ON THE SURJECTIVITY OF CERTAIN MAPS

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Abstract. We prove in this article the surjectivity of three maps. We prove in Theorem 1 the surjectivity of the chinese remainder reduction map associated to projective space of an ideal with a given factorization into ideals whose radicals are pairwise distinct maximal ideals. In Theorem 2 we prove the surjectivity of the reduction map of the strong approximation type for a ring quotiented by an ideal which satisfies unital set condition. In Theorem 3 we prove for a dedekind domain, for $k \geq 2$, the map from k-dimensional special linear group to the product of projective spaces of $k-$mutually comaximal ideals associating the $k-$rows or $k-$columns is surjective. Finally this article leads to three interesting questions 1, 2, 3 mentioned in the introduction section.

1. Introduction

For any commutative ring R with unity and an ideal $\mathcal{I} = \underset{\alpha}{\cap}\mathcal{Q}_\alpha$ with $rad(\mathcal{Q}_\alpha) = \mathcal{M}_\alpha$ a maximal ideal which are pairwise distinct i.e. $\mathcal{M}_\alpha \neq \mathcal{M}_\beta$ for $\alpha \neq \beta$ we associate a $k-$dimensional projective space for any positive integer $k > 0$.

Here in this article we prove the following three main results. The first main result concerns the surjectivity of the chinese remainder reduction map associated to a projective space of an ideal with a given comaximal ideal factorization which is stated as:

Theorem 1. *Let R be a commutative ring with unity. Let $\mathcal{I} = \mathcal{Q}_1\mathcal{Q}_2\ldots\mathcal{Q}_k$ where $rad(\mathcal{Q}_k) = \mathcal{M}_k$ are pairwise distinct maximal ideals in R. Then the Chinese Remainder Reduction Map associated to the Projective Space*

$$\mathbb{PF}^{l+1}_{\mathcal{I}} \longrightarrow \mathbb{PF}^{l+1}_{\mathcal{Q}_1} \times \mathbb{PF}^{1+1}_{\mathcal{Q}_2} \times \ldots \times \mathbb{PF}^{1+1}_{\mathcal{Q}_k}$$

is surjective.

We also give a counter example in Section 6.3 where the surjectivity does not hold in the case of projective spaces associated to a product of two prime ideals each of which cannot be expressed as a finite intersection of ideals whose radicals are pairwise distinct maximal ideals.

The second main result is a result of strong approximation type. Here we give a criterion called the Unital Set Condition which is given in Definition 7 and prove the following surjectivity theorem which is stated as:

Theorem 2. *Let R be a commutative ring with unity. Let $\mathcal{I} \subset R$ be an ideal which satisfies the Unital Set Condition 7. Then the reduction map*

$$SL_k(R) \longrightarrow SL_k(\frac{R}{\mathcal{I}})$$

is surjective.

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A survey of results on Strong Approximation can be found in [1]. The third main result concerns the surjectivity of another map from the group $SL_k(R)$ to a product of $k-$projective spaces associated to $k-$pairwise comaximal ideals. Before we state the main theorem we need a definition.

Definition 1. *Let R be a commutative ring with unity. Suppose the ring R satisfies the following four properties.*

- *(Property 1): For each maximal ideal $\mathcal{M}$ we have $\mathcal{M}^i \neq \mathcal{M}^{i+1}$ for all $i \geq 0$.*
- *(Property 2): $\bigcap\limits_{n\geq 0} \mathcal{M}^i = (0)$.*
- *(Property 3): $dim_{\frac{R}{\mathcal{M}}}(\frac{\mathcal{M}^i}{\mathcal{M}^{i+1}}) = 1$.*

The examples of such rings are Integers, Principal Ideal Domains, Discrete Valuations Rings, Dedekind domains (which also includes their localizations at any multiplicatively closed set). One can actually show that a ring R satisifies these properties if and only if it is a dedekind domain. So as a consequence the ring R also satisifes the following property

- *(Property 4): Every non-zero element $r \in R$ is contained in finitely many maximal ideals.*

The theorem is stated as:

Theorem 3. *Let R be a commutative ring with unity. Suppose R is a Dedekind domain (refer Definition 1). Let $M(R)$ be the monoid generated by maximal ideals in R. Let $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_k \in M(R)$ be $k-$ pairwise co-maximal ideals. Let $k \geq 2$ be a positive integer. Consider*

$$SL_k(R) = \{A = [a_{ij}]_{k\times k} \in M_{k\times k}(R) \mid det(A) = 1\}.$$

Then the maps

$$\sigma_1, \sigma_2 : SL_k(R) \longrightarrow \mathbb{PF}^{k-1}_{\mathcal{I}_1} \times \mathbb{PF}^{k-1}_{\mathcal{I}_2} \times \ldots \times \mathbb{PF}^{k-1}_{\mathcal{I}_k}$$

given by

$$\sigma_1 : (A) = ([a_{11} : a_{12} : \ldots : a_{1k}], [a_{21} : a_{22} : \ldots : a_{2k}], \ldots, [a_{k1} : a_{k2} : \ldots : a_{kk}]),$$
$$\sigma_2 : (A) = ([a_{11} : a_{21} : \ldots : a_{k1}], [a_{12} : a_{22} : \ldots : a_{k2}], \ldots, [a_{1k} : a_{2k} : \ldots : a_{kk}])$$

are surjective.

Then as a consequence of this Theorem 3 we prove in Theorem 16 another surjectivity theorem where we consider rectangular matrices with entries in a ring R with highest dimensional minors forming a unital set.

This article leads to the following three open questions.

