Let $\widehat{G}$ be the group of linear transformations of V that preserve f modulo a scalar. In general, $\widehat{G}$ is larger than G. If so, then $\widehat{G}$ fuses the orbits of G on $\mathrm{PG}(V)$ in larger orbits. Let $\widehat{\mathcal{P}}_1$ be the $\widehat{G}$-orbit containing $\mathcal{P}_1$ and define an incidence structure $\widehat{\mathcal{G}}$ on it in the same way as $\mathcal{G}$ on $\mathcal{P}_1$. Then $\widehat{\mathcal{G}}$ is the disjoint union of copies of $\mathcal{G}$, one copy for each of the orbits of G fused in $\widehat{\mathcal{P}}_1$.

Problem 5.9 Suppose that $\mathbb{F}$ is a quadratic extension of a field $\mathbb{F}_0$. Is it possible to modify the construction of $\mathcal{G}$ in such a way as to obtain an $Af.F_4$-geometry for the twisted group ${}^2E_6(\mathbb{F}_0)$ with F_4-residues isomorphic to the $F_4(\mathbb{F}_0)$-building?

5.3 More $Af.F_4$-geometries

5.3.1 Affine extensions of buildings of type $F_4(\mathbb{F})$

Let $\mathcal{F}$ be the $F_4(\mathbb{F})$-building, with points and lines chosen as in Subsection 5.1, let $\mathcal{F}^*$ be its dual and $e : \mathcal{F} \to H_\iota$ and $e^* : \mathcal{F}^* \to W'$ the 26- and 52-dimensional embeddings considered in Section 5.1. The affine extensions $\mathrm{Ex}_e(\mathcal{F})$ and $\mathrm{Ex}_{e^*}(\mathcal{F}^*)$ are flag-transitive $Af.F_4$-geometries. Take $\{0, 1, 2, 3, 4\}$ as the type-set for both $\mathrm{Ex}_e(\mathcal{F})$ and $\mathrm{Ex}_{e^*}(\mathcal{F}^*)$, with 0 standing for points and the types 4, 3, 2 and 1 of $\mathcal{F}^*$ replaced by 1, 2, 3 and 4:

$(Af.F_4)$ 

The $\{1, 2, 3, 4\}$-residues of $\mathrm{Ex}_e(\mathcal{F})$ are isomorphic to the affine extension of the $\mathrm{Sp}(6, \mathbb{F})$-polar space, naturally embedded in $V(6, \mathbb{F})$. This extension is isomorphic to the minimal standard quotient of the affine polar space obtained by removing a singular hyperplane from the $\mathrm{Sp}(8, \mathbb{F})$-polar space. On the other hand, the $\{1, 2, 3, 4\}$-residues of $\mathrm{Ex}_{e^*}(\mathcal{F}^*)$ are isomorphic to the affine extension of the $O(7, \mathbb{F})$-polar space. This extension is isomorphic to the affine polar space obtained by removing a singular hyperplane from the $O(9, \mathbb{F})$-polar space. Note that this affine polar space does not admit any proper standard quotient.

If $\mathrm{char}(\mathbb{F}) = 2$ and $\mathbb{F}$ is perfect then $\mathcal{F} \cong \mathcal{F}^*$ and the embedding e is a morphic image of e^*. Consequently, $\mathrm{Ex}_{e^*}(\mathcal{F}^*)$ is a 2-cover of $\mathrm{Ex}_e(\mathcal{F})$ (see the comments at the end of Subsection 5.1).

In particular, let $\mathbb{F} = \mathrm{GF}(2)$. Then $\mathrm{Ex}_e(\mathcal{F}) = \mathcal{G}(2^{26}F_4(2))$. The latter is simply connected (Subsection 3.4) but it is not 2-simply connected, since

$\mathrm{Ex}_{e^*}(\mathcal{F}^*)$ is a proper 2-cover of it. Clearly, property (TR) of Section 2 fails to hold in $\mathrm{Ex}_{e^*}(\mathcal{F}^*)$.

Problem 5.10 Determine the universal 2-cover of $\mathcal{G}(2^{26}F_4(2))$. This is the same as determining the universal representation group of the point-line geometry of the $F_4(2)$-building (see Proposition 6.2).

Problem 5.11 Are $\mathrm{Ex}_e(\mathcal{F})$ and $\mathrm{Ex}_{e^*}(\mathcal{F}^*)$ simply connected for any choice of the field $\mathbb{F}$? What about their universal 2-covers? (Compare Propositions 6.1 and 6.2.)

5.3.2 More affine extensions

Let $\mathcal{F}$ be the metasymplectic space of the $D_4(\mathbb{F})$-building. It is well known that $\mathcal{F}$ admits a projective embedding e of vector dimension 27 or 28, according to whether char($\mathbb{F}$) is 2 or different from 2. In any case, $\mathrm{Ex}_e(\mathcal{F})$ is a flag-transitive $Af.F_4$-geometry. If $\mathbb{F} = \mathrm{GF}(2)$ then $\mathrm{Ex}_e(\mathcal{F})$ has the same orders as $\mathcal{G}(Fi_{22})$ and $\mathcal{G}(3^{\cdot}Fi_{22})$, but it has nothing to do with either of them. Needless to say, (TR) fails to hold in $\mathrm{Ex}_e(\mathcal{F})$.

Suppose now that $\mathbb{F}$ is a quadratic extension of a subfield $\mathbb{F}_0$ and let $\mathcal{E}$ be the $E_6(\mathbb{F})$-building. It is well known that $\mathcal{E}$ contains a subgeometry $\mathcal{F}_{\mathrm{tw}}$ isomorphic to the F_4-building of twisted type ${}^2E_6(\mathbb{F}_0)$. Regarding $\mathcal{F}_{\mathrm{tw}}$ as a poset with set of types $\{1,2,3,4\}$ as we have done for $\mathcal{F}$, the $\{1,2,3\}$-residues of $\mathcal{F}_{\mathrm{tw}}$ are isomorphic to the polar space associated to $\mathrm{PSU}(6,\mathbb{F})$ while $\{4,3,2\}$-residues are isomorphic to the $O^-(8,\mathbb{F}_0)$-polar space. The 27-dimensional embedding ε of $\mathcal{E}$ induces a projective embedding e_{tw} of $\mathcal{F}_{\mathrm{tw}}$. The extension $\mathrm{Ex}_{e_{\mathrm{tw}}}(\mathcal{F}_{\mathrm{tw}})$ is a flag-transitive $Af.F_4$-geometry.