Question 1. *Let R be a commutative ring with unity. Suppose R is a dedekind domain (refer Definition 1). Let $M(R)$ be the monoid generated by maximal ideals in R. Let $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_k \in M(R)$ be $k-$ pairwise co-maximal ideals. Let $k \geq 2$ be a positive integer. Let $G_k(R) \subset SL_k(R)$ be a subgroup. Under what conditions on $G_k(R)$ are the maps*

$$\sigma_1, \sigma_2 : G_k(R) \longrightarrow \mathbb{PF}^{k-1}_{\mathcal{I}_1} \times \mathbb{PF}^{k-1}_{\mathcal{I}_2} \times \ldots \times \mathbb{PF}^{k-1}_{\mathcal{I}_k}$$

given by

$$\sigma_1 : (A) = ([a_{11} : a_{12} : \ldots : a_{1k}], [a_{21} : a_{22} : \ldots : a_{2k}], \ldots, [a_{k1} : a_{k2} : \ldots : a_{kk}]),$$
$$\sigma_2 : (A) = ([a_{11} : a_{21} : \ldots : a_{k1}], [a_{12} : a_{22} : \ldots : a_{k2}], \ldots, [a_{1k} : a_{2k} : \ldots : a_{kk}])$$

surjective?

The second question is concerning surjectivity of the map where the equation is different from the defining equation of $SL_k(R) \subset M_{k\times k}(R)$. Before stating the following open question we mention that we prove another Surjectivity Theorem 17 for the Sum-Product equation in Section 12. Now we state the question concerning general varieties in a slightly general context:

Question 2. *Let R be a commutative ring with unity. Suppose R is a Dedekind Domain (refer Definition 1). Let $M(R)$ be the monoid generated by maximal ideals in R. Let $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_k \in M(R)$ be $k-$ pairwise co-maximal ideals. Let $k \geq 2$ be a positive integer. Let $M_{k\times k}(R)$ be the set of $k \times k$ matrices with entries in R. Let $f : M_{k\times k}(R) \longrightarrow R$ be a polynomial function in the entries. Suppose $f(g = [g_{ij}]_{k\times k}) = 0$ implies each row of g is unital. Let $V_f(R) = \{x = [x_{ij}] \in M_{k\times k}(R) \mid$ such that $f(x_{11}, x_{12}, \ldots, x_{kk}) = 0\}$. For what equations $f = 0$ is the map*

$$\sigma_1 : V_f(R) \longrightarrow \mathbb{PF}^{k-1}_{\mathcal{I}_1} \times \mathbb{PF}^{k-1}_{\mathcal{I}_2} \times \ldots \times \mathbb{PF}^{k-1}_{\mathcal{I}_k}$$

given by

$$\sigma_1 : (A) = ([a_{11} : a_{12} : \ldots : a_{1k}], [a_{21} : a_{22} : \ldots : a_{2k}], \ldots, [a_{k1} : a_{k2} : \ldots : a_{kk}])$$

surjective?

The third question is the following.

Question 3. *(Open Question:) Classify geometically defined spaces which are actually full Projective Spaces associated to an ideal in a ring.*

Here we remark on the Projective Space associated to the ideal as an application of Chinese Remainder Reduction Isomorphism.

Remark 1. *This remark concerns the question as to what spaces can be considered as projective spaces associated to ideals. The following are some examples.*

- *Let $\mathbb{K}$ be an algebraically closed field. Then we know via segre embedding the space is $(\mathbb{PF}^k_{\mathbb{C}})^n = \mathbb{PF}^k_{\mathbb{C}} \times \mathbb{PF}^k_{\mathbb{C}} \times \ldots \times \mathbb{PF}^k_{\mathbb{C}}$ is a projective algebraic variety in a suitable high dimensional projective space. However it is also a projective space associated to an ideal. Suppose if R is a commutative ring with unity and $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ are ideals all whose quotients are isomorphic to $\mathbb{C}$ then $(\mathbb{PF}^k_{\mathbb{C}})^n = \mathbb{PF}^k_{\mathcal{I}}$ where $\mathcal{I} = \prod\limits_{i=1}^{n} \mathcal{M}_i$ via CR-Reduction isomorphism.*
- *The fields need not be the same as in the above case. If $\mathbb{K}_1, \mathbb{K}_2, \ldots, \mathbb{K}_r$ are $r-$fields and if $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_r$ are pairwise comaximal ideals in R with $\frac{R}{\mathcal{M}_i} = \mathbb{K}_i$ then $\prod\limits_{i=1}^{r} \mathbb{PF}^k_{\mathbb{K}_i} \cong \mathbb{PF}^k_{\mathcal{J}}$ where $\mathcal{J} = \prod\limits_{i=1}^{r} \mathcal{M}_i$ via CR-reduction isomorphism. For example*

  $$\mathbb{PF}^2_{\mathbb{R}} \times \mathbb{PF}^2_{\mathbb{C}} \cong \mathbb{PF}^2_{(x(x^2+1))}$$

  where $R = \mathbb{R}[x], \mathcal{M}_1 = (x), \mathcal{M}_2 = (x^2 + 1)$.

2. A Fundamental Lemma on Arithmetic Progressions

In this section we prove the Fundamental Lemma on Arithmetic Progressions for Integers, Dedekind Domains and Schemes.