Problem 5.12 The 78-dimensional embedding $e^* : \mathcal{E} \to W$ induces a representation of the dual $\mathcal{F}^*_{\mathrm{tw}}$ of $\mathcal{F}_{\mathrm{tw}}$ in a subspace S of W. The lines of $\mathcal{F}_{\mathrm{tw}}$ are mapped by e^* into lines of $\mathrm{PG}(S)$, but not onto them. Can we find a Baer subgeometry $\mathcal{S}_0$ of $\mathrm{PG}(S)$ such that e^*_{tw} induces a projective embedding $e^*_0 : \mathcal{F}^*_{\mathrm{tw}} \to \mathcal{S}_0$?

If $\mathcal{S}_0$ existed, then $\mathrm{Ex}_{e^*_0}(\mathcal{F}^*_{\mathrm{tw}})$ would be an $Af.F_4$-geometry with point-residues isomorphic to $\mathcal{F}^*_{\mathrm{tw}}$. In particular, for $\mathbb{F} = \mathrm{GF}(4)$ the extension $\mathrm{Ex}_{e^*_0}(\mathcal{F}^*_{\mathrm{tw}})$ would have the same orders as $\mathcal{G}(B)$, but (TR) would not hold in it.

Part II
Various basics

6 Representations, projective embeddings, affine extensions and hyperplane complements

Throughout this section $\mathcal{G}$ is a geometry with a string-shaped diagram of rank $n \geq 2$ over the set of types $\{1,2,...,n\}$. We stick to all conventions stated in

Subsection 1.2 for geometries of this kind. We do not assume (IP) on $\mathcal{G}$, but we assume that (IP) holds in the point-line system $\mathcal{G}_{|1,2}$ of $\mathcal{G}$, as it is customary in the literature on embeddings and hyperplanes.

Leaving geometries with non-string diagrams out of our exposition is not really restrictive. Indeed, when considering embeddings or hyperplanes of a geometry belonging to a non-string diagram, we actually deal with its grassmann geometry with respect to a given type [19, Chapter 5]. The diagram of a grassmann geometry is indeed string-shaped.

6.1 Representations and extensions

Following [20], we choose a very general setting for our definitions. We shall turn later to the definitions usually given in the literature.

Given a group R, let $S(R)$ be its subgroup lattice. A *complete representation* of $\mathcal{G}$ in R (a *representation* of $\mathcal{G}$ in R, for short) is a mapping $\rho : \mathcal{G} \to S(R)$ satisfying all the following:

(R1) $\langle \rho(p) \rangle_{p \in P(\mathcal{G})} = R$;

(R2) $\langle \rho(p) \rangle_{p \in P(x)} = \rho(x)$ for every $x \in \mathcal{G}$;

(R3) $\rho(p) \cap \rho(q) = 1$ for any two distinct points $p, q \in P(\mathcal{G})$.

Note that, regarded $\mathcal{G}$ and $S(R)$ as posets, (R2) implies that ρ is a morphism from $\mathcal{G}$ to $S(R)$. Henceforth we write $\rho : \mathcal{G} \to R$ as a shortening of the phrase "ρ is a representation of $\mathcal{G}$ in R".

Given a representation $\rho : \mathcal{G} \to R$, the *extension* $\mathrm{Ex}_\rho(\mathcal{G})$ of $\mathcal{G}$ via ρ (called 'expansion' in [19], [20] and [21]) is the geometry over the set of types $\{0, 1, ..., n\}$ defined as follows: the elements of R are the 0-elements of $\mathrm{Ex}_\rho(\mathcal{G})$ while for $i = 1, 2, ..., n$ the elements of $\mathrm{Ex}_\rho(\mathcal{G})$ of type i are the pairs $(a\rho(x), x)$ where x is an i-element of $\mathcal{G}$ and $a\rho(x)$, called the *support* of $(a\rho(x), x)$, is a coset of the subgroup $\rho(x)$ of R. Incidence is defined as follows: a 0-element a and an i-element $(b\rho(x), x)$ are incident if and only if $a \in b\rho(x)$; two elements $(a\rho(x), x)$ and $(b\rho(y), y)$ with $1 \leq \tau(x) < \tau(y) \leq n$ are incident if and only if $x < y$ in $\mathcal{G}$ and $a\rho(x) \subseteq b\rho(y)$. The residual connectedness of $\mathrm{Ex}_\rho(\mathcal{G})$ follows from (R1) and (R2). The diagram of $\mathrm{Ex}_\rho(\mathcal{G})$ is still string-shaped. The residues of the 0-elements of $\mathrm{Ex}_\rho(\mathcal{G})$ are isomorphic to $\mathcal{G}$.

The $\{0, 1\}$-residues of $\mathrm{Ex}_\rho(\mathcal{G})$ are partial linear spaces, by (R3). In order to say more on them we need to know more on the mappings induced by ρ on the lines of $\mathcal{G}$. For instance, given $l \in L(\mathcal{G})$, suppose that $\rho(l)$ is abelian and $\langle \rho(p), \rho(q) \rangle = \rho(l)$ for any two points $p, q \in P(l)$. Then the lower residue of $\rho(l)$ in $\mathrm{Ex}_\rho(\mathcal{G})$ is a net. If moreover $\rho(l) = \cup_{p \in P(l)} \rho(p)$ then that residue is an affine plane.

When (IP) holds in $\mathcal{G}$ and ρ satisfies the following:

(R4) for every choice of $p \in P(\mathcal{G})$ and $x \in \mathcal{G}$, if $\rho(p) \leq \rho(x)$ then $p \in P(x)$, then ρ induces an isomorphism from the poset $\mathcal{G}$ to the poset induced by $S(R)$ on $\rho(\mathcal{G}) = \{\rho(x)\}_{x\in\mathcal{G}}$. In this case the elements of $\mathrm{Ex}_\rho(\mathcal{G})$ of type > 0 bijectively correspond to their supports, whence we can simplify the definition of $\mathrm{Ex}_\rho(\mathcal{G})$ by taking cosets $a\rho(x)$ instead of pairs $(a\rho(x), x)$ as elements and inclusion as the incidence relation.