2.1. **Fundamental Lemma on Arithmetic progressions for Integers.**

Theorem 4 (A Fundamental Lemma on Arithmetic Progressions for Integers)**.** *Let $a, b \in \mathbb{Z}$ be integers with $(a) + (b) = 1$. Consider the set $\{a + nb \mid n \in \mathbb{Z}\}$. Let $m \in \mathbb{Z}$ be any non-zero integer. Then there exists an $n_0 \in \mathbb{Z}$ and an element of the form $a + n_0 b$ such that $gcd(a + n_0 b, m) = 1$.*

Proof. Assume a, b are both non-zero. Otherwise the Theorem 4 is trivial. Let $q_1, q_2, q_3, \ldots, q_t$ are the distinct prime factors of m. Suppose $q \mid gcd(m, b)$ then $q \nmid a+nb$ for all $n \in \mathbb{Z}$. Such prime factors q need not be considered. Let $q \mid m, q \nmid b$ then there exists $t_q \in \mathbb{Z}$ such that the exact set of elements in the given arithmetic progression divisible by q is given by

$$\ldots, a + (t_q - 2q)b, a + (t_q - q)b, a + t_q b, a + (t_q + q)b, a + (t_q + 2q)b \ldots$$

Since there are finitely many such prime factors for m which do not divide b we get a set of congruence conditions for the multiples of b as $n \equiv t_q \ mod\ q$. In order to get an n_0 we solve a different set of congruence conditions for each such prime factor say for example $n \equiv t_q + 1 \ mod\ q$. By Chinese Remainder Theorem we have such solutions n_0 for n which therefore satisfy $gcd(a + n_0 b, m) = 1$. □

2.2. **Fundamental Lemma on Arithmetic progressions for Dedekind Domains.**

Theorem 5 (A Fundamental Lemma on Arithmetic Progressions for Dedekind Domains)**.** *Let $\mathcal{O}$ be a dedekind domain. Let $a, b \in \mathcal{O}$ such that sum of the ideals $(a) + (b) = \mathcal{O}$. Consider the set $\mathcal{A} = \{a+nb \mid n \in \mathcal{O}\}$. Let $\mathcal{M} \subset \mathcal{O}$ be any nonzero ideal. Then there exists an $n_0 \in \mathcal{O}$ and an element $a+n_0 b \in \mathcal{A}$ such that the sum of the ideals $(a+n_0 b)+\mathcal{M} = \mathcal{O}$.*

Proof. Assume a, b are both non-zero as otherwise the Theorem 5 is trivial. Let the ideal $\mathcal{M} = \mathcal{Q}_1^{r_1}\mathcal{Q}_2^{r_2}\ldots\mathcal{Q}_t^{r_t}$ be the unique factorization into prime ideals. Suppose $\mathcal{Q} \in \{\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_t\}$ and $\mathcal{Q} \supset \mathcal{M} + (b)$ then $a + nb \notin \mathcal{Q}$ for all $n \in \mathcal{O}$ because otherwise both $a, b \in \mathcal{Q}$ which is a contradiction. Such prime ideals $\mathcal{Q}$ need not be considered.

Let $\mathcal{M} \subset \mathcal{Q}$ and $b \notin \mathcal{Q}$ then there exists $t_\mathcal{Q} \in \mathcal{O}$ such that

$$\{t \mid a + tb \in \mathcal{Q}\} = t_\mathcal{Q} + \mathcal{Q}$$

an arithmetic progression. This can be proved as follows. First of all since $b \notin \mathcal{Q}$ we have $(b) + \mathcal{Q} = \mathcal{O}$. So there exists $t_\mathcal{Q}$ such that $a + t_\mathcal{Q} b \in \mathcal{Q}$. If $a + tb \in \mathcal{Q}$ then $(t - t_\mathcal{Q})b \in \mathcal{Q}$. So $t \in t_\mathcal{Q} + \mathcal{Q}$.

Since there are finitely many such prime ideals $\mathcal{Q}$ in the factorization of $\mathcal{M}$ such that $b \notin \mathcal{Q}$ we get a set of congruence conditions for the multiples of b as $n \equiv t_\mathcal{Q} \mod \mathcal{Q}$. In order to get an n_0 we solve a different set of congruence conditions for each such prime ideal factors say for example $n \equiv t_\mathcal{Q} + 1 \mod \mathcal{Q}$. By Chinese Remainder Theorem we have such solutions n_0 for n which therefore satisfy $a + n_0 b \notin \mathcal{Q}$ for all primes ideal factors $\mathcal{Q} \in \{\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_t\}$ and hence the sum of the ideals $(a + n_0 b) + \mathcal{M} = \mathcal{O}$.

This proves the Fundamental Lemma 5 on Arithmetic Progressions. □

2.3. **Fundamental Lemma on Arithmetic progressions for Schemes.**

Theorem 6. *Let X be a scheme. Let $Y \subset X$ be an affine subscheme. Let $f, g \in \mathcal{O}(Y)$ be two regular functions on Y such that the unit regular function $\mathbb{1}_Y \in (f, g) \subset \mathcal{O}(Y)$. Let $E \subset Y$ be any finite set of closed points. Then there exists a regular function $a \in \mathcal{O}(Y)$ such that $f + ag$ is a non-zero element in the residue field $k(\mathcal{M}) = \frac{\mathcal{O}(Y)_\mathcal{M}}{\mathcal{M}_\mathcal{M}} = \frac{\mathcal{O}(Y)}{\mathcal{M}}$ at every $\mathcal{M} \in E$.*

Proof. Let the set of closed points be given by $E = \{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_t\}$. If g vanishes in the residue field at $\mathcal{M}_i$ then for all regular functions $a \in \mathcal{O}(Y), f + ag$ does not vanish in the residue field at $\mathcal{M}_i$. Otherwise both $f, g \in \mathcal{M}_i$ which is a contradiction to $\mathbb{1}_Y \in (f, g)$.

Now consider the finitely many maximal ideals $\mathcal{M} \in E$ such that $g \notin \mathcal{M}$. Then there exists $t_{\mathcal{M}}$ such that the set

$$\{t \mid f + tg \in \mathcal{M}\} = t_{\mathcal{M}} + \mathcal{M}$$

a complete arithmetic progression. This can be proved as follows. First of all since $g \notin \mathcal{M}$ we have $(g) + \mathcal{M} = (\mathbb{1}_Y)$. So there exists $t_{\mathcal{M}}$ such that $f + t_{\mathcal{M}} g \in \mathcal{M}$. Now if $f + tg \in \mathcal{M}$ then $(t - t_{\mathcal{M}})g \in \mathcal{M}$. Hence $t \in t_{\mathcal{M}} + \mathcal{M}$.