Given two representations $\rho_1 : \mathcal{G} \to R_1$ and $\rho_2 : \mathcal{G} \to R_2$, a *morphism* from ρ_1 to ρ_2 is a homomorphism $f : R_1 \to R_2$ such that, for every $x \in \mathcal{G}$, f induces an isomorphism from $\rho_1(x)$ to $\rho_2(x)$. Note that f is surjective, by (R1). Every morphism from ρ_1 to ρ_2 induces a covering from $\mathrm{Ex}_{\rho_1}(\mathcal{G})$ to $\mathrm{Ex}_{\rho_2}(\mathcal{G})$. Moreover, let $\widetilde{R}$ be the universal completion of the amalgam $\mathcal{A}_\rho(\mathcal{G}) := \{\rho(x)\}_{x\in\mathcal{G}}$, where we only take a record of the inclusions $\rho(x) < \rho(y)$ for $x < y$. Let $\tilde{\rho}$ be the representation of $\mathcal{G}$ in $\widetilde{R}$ naturally induced by ρ, namely $\tilde{\rho}(x) = \rho(x)$ for every $x \in \mathcal{G}$, with $\rho(x)$ being now regarded as a subgroup of $\widetilde{R}$. The canonical projection $\tilde{f} : \widetilde{R} \to R$ is a morphism from $\tilde{\rho}$ to ρ and, for every morphism of representations $f : \bar{\rho} \to \rho$ there is unique morphism $g : \tilde{\rho} \to \bar{\rho}$ such that $\tilde{f} = f \circ g$. This property uniquely determines $\tilde{\rho}$ up to isomorphism. The following also holds (see [20]):

Proposition 6.1 *The extension* $\mathrm{Ex}_{\tilde{\rho}}(\mathcal{G})$ *is the universal cover of* $\mathrm{Ex}_\rho(\mathcal{G})$*.*

We call $\tilde{\rho}$ the *hull* of ρ, also the *abstract hull* of ρ when we need to distinguish it from linear hulls, to be defined later.

A representation $\rho : \mathcal{G} \to R$ is said to be *abelian* if R is abelian. If ρ is abelian then we can also consider its *abelian hull*, defining it as the composition of $\tilde{\rho}$ with the canonical projection of $\widetilde{R}$ onto $\widetilde{R}/\widetilde{R}'$, where $\widetilde{R}'$ is the commutator subgroup of $\widetilde{R}$.

With $\mathcal{G}$ and R as above, a *point-line representation* of $\mathcal{G}$ in R is a (complete) representation $e : \mathcal{G}_{|1,2} \to R$. Point-line representations may also be called representations for short, when there is no danger of confusion with complete representations. If e is a point-line representation of $\mathcal{G}$, we take the liberty of writing $e : \mathcal{G} \to R$ instead of $e : \mathcal{G}_{|1,2} \to R$, provided that no ambiguity arises.

Let $e : \mathcal{G} \to R$ be a point-line representation of $\mathcal{G}$. We can extend e to a complete representation $\rho_e : \mathcal{G} \to R$ by setting $\rho_e(x) := \langle e(p)\rangle_{p\in P(x)}$, according to (R2). We call ρ_e the *completion* of e.

Clearly, if $f : \bar{e} \to e$ is a morphism of point-line representations then f induces a morphism from the completion $\rho_{\bar{e}}$ of $\bar{e}$ to the completion ρ_e of e. So, if e is its own hull then ρ_e is its own hull as well, but the converse fails to hold in general. Consequently, if $\widetilde{\rho_e}$ and $\rho_{\tilde{e}}$ are the hull of ρ_e and the completion of the hull $\tilde{e}$ of e respectively, then $\mathrm{Ex}_{\rho_{\tilde{e}}}(\mathcal{G})$ is a 2-cover of $\mathrm{Ext}_{\widetilde{\rho_e}}(\mathcal{G})$. Nevertheless, $\mathrm{Ext}_{\widetilde{\rho_e}}(\mathcal{G})$ is simply connected by Proposition 6.1. Note that $\mathrm{Ex}_{\rho_{\tilde{e}}}(\mathcal{G})$ is also simply connected (but possibly not 2-simply connected), because $\rho_{\tilde{e}}$ is its own hull. By combining Proposition 6.1 with [18, Theorem 1] we can say something more.

Proposition 6.2 *Let e be a point-line representation of $\mathcal{G}$. Suppose that $\mathcal{G}$ has rank $n \geq 3$ and that $\mathcal{G}$ as well as all of its residues of rank at least 3 are simply* *connected. Let* $\tilde{e}$ *be the hull of* e*. Then* $\mathrm{Ex}_{\tilde{e}}(\mathcal{G}_{|1,2})$ *is a truncation of the universal 2-cover of* $\mathrm{Ex}_e(\mathcal{G})$*. If moreover* $\mathcal{G}$ *satisfies* (IP) *then* $\mathrm{Ex}_{\rho_{\tilde{e}}}(\mathcal{G})$ *is the universal 2-cover of* $\mathrm{Ex}_e(\mathcal{G})$*.*

6.2 $GF(2)$-representations

Suppose that all lines of $\mathcal{G}$ have exactly three points. A GF(2)-*representation* of $\mathcal{G}$ is a point-line representation $e : \mathcal{G} \to R$ such that

(R5) $e(p)$ has order 2 for every point $p \in P(\mathcal{G})$ and $e(l)$ is elementary abelian of order 2^2 for every line $l \in L(\mathcal{G})$.

We refer the reader to Ivanov and Shpectorov [13, Chapter 2] for properties of these representations. We only make a few remarks here.

Let $e : \mathcal{G} \to R$ be a GF(2)-representation and let ρ_e be its completion. Then the $\{0,1\}$-residues of $\mathrm{Ex}_{\rho_e}(\mathcal{G})$ are affine planes of order 2.