Since there are finitely such maximal ideals $\mathcal{M}$ such that $g \notin \mathcal{M}$ in the set E we get a finite set of congruence conditions for the multiples a of g as $a \equiv t_{\mathcal{M}} \mod \mathcal{M}$. In order to get an a_0 we solve a different set of congruence conditions for each such maximal ideal in E say for example $a \equiv t_{\mathcal{M}} + 1 \mod \mathcal{M}$. By Chinese Remainder Theorem we have such solutions a_0 for a which therefore satisfy $f + a_0 g \notin \mathcal{M}$ for all maximal ideals $\mathcal{M} \in E$ and hence the regular function $f + n_0 g$ does not vanish in the residue field $k(\mathcal{M})$ for every $\mathcal{M} \in E$. This proves the Theorem 6. □

3. A Theorem on Ideal Avoidance

In this section first we prove below the Order Prescription Lemma 1 before stating the Theorem 7 on Ideal Avoidance.

Lemma 1 (Order Prescription Lemma)**.** *Let R be a commutative ring with unity. Let $\{\mathcal{M}_i : 1 \leq i \leq t\}$ be a finite set of maximal ideals. For each $1 \leq i \leq t$ let $\mathcal{M}_i^{m_i} \supset \mathcal{I}$ but $\mathcal{M}_i^{m_i+1} \not\supseteq \mathcal{I}$ then there exists a function $f \in \mathcal{I}$ such that $f \in \mathcal{I} \backslash \bigcup_{i=1}^{t} \mathcal{I}\mathcal{M}_i$. In particular*

$$f \in \mathcal{M}_i^{m_i} \backslash \mathcal{M}_i^{m_i+1} \cap \mathcal{I} \text{ for } 1 \leq i \leq t.$$

Proof. Let $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_r$ be the finite set of maximal ideals for which $m_i = 0$ and let $\mathcal{M}_{r+1}, \mathcal{M}_{r+2}, \ldots, \mathcal{M}_t$ be the remaining ideals for which $m_i > 0$. So first we observe that for $1 \leq j \leq r, \mathcal{M}_j$ does not contain $\mathcal{I}\left(\prod_{i=1, i \neq j}^{t} \mathcal{M}_i\right)$. So there exists $g_j \in \mathcal{I}\left(\prod_{i=1, i \neq j}^{t} \mathcal{M}_i\right)$ with $g_j \notin \mathcal{M}_j$. Then $g = \sum_{i=1}^{r} g_i \in \mathcal{I}, g \notin \mathcal{M}_j$ for $j = 1, 2, \ldots, r$. Let $f_i \in \mathcal{I} \backslash \mathcal{M}_i^{m_i+1}$ for $i \geq (r+1)$. Let $f_{ij} \in \mathcal{M}_j \backslash \mathcal{M}_i$. Then we observe that

$$f = g + \sum_{i > r, g \in \mathcal{M}_i^{m_i+1}} \left(f_i \prod_{j \neq i} f_{ij}^{m_j+1}\right) \in \left(\mathcal{I} \bigcap_{i=1}^{t} (\mathcal{M}_i^{m_i} \backslash \mathcal{M}_i^{m_i+1})\right) \backslash \left(\bigcup_{i=1}^{t} \mathcal{I}\mathcal{M}_i\right)$$

Taking this f, the Lemma 1 follows. □

Theorem 7 (A Theorem on Ideal Avoidance)**.**

Let R be a commutative ring with unity. Suppose for every maximal ideal $\mathcal{M}$, $\bigcap_{i=1}^{\infty} \mathcal{M}^i = (0)$. Let $\mathcal{I} \subset R$ be an ideal. Let $\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_r \subset R$ be r proper ideals (not the ring itself) such that

$$\mathcal{I} = \bigcup_{i=1}^{r} \mathcal{I}\mathcal{J}_i.$$

Then $\mathcal{I} = (0)$.

Proof. Replace the set of ideals $\{\mathcal{J}_i : 1 \leq i \leq r\}$ by a finite set of maximal ideals $\{\mathcal{M}_i : 1 \leq i \leq s\}$ such that each maximal ideal $\mathcal{M}_i$ contains some ideal $\mathcal{J}_j$ for some j and for any ideal $\mathcal{J}_i$ there exists a maximal ideal $\mathcal{M}_j$ such that $\mathcal{M}_j \supset \mathcal{J}_i$. Then we have

$$\mathcal{I} = \bigcup_{i=1}^{s} \mathcal{I}\mathcal{M}_i$$

Before applying Order Prescription Lemma 1 for the ideal $\mathcal{I}$, if it is non-zero, we observe that a suitable choice of m_i for $\mathcal{M}_i$ exist because of the hypothesis about intersection property. So $\mathcal{I} = (0)$. This proves the Theorem 7. $\square$

4. **The Unital Lemma**

In this section we prove Unital Lemma which is useful to obtain a unit in a $k-$row unital vector via an $SL_k(\mathbb{Z})-$Elementary Transformation.