In view of (R5), the amalgam $\mathcal{A}_{1,2}(\mathcal{G}) := \mathcal{A}_e(\mathcal{G}_{|1,2}) = \{e(x)\}_{x \in \mathcal{G}_{|1,2}}$ is completely determined by $\mathcal{G}_{|1,2}$ itself, with no need of any further information on e. Hence all GF(2)-representations of $\mathcal{G}$ have the same hull, which is called the *universal representation* of $\mathcal{G}$. The universal completion of $\mathcal{A}_{1,2}(\mathcal{G})$ is called the (*universal*) *representation group* of $\mathcal{G}$. When R is abelian, the abelian hull of e is called the (*universal*) *representation module* of $\mathcal{G}$.

Remark 6.3 We warn the reader that, in the literature, it is customary to consider the amalgam $\mathcal{A}_{1,2}(\mathcal{G})$ and its universal completion even if $\mathcal{G}$ admits no GF(2)-representation, as when the universal completion of $\mathcal{A}_{1,2}(\mathcal{G})$ collapses to 1, for instance. However, in this paper we never meet situations like these.

6.3 Projective embeddings and affine extensions

Let V be a vector space over a field $\mathbb{F}$ and let $e : \mathcal{G} \to V$ be a point-line representation of $\mathcal{G}$ in the additive group of V. We say that e is a *projective embedding* (defined over $\mathbb{F}$) if it satisfies the followings:

(E1) $e(p)$ is 1-dimensional vector subspace of V for every point $p \in P(\mathcal{G})$;

(E2) for every line $l \in L(\mathcal{G})$, $e(l)$ is a 2-dimensional subspace of V and $e(l) = \cup_{p \in P(l)} e(p)$.

In view of the above, a linear representation $e : \mathcal{G} \to V$ can also be regarded as a mapping from $\mathcal{G}$ to the set of points and lines of $\mathrm{PG}(V)$. Accordingly, we also write $e : \mathcal{G} \to \mathrm{PG}(V)$ instead of $e : \mathcal{G} \to V$.

Let ρ_e be the completion of e. The extension $\mathrm{Ex}_{\rho_e}(\mathcal{G})$ (also denoted by $\mathrm{Ext}_e(\mathcal{G})$ for short) is called the *affine extension* of $\mathcal{G}$ via e. The $\{0,1\}$-residues of $\mathrm{Ex}_{\rho_e}(\mathcal{G})$ are affine planes.

Given two projective embeddings $e_1 : \mathcal{G} \to \mathrm{PG}(V_1)$ and $e_2 : \mathcal{G} \to \mathrm{PG}(V_2)$ defined over the same field $\mathbb{F}$, a (*projective*) *morphism* from e_1 to e_2 is semilinear mapping $f : V_1 \to V_2$ such that $f \circ e_1 = e_2$. Clearly, morphisms of projective embeddings are also morphisms of representations as defined in Subsection 6.1.

A linear analogue of the abstract hull $\tilde{e}$ of e can also be defined. The elements of the amalgam $\mathcal{A}_e(\mathcal{G}_{|1,2})$ are $\mathbb{F}$-vector spaces. Let $\widehat{V}$ be the largest $\mathbb{F}$-vector space in which $\mathcal{A}_e(\mathcal{G}_{|1,2})$ can be embedded. Then the mapping $\hat{e} : \mathcal{G} \to \widehat{V}$ defined by the clause $\hat{e}(x) = e(x)$ for $x \in \mathcal{G}_{|1,2}$ but with $e(x)$ regarded as a subspace of $\widehat{V}$, is a projective embedding of $\mathcal{G}$ in $\widehat{V}$ and the canonical projection $\hat{f} : \widehat{V} \to V$ is a morphism from $\hat{e}$ to e. The pair $(\hat{e}, \hat{f})$ satisfies the following universal property: for every morphism of projective embeddings $f : \bar{e} \to e$ there is unique projective morphism $g : \hat{e} \to \bar{e}$ such that $\hat{f} = f \circ g$. We call $\hat{e}$ the (*linear*) *hull* of e. Clearly, $\hat{e}$ is also a point-line representation of $\mathcal{G}$. Hence a morphism of representations exists from the abstract hull $\tilde{e}$ of e to $\hat{e}$.

The category of projective embeddings of a geometry $\mathcal{G}$ often admits an initial object, called the *absolutely universal* embedding of $\mathcal{G}$ (see Kasikova and Shult [16]. In general, that object is not initial in the category of point-line representations of $\mathcal{G}$.

Remark 6.4 Suppose that all lines of $\mathcal{G}$ have precisely three points and that $\mathcal{G}$ admits a projective embedding. Then the projective embeddings of $\mathcal{G}$ are just the abelian GF(2)-representations of $\mathcal{G}$. The universal representation module of $\mathcal{G}$ affords the absolutely universal embedding of $\mathcal{G}$.

6.4 Subspaces, hyperplanes and their complements

A (proper) *subspace* of $\mathcal{G}$ is a (proper) subset S of $P(\mathcal{G})$ such that every line of $\mathcal{G}$ either is fully contained in S or it meets S in at most one point. A subspace S is said to be *totally singular* (also just *singular*, for short) if all of its points are pairwise collinear.

A proper subspace of $\mathcal{G}$ meeting every line of $\mathcal{G}$ non-trivially is a *hyperplane* of $\mathcal{G}$. Given a hyperplane H of $\mathcal{G}$, the *complement* $\mathcal{G} \setminus H$ of H in $\mathcal{G}$ is the substructure of $\mathcal{G}$ formed by the points exterior to H and the elements $x \in \mathcal{G}$ such that $P(x) \not\subseteq H$, with the incidence relation and the type function inherited from $\mathcal{G}$.

In general $\mathcal{G} \setminus H$ is not residually connected (not even connected). However, in many interesting cases $\mathcal{G} \setminus H$ is residually connected. Let this be the case. Then $\mathcal{G} \setminus H$ is a geometry. The residues of the points of $\mathcal{G} \setminus H$ are the same as in $\mathcal{G}$. If x is an element of $\mathcal{G} \setminus H$ of type $\tau(x) > 2$, then $H \cap P(x)$ is a hyperplane of $\mathrm{Res}^-_{\mathcal{G}}(x)$ and $\mathrm{Res}^-_{\mathcal{G}\setminus H}(x)$ is the complement of $P(x) \cap H$ in $\mathrm{Res}^-_{\mathcal{G}}(x)$. For instance, if $\mathrm{Res}^-_{\mathcal{G}}(x)$ is a projective geometry then $\mathrm{Res}^-_{\mathcal{G}\setminus H}(x)$ is an affine geometry.