Theorem 8. *Let R be a commutative ring with unity. Let $k \geq 2$ be a positive integer. Let $\{a_1, a_2, \ldots, a_k\} \subset R$ be a unital set i.e. $\sum_{i=1}^{k}(a_i) = \mathcal{O}$. Let $\mathcal{J} \subset R$ be an ideal contained in only finitely many maximal ideals. Then there exist $A \in (a_2, \ldots, a_k)$ such that $a_1 + A$ is a unit mod $\mathcal{J}$.*

Proof. Let $\{\mathcal{M}_i : 1 \leq i \leq t\}$ be the finite set of maximal ideals containing in $\mathcal{J}$. For example $\mathcal{J}$ could be a product of maximal ideals. Since the set $\{a_1, a_2, \ldots, a_k\}$ is unital there exists $d \in (a_2, a_3, \ldots, a_k)$ such that $(a_1) + (d) = (1)$. Now we apply the Fundamental Lemma on Arithmetic Progressions for Schemes 6 where $X = Y = Spec(R)$, $E = \{\mathcal{M}_i : 1 \leq i \leq t\}$ to conclude that there exists $n_0 \in R$ such that $A = n_0 d$ and $a_1 + A = a_1 + n_0 d \notin \mathcal{M}_i$ for $1 \leq i \leq t$. This proves the Theorem 8. $\square$

Lemma 2. *Let R be a commutative ring with unity. Let $k \geq 2$ be a positive integer. Let $\{a_1, a_2, \ldots, a_k\} \subset R$ be a unital set i.e. $\sum_{i=1}^{k}(a_i) = \mathcal{O}$. Let E be a finite set of maximal ideals in R. Then there exist $A \in (a_2, \ldots, a_k)$ such that $a_1 + A \notin \mathcal{M}$ for all $\mathcal{M} \in E$.*

Proof. The proof is essentially similar to the previous Theorem 8. $\square$

5. **Projective Spaces over Arbitrary Commutative Rings with Identity**

In this section we define projective spaces associated to certain classes

$$\mathcal{I}\mathcal{I}(R)^*, \mathcal{RAD}(R)^*, \mathcal{RADINF}(R)^*$$

of ideals over arbitrary commutative rings with unity.

Definition 2. *Let R be a commutative ring with identity. Let us define the set of non-zero ideal integers denoted by*

$$\mathcal{I}\mathcal{I}(R)^* = \{\mathcal{I} \subset \mathcal{R} \mid \mathcal{I} \textit{ is a product of its maximal ideals }\}.$$

and $\mathcal{I}\mathcal{I}(R) = \mathcal{I}\mathcal{I}(R)^* \cup \{(0)\}$.

Definition 3. *Let R be a commutative ring with identity. Let us define the set of non-zero ideals denoted by*

$$\mathcal{RAD}(R)^* = \{\mathcal{I} \subset \mathcal{R} \mid \mathcal{I} \textit{ is a product of its ideals whose radicals are distinct maximal ideals }\}.$$

and $\mathcal{RAD}(R) = \mathcal{RAD}(R)^* \cup \{(0)\}$. Clearly $\mathcal{RAD}(R) \supset \mathcal{I}\mathcal{I}(R)$.

Definition 4. *Let R be a commutative ring with identity. Let us define the set of non-zero ideals denoted by*

$\mathcal{RADINF}(R)^* = \{\mathcal{I} \subset \mathcal{R} \mid \mathcal{I}$ *is an arbitrary intersection of its ideals whose radicals are all distinct maximal ideals* $\}$.

and $\mathcal{RADINF}(R) = \mathcal{RADINF}(R)^* \cup \{(0)\}$. *Clearly* $\mathcal{RADINF}(R) \supset \mathcal{RAD}(R) \supset \mathcal{II}(R)$.

Definition 5. *Let R be a commutative ring with identity. Let $0 \neq \mathcal{I} \subset R$ be an nonzero ideal such that $\mathcal{I} \in \mathcal{RADINF}(R)$. Let $(a_0, a_1, a_2, \ldots, a_k)$, $(b_0, b_1, b_2, \ldots, b_k) \in R^{k+1}$. Suppose each of the sets $\{a_0, a_1, a_2, \ldots, a_k\}$, $\{b_0, b_1, b_2, \ldots, b_k\}$ generate the unit ideal R. We say*

$$(a_0, a_1, a_2, \ldots, a_k) \sim_{GR} (b_0, b_1, b_2, \ldots, b_k)$$

if and only if $a_ib_j - a_jb_i \in \mathcal{I}$ for $0 \leq i < j \leq k$. This relation $\sim_{GR}$ is an equivalence relation (See Lemma 3). The equivalence class of $(a_0, a_1, a_2, \ldots, a_k)$ is denoted by $[a_0 : a_1 : a_2 : \ldots : a_k]$. Define the $k-$dimensional projective space corresponding to $\mathcal{I}$ denoted by

$\mathbb{PF}^k_{\mathcal{I}} = \{[a_0 : a_1 : a_2 : \ldots : a_k] \mid$ *the set* $\{a_0, a_1, a_2, \ldots, a_k\} \subset R$ *generates the unit ideal* $= R\}$.

Note here we can have elements $\{a_0, a_1, a_2, \ldots, a_k\}$ where each a_i is not a unit $\mod \mathcal{I}$.

Lemma 3. *Using the notation in Definition 5, the relation $\sim_{GR}$ is an equivalence relation.*