If $\mathcal{G}$ admits a projective embedding $e : \mathcal{G} \to \mathrm{PG}(V)$, then for every projective hyperplane U of $\mathrm{PG}(V)$ the preimage $e^{-1}(U \cap e(P(\mathcal{G})))$ is a hyperplane of $\mathcal{G}$. In many embeddable geometries (as classical polar spaces, for instance) all hyperplanes arise in this way, provided that e is its own linear hull.

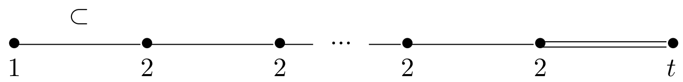
7 Affine polar spaces and their quotients

7.1 Affine polar spaces

Let $\mathcal{P}$ be a non-degenerate polar space of finite rank $n \geq 2$. We assume that all lines of $\mathcal{P}$ have at least three points. Let H be a hyperplane of $\mathcal{P}$. Its complement $\mathcal{P} \setminus H$ is residually connected. When $n \geq 3$ it belongs to the following diagram:

(Af.C$_{n-1}$) 

When all lines of $\mathcal{P}$ have exactly 3 points, the diagram of $\mathcal{P} \setminus H$ can also be drawn as follows:

(c.C$_{n-1}$) 

We call $\mathcal{P} \setminus H$ an *affine polar space* of rank n (*affine generalized quadrangle* when $n = 2$). The next proposition is well known (and easy to prove):

Proposition 7.1 *Property* (IP) *holds in every affine polar space.*

The following is also well known (see [19, Proposition 12.50]).

Proposition 7.2 *Affine polar spaces of rank $n > 2$ are simply connected.*

Let H be a hyperplane of $\mathcal{P}$. The polar space $\mathcal{P}$ induces a possibly degenerate polar space $\mathcal{H}$ on H, of rank n or $n-1$. The polar space $\mathcal{H}$ is degenerate if and only if $H = p^{\perp}$ for a given point p of $\mathcal{P}$ (where $\perp$ stands for the collinearity relation of $\mathcal{P}$, as usual). In this case the hyperplane H is said to be *singular* with p as its *deepest point* (but we warn the reader that a singular hyperplane is not a singular subspace in the sense of Subsection 6.4).

In many cases (in all cases when $\mathcal{P}$ is finite), the isomorphism type of $\mathcal{H}$ uniquely determines H up to automorphisms of $\mathcal{P}$. Hence it also determines $\mathcal{P} \setminus H$ up to isomorphisms. In these cases, a symbol denoting the isomorphism type of $\mathcal{H}$ can be used to recall what H is. For instance, if $\mathcal{H}$ is isomorphic to the polar space associated to the symplectic group $S_6(2)$ then we say that H is of $S_6(2)$-type or that it is an $S_6(2)$-hyperplane.

7.2 Standard quotients of affine polar spaces

In general, an affine polar space admits several quotients. Those that satisfy (IP) are called *standard quotients* in [6]. They can be described as follows. Given $\mathcal{P}$ and H as above, let Θ_H be the equivalence relation on the set of points of $\mathcal{P} \setminus H$ defined as follows: two points a and b of $\mathcal{P} \setminus H$ correspond in Θ_H if and only if $a^{\perp} \cap H = b^{\perp} \cap H$. Then Θ_H can be extended in a natural way to an equivalence relation on the set of elements of $\mathcal{P} \setminus H$ of any type and we can consider the quotient $(\mathcal{P} \setminus H)/\Theta_H$ of $\mathcal{P} \setminus H$ by the relation Θ_H thus extended. This quotient is the minimal standard quotient of $\mathcal{P} \setminus H$. Any other standard quotient of $\mathcal{P} \setminus H$ can be obtained as $(\mathcal{P} \setminus H)/\Theta$ for a refinement Θ of Θ_H (see [6]). The following is the main result of Cuypers and Pasini [6]:

Proposition 7.3 *Every geometry belonging to the diagram* $\mathrm{Af.C}_{n-1}$ *($n \geq 3$) and satisfying* (IP) *is a standard quotient of an affine polar space.*

In all cases we are aware of, the stabilizer of H in $\mathrm{Aut}(\mathcal{P})$ acts flag-transitively on $\mathcal{P} \setminus H$. It also induces a flag-transitive group on the minimal standard quotient $(\mathcal{P} \setminus H)/\Theta_H$. Perhaps this is true in any case. Anyway, it is always true when $\mathcal{P}$ is finite.

The collinearity graph of an affine polar space has diameter at most 3 (sometimes 2) while minimal standard quotients of affine polar spaces have diameter at most 2 (sometimes just 1). Clearly, a standard quotient of an affine polar space is minimal if and only if its collinearity graph has diameter at most 2. So, if an affine polar space has diameter 2, then it does not admit any proper standard quotient.

Example 7.4 Let $\mathcal{Q}$ be the $O_5(2)$-generalized quadrangle and H a hyperplane of $\mathcal{Q}$. Then H can be either singular or an ovoid or a (3×3)-grid. If H is singular then $\mathcal{Q} \setminus H$ is the vertex-edge system of the cube. If H is an ovoid then $\mathcal{Q} \setminus H$ is the Petersen graph. If H is a grid then $\mathcal{P} \setminus H$ is a dual grid. If H is non-singular then $\mathcal{Q} \setminus H$ admits no proper quotient. If H be singular then $\mathcal{Q} \setminus H$ admits a 2-fold quotient, isomorphic to the affine plane of order 2.

7.3 Minimal standard quotients as affine extensions or tangent geometries

Given a projective embedding $e : \mathcal{P}_0 \to \mathrm{PG}(V)$ of a polar space $\mathcal{P}_0$, suppose that for any two non-collinear points x and y of $\mathcal{P}_0$ the set $e(\{x,y\}^{\perp\perp})$ spans a line of $\mathrm{PG}(V)$. Then the affine extension $\mathrm{Ex}_e(\mathcal{P}_0)$ is a minimal standard quotient of an affine polar space. Indeed, let $\mathcal{P}$ be a polar space with point-residues isomorphic to $\mathcal{P}_0$ and let $H = p^\perp$ for a point p of $\mathcal{P}$. Then $\mathrm{Ex}_e(\mathcal{P}_0) \cong (\mathcal{P} \setminus H)/\Theta_H$.