Proof. The relation is reflexive and symmetric. We need to prove transitivity. Suppose $(a_0, a_1, a_2, \ldots, a_k)$,$(b_0, b_1, b_2, \ldots, b_k)$,$(c_0, c_1, c_2, \ldots, c_k) \in R^{k+1}$ and each of the sets $\{a_0, a_1, a_2, \ldots, a_k\}$,$\{b_0, b_1, b_2, \ldots, b_k\}$,$\{c_0, c_1, c_2, \ldots, c_k\}$ generate the unit ideal R. First consider the case when $\mathcal{I} \in \mathcal{RADINF}(R)$ is an ideal whose radical is a maximal ideal $\mathcal{M}$. Suppose $(a_i : 0 \leq i \leq k) \sim_{GR} (b_i : 0 \leq i \leq k), (a_i : 0 \leq i \leq k) \sim_{GR} (c_i : 0 \leq i \leq k)$. Suppose without loss of generality $a_1 \notin \mathcal{M}$. So a_1 is a unit $\mod \mathcal{I}$. We assume $a_1 = 1$. Now for any $0 \leq i < j \leq k$ we have $b_ic_j = a_1b_ic_j \equiv b_1a_ic_j \equiv b_1c_ia_j = b_1a_jc_i \equiv a_1b_jc_i = b_jc_i \mod \mathcal{I}$. Hence the transitivity follows for $\mathcal{I}$. Since every ideal $\mathcal{I} \in \mathcal{RADINF}(R)$ is an intersection of ideals with distinct radical maximal ideals, the Lemma 3 follows for any nonzero ideal $\mathcal{I} \in \mathcal{RADINF}(R)$. □

6. On Surjectivity of the Chinese Remainder Reduction Map

6.1. $SL_{k+1}-$Invariance of the Image of the Chinese Remainder Reduction Map.

Definition 6 ($SL_{k+1}-$action)**.** *Let R be a commutative ring with unity. Let*

$$\mathcal{I} \in \mathcal{RADINF}(R)^*.$$

There is a well defined left action of $SL_{k+1}(R)$ as follows. Let $g \in SL_{k+1}(R)$. Define

$$L_g = r_{g^{-1}} : \mathbb{PF}^k_{\mathcal{I}} \longrightarrow \mathbb{PF}^k_{\mathcal{I}}$$

given by $L_g([a_0 : a_1 : a_2 \ldots : a_k]) = g \bullet ([a_0 : a_1 : a_2 \ldots : a_k]) = r_{g^{-1}}([a_0 : a_1 : a_2 \ldots : a_k]) = [b_0 : b_1 : b_2 : \ldots : b_k]$ where

$$(b_0, b_1, b_2, \ldots, b_k) = (a_0, a_1, a_2, \ldots, a_k)g^{-1}.$$

This action can be extended to a product of such projective spaces.

Lemma 4 ($SL_{k+1}-$Invariance of the Image)**.** *Let R be a commutative ring with unity. Let $\mathcal{I}_i \in \mathcal{RADINF}(R)^* : 1 \leq i \leq n$ be finitely many pairwise co-maximal ideals in R. Let*

$$\mathcal{I} = \prod_{i=1}^{n} \mathcal{I}_i.$$

The image of the chinese remainder reduction map is a union of $SL_{k+1}-$orbits.

Proof. If

$$\sigma : \mathbb{PF}^k_{\mathcal{I}} \longrightarrow \prod_{i=1}^{n} \mathbb{PF}^k_{\mathcal{I}_i}$$

then the chinese remainder reduction map σ is always $SL_{k+1}-$invariant in the sense that for any $g \in SL_{k+1}(R)$ we have

$$g \bullet \sigma([a_0 : a_1 : a_2 \ldots : a_k]) = \sigma(g \bullet [a_0 : a_1 : a_2 \ldots : a_k]).$$

Hence this theorem follows. $\square$

Note 1. *Let* $\tilde{S}L_{k+1}(R) = \{A \in M_{k+1}(R) \mid det(A) = \pm 1\}$. *We can similarly conclude like in Lemma 4 that the image of the Chinese Remainder Reduction Map is* $\tilde{S}L_{k+1}(R)-$ *invariant and it is a union of* $\tilde{S}L_{k+1}(R)-$*orbits.*

6.2. **Surjectivity of the Chinese Remainder Reduction Map.** Here in this section we prove the first main Theorem 1 of this article.

Proof. The theorem holds for $k = 1$ and any $l > 0$ as there is nothing to prove. Now we prove by induction on k. Let

$$([a_{10}, a_{11}, \ldots, a_{1l}], \ldots, [a_{k0}, a_{k1}, \ldots, a_{kl}]) \in \mathbb{PF}^{l+1}_{\mathcal{Q}_1} \times \mathbb{PF}^{1+1}_{\mathcal{Q}_2} \times \ldots \times \mathbb{PF}^{1+1}_{\mathcal{Q}_k}$$

By induction we have an element $[b_0 : b_1 : b_2 : \ldots : b_l] \in \mathbb{PF}^{l+1}_{\mathcal{Q}_2\mathcal{Q}_3\ldots\mathcal{Q}_k}$ representing the last $k-1$ elements. Now consider the matrix

$$A = \begin{pmatrix} \mathcal{Q}_1 \longrightarrow & a_0 & a_1 & \cdots & a_{l-1} & a_l \\ \mathcal{Q}_2 \ldots \mathcal{Q}_k \longrightarrow & b_0 & b_1 & \cdots & b_{l-1} & b_l \end{pmatrix}$$

Now one of the elements in the first row is not in $\mathcal{M}_1$. By finding inverse of this element modulo $\mathcal{Q}_1$ and hence by a suitable application of $\tilde{S}L_{l+1}(R)$ matrix the matrix A can be transformed to the following matrix B where we replace the unique non-zero entry in the first row by 1.

$$B = \begin{pmatrix} \mathcal{Q}_1 \longrightarrow & 1 & 0 & \cdots & 0 & 0 \\ \mathcal{Q}_2 \ldots \mathcal{Q}_k \longrightarrow & c_0 & c_1 & \cdots & c_{l-1} & c_l \end{pmatrix}$$

If c_0 is a unit $\mod \mathcal{Q}_2 \ldots \mathcal{Q}_k$ then we are done as this reduces to ordinary chinese remainder theorem. Otherwise suppose

$$c_0 \in \mathcal{M}_2\mathcal{M}_3 \ldots \mathcal{M}_r \backslash \mathcal{M}_{r+1}\mathcal{M}_{r+2} \ldots \mathcal{M}_k.$$

Let $\sum_{i=0}^{l} c_i x_i = 1$. Now consider any element $a \in \mathcal{M}_{r+1} \ldots \mathcal{M}_k \backslash (\mathcal{M}_2 \ldots \mathcal{M}_r) \neq \emptyset$. Then the matrix

$$C = \begin{pmatrix} \mathcal{Q}_1 \longrightarrow & 1 & 0 & \cdots & 0 & 0 \\ \mathcal{Q}_2 \ldots \mathcal{Q}_k \longrightarrow & c_0 + \sum_{i=1}^{l} a c_i x_i = a + c_0(1 - a x_0) & c_1 & \cdots & c_{l-1} & c_l \end{pmatrix}$$

is obtained from B by $\tilde{S}L_{l+1}(R)-$ matrix. Now the element

$$a + c_0(1 - a x_0) \notin \mathcal{M}_2 \cup \ldots \cup \mathcal{M}_k.$$

Let $u \in R$ be such that $u(a + c_0(1 - a x_0)) \equiv 1 \mod \prod_{i=2}^{k} \mathcal{Q}_i$. Then the matrix C represents the same elements as the matrix D.

$$D = \begin{pmatrix} \mathcal{Q}_1 \longrightarrow & 1 & 0 & \cdots & 0 & 0 \\ \mathcal{Q}_2 \ldots \mathcal{Q}_k \longrightarrow & 1 & uc_1 & \cdots & uc_{l-1} & uc_l \end{pmatrix}$$

The elements in the matrix C is in the image of CR-reduction map by the usual Chinese Remainder Theorem.

Hence the induction step is completed and the Theorem 1 follows. □

6.3. **A Counter Example where Surjectivity need not hold.**

Example 1 (Construction of a Counter Example for Surjectivity in One Dimension)**.** *Let $R = \mathbb{K}[x,y]$ where $\mathbb{K}$ is a field. Consider the prime ideals $\mathcal{P}_1 = (x-1)$, $\mathcal{P}_2 = (y-1)$. We note that these are not finite intersection of ideals whose radicals are maximal ideals because there are infinitely many maximal ideals containing each of these prime ideals. However here we observe that $\mathcal{P}_1\mathcal{P}_2 = \mathcal{P}_1 \cap \mathcal{P}_2$ by unique factorization domain property and the projective spaces $\mathbb{PF}^1_{\mathcal{P}_1}, \mathbb{PF}^1_{\mathcal{P}_2}, \mathbb{PF}^1_{\mathcal{P}_1\mathcal{P}_2}$ makes sense as the relation*

$$\sim_{\mathcal{P}_1}, \sim_{\mathcal{P}_2}, \sim_{\mathcal{P}_1\mathcal{P}_2}$$

are all also equivalence relations. Here let $a,b,c,d \in R$ be such that each of the pairs $(a,b),(c,d)$ generate a unit ideal. We say $(a,b) \sim_{\mathcal{I}} (c,d)$ if and only if $ad - bc \in \mathcal{I}$ where $\mathcal{I} = \mathcal{P}_1$ or $\mathcal{P}_2$ or $\mathcal{P}_1\mathcal{P}_2$.
Now consider the Chinese Remainder Reduction map

$$\mathbb{PF}^1_{\mathcal{P}_1\mathcal{P}_2} \longrightarrow \mathbb{PF}^1_{\mathcal{P}_1} \times \mathbb{PF}^1_{\mathcal{P}_2}$$

This map is not surjective.
Consider the element $([1:0],[0:1]) \in \mathbb{PF}^1_{\mathcal{P}_1} \times \mathbb{PF}^1_{\mathcal{P}_2}$. If $a,b \in R$ represent this element via congruence conditions then we get

$$a \equiv 1 \mod (x-1), a \equiv 0 \mod (y-1)$$
$$b \equiv 0 \mod (x-1), b \equiv 1 \mod (y-1)$$

So we get $a = (y-1)t$ and $a - 1 = -(x-1)u$. So we get that $(y-1)t + (x-1)u = 1$ which yields a contradiction if we substitute $x = 1, y = 1$. There is no such $''a''$ and similarly there is no such $''b''$ as well. So via congruences we cannot obtain a representing element pair (a,b).
Now let $a,b \in R$ generate a unit ideal such that $[a:b] = [1:0] \in \mathbb{PF}^1_{\mathcal{P}_1}$ and $[a:b] = [0:1] \in \mathbb{PF}^1_{\mathcal{P}_2}$ then $(x-1) \mid b, (y-1) \mid a$. So we have the ideal $(a,b) \subset (x-1, y-1)$ which is impossible.
This proves that the Chinese Remainder Reduction map is not surjective.

7. **Surjectivity of the map $SL_k(R) \longrightarrow SL_k(\frac{R}{\mathcal{I}})$ and the Unital Set Condition with respect to an Ideal**

Question 4. *In this section we answer the question: When is the reduction map*

$$SL_k(R) \longrightarrow SL_k(\frac{R}{\mathcal{I}})$$

surjective?

Definition 7 (Unital Set Condition with respect to an Ideal)**.** *Let R be a commutative ring with unity. Let $\mathcal{I} \subset \mathcal{R}$ be an ideal. We say $\mathcal{I}$ satisfies unital set condition USC if for every unital set $\{a_1, a_2, \ldots, a_k\} \subset R$ with $k \geq 2$, there exists an element $j \in (a_2, \ldots, a_k)$ such that $a_1 + j$ is a unit modulo $\mathcal{I}$.*

Now we prove the second main Theorem 2 of our article.