We now turn to tangent geometries. An abstract definition of tangent geometries is stated in [6], according to which tangent geometries of polar spaces and minimal standard quotients of affine polar spaces are ultimately the same thing, but we are not interest in it here. A more concrete although slightly less general definition is also given in [6], which we shall now recall.

Let H be a hyperplane of $\mathcal{P}$ and let $\mathcal{H}$ be the polar space induced by $\mathcal{P}$ on H. Let $e : \mathcal{H} \to V$ be a projective embedding of $\mathcal{H}$. For instance, e can be induced by an embedding of $\mathcal{P}$, but this is not essential for the sequel. Assume that all non-singular hyperplanes of $\mathcal{H}$ arise from e as explained in the final paragraph of Subsection 6.4. According to this assumption, for every non-singular hyperplane K of $\mathcal{H}$, $\langle e(K)\rangle^{\perp_e}$ is a non-singular point of $\mathrm{PG}(V)$ and $\langle e(K)\rangle^{\perp_e\perp_e} = \langle e(K)\rangle$, where $\perp_e$ is the orthogonality relation associated to $e(\mathcal{H})$ in V. We call $\langle e(K)\rangle^{\perp_e}$

the *pole* of K. Thus, the non-singular hyperplanes of $\mathcal{H}$ bijectively correspond to their poles. Let $\mathcal{T}_{e,0}(\mathcal{H})$ be the set of the poles of the non-singular hyperplanes of $\mathcal{H}$. For $i > 0$ let $\mathcal{T}_{e,i}(\mathcal{H})$ be the set of the i-dimensional subspaces X of $\mathrm{PG}(V)$ such that X meets the image $e(H)$ of H in an $(i-1)$-dimensional projective subspace contained in $X^{\perp_e}$ and $X \setminus e(H) \subseteq \mathcal{T}_{e,0}(\mathcal{H})$. Clearly, $\mathcal{T}_{e,i}(\mathcal{H}) = \emptyset$ if i is too large. Let m be the largest i such that $\mathcal{T}_{e,i}(\mathcal{H}) \neq \emptyset$. Then $\mathcal{T}_e(\mathcal{H}) := \cup_{i=0}^{m}\mathcal{T}_{e,i}(\mathcal{H})$ equipped with the natural incidence relation (namely inclusion) is an incidence structure over the set of types $\{0, 1, ..., m\}$. In general, $\mathcal{T}_e(\mathcal{H})$ is not connected. However, its connected components are residually connected. We call them *projective tangent geometries* of $\mathcal{H}$ at e, also *tangent geometries* of $\mathcal{H}$ for short. As proved in [6], the quotient $(\mathcal{P} \setminus H)/\Theta_H$ is isomorphic to one of these tangent geometries. The other tangent geometries of $\mathcal{H}$ at e, if any, are also isomorphic to minimal standard quotients of affine polar spaces, but they might arise from polar spaces quite different from $\mathcal{P}$.

8 Shrinkings and geometries at infinity

8.1 Shrinkings

The earliest explicit mention of shrinkings can be found in a paper by Stroth and Wiedorn [25], but that idea is implicit in other papers too (e.g. Ivanov [9], Ivanov and Wiedorn [14], Ivanov, Pasechnik and Shpectorov [15]). That construction has been later generalized by Pasini and Wiedorn [22]. Less ambitious expositions, closer to the setting of [25], are given in [21] and [26]. In the sequel we place ourselves at a level of generality intermediate between [25] and [22].

Let $\mathcal{G}$ be a geometry of rank $n+1$ with string-shaped diagram and $0, 1, ..., n$ as types, elements of type 0, 1 and 2 being called points, lines and planes respectively. We assume that the lower residues of the planes of $\mathcal{G}$ are affine planes.

Let Λ_0 be the binary relation on the line-set $L(\mathcal{G})$ of $\mathcal{G}$ defined as follows: two lines $l, m \in L(\mathcal{G})$ correspond in Λ_0 if and only if they are parallel in the lower residue of a plane of $\mathcal{G}$. Let Λ be the transitive closure of Λ_0. Similarly, given an element $x \in \mathcal{G}$ of type $\tau(x) > 1$, let $L(x)$ be the set of lines of $\mathcal{G}$ incident with x and let $\Lambda_{0,x}$ be defined on $L(x)$ by declaring that $(l, m) \in \Lambda_{0,x}$ precisely when l and m are parallel in a $\{0, 1\}$-residue of $\mathrm{Res}^-(x)$. Let Λ_x be the transitive closure of $\Lambda_{0,x}$. The relation Λ_x is a possibly proper refinement of the relation induced by Λ on $L(x)$. In particular, when $\tau(x) = 2$ then Λ_x is the usual parallelism of the affine plane $\mathrm{Res}^-(x)$, but it can happen that Λ induces the trivial relation on $L(x)$.

Let C be one of the equivalence classes of Λ. We define a geometry $\mathrm{Shr}(C)$ with $\{1, 2, ..., n\}$ as the set of types and C as the set of 1-elements. For every $i = 2, 3, ..., n$, the i-elements of $\mathrm{Shr}(C)$ are the pairs (C_x, x) where x is an i-element of $\mathcal{G}$, which we call the *support* of (C_x, x), and C_x is a class of Λ_x such that $C_x \subseteq C$. A 1-element $l \in C$ is incident with (C_x, x) if and only if $l \in C_x$. Two elements (C_x, x) and (C_y, y) of type i and j respectively, with $2 \leq i < j$, are incident in $\mathrm{Shr}(C)$ if and only if their supports x and y are incident in $\mathcal{G}$ and $C_x \subseteq C_y$. We call $\mathrm{Shr}(C)$ a *shrinking* of $\mathcal{G}$.

For every 1-element $l \in C$ of $\mathrm{Shr}(C)$, the residue of l in $\mathrm{Shr}(C)$ is isomorphic to the upper residue of l in $\mathcal{G}$. For every type $k = 2, 3, ..., n$, the $\{1, 2, ..., k\}$-residues of $\mathrm{Shr}(C)$ are shrinkings of $\{0, 1, 2, ..., k\}$-residues of $\mathcal{G}$. In particular, the $\{1, 2\}$-residues of $\mathrm{Shr}(C)$ are shrinkings of the $\{0, 1, 2\}$-residues of $\mathcal{G}$.

If $\mathcal{G}$ satisfies (IP) then (IP) also holds in $\mathrm{Shr}(C)$. In many cases the following also holds:

(RS) (**Residual Stability of Parallelism**.) For every element $x \in \mathcal{G}$ with type $\tau(x) > 2$, the relation Λ induces Λ_x on $L(x)$.

If (RS) holds, then the class C_x is uniquely determined by x and C. So, we can replace the elements of $\mathrm{Shr}(C)$ with their supports (with the convention that a line $l \in C$ is its own support), thus regarding $\mathrm{Shr}(C)$ as a subgeometry of $\mathcal{G}$.

In general, the isomorphism type of a shrinking $\mathrm{Shr}(C)$ depends on the choice of the class C, but when $\mathrm{Aut}(\mathcal{G})$ acts transitively on $L(\mathcal{G})$ then all shrinkings of $\mathcal{G}$ are mutually isomorphic. In this case it is customary to call $\mathrm{Shr}(C)$ *the* shrinking of $\mathcal{G}$, denoting it by the symbol $\mathrm{Shr}(\mathcal{G})$.

Clearly, if $\mathcal{G}$ is flag-transitive then $\mathrm{Shr}(\mathcal{G})$ is flag-transitive.

Example 8.1 The shrinkings of a $(k + 1)$-dimensional affine geometry are k-dimensional affine geometries. So, if the $\{0, 1, ..., k\}$-residues of $\mathcal{G}$ are affine geometries then the $\{1, 2, ..., k\}$-residues of $\mathrm{Shr}(C)$ are still affine geometries.

Example 8.2 The shrinkings of an affine polar space of rank $n + 1 \geq 3$ are affine polar spaces of rank n (Pasini and Wiedorn [22, Proposition 8.1]). More generally, the shrinkings of a (minimal) standard quotient of an affine polar space of rank $n + 1$ are (minimal) standard quotients of affine polar spaces of rank n.

For instance, let $\mathcal{P}$ be the $O_7(2)$-polar space and put $\mathcal{G} := \mathcal{P} \setminus H$, for a hyperplane H of $\mathcal{P}$. The hyperplane H can be singular or isomorphic to either the $O_6^-(2)$-quadrangle or the $O_6^+(2)$-polar space. If H is singular then $\mathrm{Shr}(\mathcal{G})$ is the cube-graph, if H is the $O_6^-(2)$-quadrangle then $\mathrm{Shr}(\mathcal{G})$ is the Petersen graph and if H is the $O_6^+(2)$-polar space then $\mathrm{Shr}(\mathcal{G})$ is a dual grid (compare Example 7.4). In the first case $\mathcal{G}$ admits a 2-fold quotient, the shrinkings of which are affine planes of order 2.

Example 8.3 Let e be a projective embedding of a geometry $\mathcal{G}_0$ with string-shaped diagram. Suppose that e admits completion and let $\mathcal{G} = \mathrm{Ex}_e(\mathcal{G}_0)$ be its affine extension. Then $\mathcal{G}$ admits shrinkings and its shrinkings are extensions of the representations induced by e on the point-residues of $\mathcal{G}_0$.

8.2 Geometries at infinity

Different definitions of geometries at infinity can be given, suited to different situations (see Pasini and Wiedorn [22]). We will consider only one of them here.

With $\mathcal{G}$ as in the previous subsection, suppose that for a type $k > 1$ the $\{0, 1, ..., k\}$-residues of $\mathcal{G}$ are $(k+1)$-dimensional affine geometries. Suppose that $\mathcal{G}$ satisfies condition (RS) of the previous subsection, so that the shrinkings of $\mathcal{G}$ are subgeometries of $\mathcal{G}$. With Λ defined as in the previous subsection, put $\Lambda_1 := \Lambda$ and, for $i = 2, ..., k$, let Λ_i be the transitive closure of the relation 'being parallel inside the same $\{0, 1, ..., i\}$-residue', defined on the set of i-elements of $\mathcal{G}$. Suppose that Λ_i satisfies a condition analogous to (RS). (We leave the precise statement of this condition for the reader.)

Then we can define a geometry $\mathcal{G}^{\infty,k}$ over the set of types $\{1, 2, ..., n\}$, which we call the *geometry at infinity* of $\mathcal{G}$ of *level* k. For $i = 1, 2, ..., k$, the i-elements of $\mathcal{G}^{\infty,k}$ are the classes of Λ_i while, for $i > k$, the i-elements of $\mathcal{G}^{\infty,k}$ are just the same as in $\mathcal{G}$. In particular, the 1-elements of $\mathcal{G}^{\infty,k}$ bijectively correspond to the shrinkings of $\mathcal{G}$. The incidence relation is defined as follows. Two elements X and Y of $\mathcal{G}^{\infty,k}$ of type $i, j \leq k$ are declared to be incident if a member of X is incident in $\mathcal{G}$ with a member of Y. Two elements x and y of $\mathcal{G}^{\infty,k}$ of type $i, j > k$ are incident in $\mathcal{G}^{\infty,k}$ if and only if they are incident as elements of $\mathcal{G}$. An element X of type $i \leq k$ is incident with an element y of type $j > k$ if and only if y is incident in $\mathcal{G}$ with at least one member of X.

If $k = n$ then $\mathcal{G}$ is an affine geometry and $\mathcal{G}^{\infty,k}$ is its projective geometry at infinity. When $k < n$ the $\{1, 2, ..., k\}$-residues of $\mathcal{G}^{\infty,k}$ are projective geometries. If moreover $k < n-1$ then the $\{k+2, ..., n\}$-residues of $\mathcal{G}^{\infty,k}$ are the same as in $\mathcal{G}$ and, for every element x of type $j > k$, $\mathrm{Res}^-_{\mathcal{G}^{\infty,k}}(x)$ is the geometry at infinity of $\mathrm{Res}^-_{\mathcal{G}}(x)$ of level k. (Note that property (RS) as well as its analogues for type $2, 3, ..., k$ are preserved when taking lower residues.)

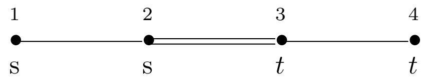
Proposition 8.4 *Let $k < n$ and suppose that no two distinct elements of $\mathcal{G}$ are incident with the same set of n-elements (as when* (IP) *holds, for instance). Then* $\mathrm{Aut}(\mathcal{G}^{\infty,k}) = \mathrm{Aut}(\mathcal{G})$*. Moreover, if* $\mathrm{Aut}(\mathcal{G})$ *acts flag-transitively on $\mathcal{G}$ then* $\mathrm{Aut}(\mathcal{G}^{\infty,k})$ *is also flag-transitive.*

Proof. The latter claim is obvious. The first claim follows from the fact that, since $k < n$, $\mathcal{G}$ and $\mathcal{G}^{\infty,k}$ have the same n-elements and, since no two elements of $\mathcal{G}$ are incident with the same set of n-elements, the action of $\mathrm{Aut}(\mathcal{G})$ on $\mathcal{G}$ is uniquely determined by its action on the set of n-elements. □

Unfortunately, Proposition 8.4 is seldom of great help. Indeed in general $\mathcal{G}^{\infty,k}$ is far more difficult to investigate than $\mathcal{G}$, due to the fact that rather odd geometries can occur as residues of $\mathcal{G}^{\infty,k}$ of type $\{k, k+1\}$ or $\{k+1, k+2\}$. However, in a few lucky cases $\mathcal{G}^{\infty,k}$ turns out to be a well known object. In those lucky cases we can exploit $\mathcal{G}^{\infty,k}$ to understand $\mathcal{G}$.

9 A lemma on flag-transitive F_4-geometries

Let $\mathcal{F}$ be an F_4-geometry with finite orders s, s, t, t, where $s > 1$.



$(1 < s < \infty, \quad 1 \leq t < \infty)$

The next lemma has been exploited in Section 2.

Lemma 9.1 *If $\mathcal{F}$ is flag-transitive then $\mathcal{F}$ is a building, except possibly when $t = 1$ and $s > 2$.*

Proof. Suppose $t > 1$. Then, by Yoshiara [28], either all C_3-residues of $\mathcal{F}$ are buildings or $s = t = 2$ and one of these residues is isomorphic to the flat C_3-geometry for the alternating group A_7. The latter case is impossible by Aschbacher [1]. Hence all C_3-residues of $\mathcal{F}$ are buildings. By Tits [24], $\mathcal{F}$ is covered by a building $\widetilde{\mathcal{F}}$. However $\mathrm{Aut}(\widetilde{\mathcal{F}})$ does not admit any normal subgroup that can define a proper quotient of $\widetilde{\mathcal{F}}$. Hence $\mathcal{F} = \widetilde{\mathcal{F}}$.

Let $t = 1$. Then the $\{1, 2, 3\}$-residues of $\mathcal{F}$ are buildings [19, Theorem 14.13]. On the other hand, $\{2, 3, 4\}$-residues arise from suitable sets of latin squares as explained by Rees [23]. It follows from Rees's description that when $s = 2$ these residues are either buildings or flat quotients of buildings. In this case all C_3-residues of $\mathcal{F}$ are covered by buildings and we obtain the conclusion as in the previous paragraph. $\square$

Remark 9.2 As far as I know, the case of $t = 1$ and $s > 2$ is still open.

10 The graph $\Phi(\mathcal{F})$ of an F_4-building $\mathcal{F}$

Let $\mathcal{F}$ be a building of type F_4 and let $\Gamma(\mathcal{F})$ be its collinearity graph. We recall that $\Gamma(\mathcal{F})$ has diameter equal to 3 and there are two kinds of pairs of points at distance 2 in it, namely symplectic pairs and special pairs. A pair $\{x, y\}$ of points at distance two in $\Gamma(\mathcal{F})$ is said to be *symplectic* if there are at least two points collinear with both x and y. If only one point exists collinear with both x and y then $\{x, y\}$ is a *special* pair. Equivalently, $\{x, y\}$ is symplectic or special according to whether it is contained in a symp or not.

The following graph, which we denote by $\Phi(\mathcal{F})$, has been exploited in Section 3 to define $c.F_4(2, t)$-geometries. The points of $\mathcal{F}$ are the vertices of $\Phi(\mathcal{F})$, two points being adjacent in $\Phi(\mathcal{F})$ when they either are collinear or form a symplectic pair. It is not so difficult to prove that $\Phi(\mathcal{F})$ has diameter equal to 2.

Clearly, the symps of $\mathcal{F}$ are maximal cliques of $\Phi(\mathcal{F})$, but not all maximal cliques of $\Phi(\mathcal{F})$ arise from symps. Nevertheless, when $\mathcal{F}$ admits finite orders then we can recover $\Gamma(\mathcal{F})$ from $\Phi(\mathcal{F})$.

Indeed let $\mathcal{F}$ be an $F_4(s, t)$-building, $s, t < \infty$. For a vertex x of $\Phi(\mathcal{F})$, let $\Phi(x)$ be the set of vertices adjacent to x in $\Phi(\mathcal{F})$, with the convention that $x \notin \Phi(x)$. Let $\{x, y\}$ be an edge of $\Phi(\mathcal{F})$. If x and y are collinear points of $\mathcal{F}$ then

$$|\Phi(x) \cap \Phi(y)| = s^2(t^2 + t + 1)(1 + st + s^2 t) + s - 1.$$

If $\{x, y\}$ is a symplectic pair then

$$|\Phi(x) \cap \Phi(y)| = s^2t^2(st+1)(t+1) + (s^2t+1)(s^2+s+1) - 2.$$

The previous two numbers are never equal. Hence $\Gamma(\mathcal{F})$ can be recovered from $\Phi(\mathcal{F})$. As the whole of $\mathcal{F}$ can be recovered from $\Gamma(\mathcal{F})$, we can also recover $\mathcal{F}$ from $\Phi(\mathcal{F})$.

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