Proof. For $k = 1$ there is nothing to prove. So assume $k > 1$. Clearly all elementary matrices $E_{ij}(r), r \in R, i \neq j$ are in the image. Now consider a diagonal matrix $diag(d_{11} = d_1, d_{22} = d_2, \ldots, d_{kk} = d_k)$ such that

$$d_1 d_2 \ldots d_k \equiv 1 \mod \mathcal{I}.$$

Let $n = d_1d_2\ldots d_k - 1 \in \mathcal{I}$.
Define a matrix

$$E = \begin{pmatrix} e_{11} & e_{12} & e_{13} & \cdots & e_{1(k-1)} & e_{1k} \\ e_{21} & e_{22} & e_{23} & \cdots & e_{2(k-1)} & e_{2k} \\ e_{31} & e_{32} & e_{33} & \cdots & e_{3(k-1)} & e_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{(k-1)1} & e_{(k-1)2} & e_{(k-1)3} & \cdots & e_{(k-1)(k-1)} & e_{(k-1)k} \\ e_{k1} & e_{k2} & e_{k3} & \cdots & e_{k(k-1)} & e_{kk} \end{pmatrix}$$

with $e_{k1} = nz, e_{12} = e_{23} = e_{34} = \ldots = e_{(k-1)k} = n$ also let

$$e_{ii} = d_i + \alpha_1^i n + \alpha_2^i n^2 + \ldots \alpha_{k-1}^i n^{k-1} \in R[\alpha_j^i : 1 \leq i \leq k, 1 \leq j \leq (k-1)]$$

be a polynomial representing a symbolic respective $n-$adic expansion modulo (n^k). Choose the rest of the entries in the matrix E to be zero. Now this matrix has determinant given by

$$e_{11}e_{22}\ldots e_{kk} - (-1)^k n^k z.$$

The sum of ideals $(e_{11}e_{22}\ldots e_{kk}) + (n^k) = (1)$ in the polynomial ring $R[\alpha_j^i : 1 \leq i \leq k, 1 \leq j \leq (k-1)]$ because $(e_{11}e_{22}\ldots e_{kk}) + (n) = (d_1d_2\ldots d_k) + (n) = (1)$ and using radical of ideals. i.e.

$$\begin{aligned} rad(A + rad(B)) &= rad(rad(A) + B) \\ &= rad(rad(A) + rad(B)) = rad(A + B) \text{ for ideals } A, B \text{ in a Ring} \end{aligned}$$

So there exist $w, \alpha \in R[\alpha_j^i : 1 \leq i \leq k, 1 \leq j \leq (k-1)]$ such that

$$\alpha e_{11}e_{22}\ldots e_{kk} + wn^k = 1.$$

If we choose for the symbols α_j^i elements of R such that

$$e_{11}e_{22}\ldots e_{kk} \equiv 1 \mod n^k$$

then we get $\alpha \equiv 1 \; mod \; n^k$. So we can solve for z so that the determinant

$$e_{11}e_{22}\ldots e_{kk} - (-1)^k n^k z = 1.$$

To solve first consider $k = 2$. If $d_1d_2 = 1 + t_1 n + t_2 n^2 + \ldots + (n^k)$ be its symbolic $n - adic$ expansion then we should have $\alpha_1^1 d_2 + \alpha_1^2 d_1 + t_1 \equiv 0 \mod n$. Such an equation is solvable say for α_1^1 or for α_1^2 as d_1, d_2 are units mod n^r for all r. To obtain a value t_1 we know that $d_1d_2 - 1 = n\tilde{t}_1$ for some $\tilde{t}_1 \in R$. So choose $t_1 = \tilde{t}_1$ and there are no remaining t_i as $k = 2$ here in this case.
For a general k. Let the symbolic $n-$adic expansions be given by

$$\begin{aligned} d_1d_2\ldots d_k &= 1 + t_1 n + t_2 n^2 + \ldots + t_k n^{k-1} + (n^k), \\ d_2d_3\ldots d_k &= s_0 + s_1 n + s_2 n^2 + \ldots + s_{k-1} n^{k-1} + (n^k) \\ e_{11} &= d_1 + \alpha_1 n + \alpha_2 n^2 + \ldots + \alpha_{k-1} n^{k-1} + (n^k). \end{aligned}$$

Fix a section $sec : \frac{R}{(n)} \longrightarrow R$. Recursively pick representative values in the image of sec in R for t_i for $i = 1, \ldots, (k-1)$, and s_i for $i = 0, \ldots, (k-1)$. Let $e_{ii} = d_i$ for all $i \geq 2$ then

$$e_{11}e_{22}\ldots e_{kk} = d_1d_2\ldots d_k + \alpha_1 n d_2 d_3 \ldots d_k + \alpha_2 n^2 d_2 d_3 \ldots d_k + \ldots + (n^k).$$

So we should have $s_0\alpha_1 + t_1 \equiv 0 \; mod \; n$. So solve for α_1 as s_0 is a unit $\mod n$. Now solve for α_2 because $s_0\alpha_2 + \ldots \equiv 0 \; mod \; n$ recursively by carrying the addendums of the previous term $s_0\alpha_1 + t_1$ which are higher powers of n and so on for the rest of the $\alpha_i's$. The α_i gets multiplied by s_0 which is a unit mod n. So solving for α_i is possible.

We have proved that the diagonal determinant one matrices in $SL_k(\frac{R}{\mathcal{I}})$ are in the image of the reduction map $\sigma : SL_k(R) \longrightarrow SL_k(\frac{R}{\mathcal{I}})$ by choosing $n = d_1 d_2 \ldots d_k - 1 \in \mathcal{I}$ for each $diag(d_1, d_2, \ldots, d_k) \in SL_k(\frac{R}{\mathcal{I}})$.
Now we prove the following claim. We note here that $k > 1$.

Claim 1. *All matrices in $SL_k(\frac{R}{\mathcal{I}})$ can be reduced to identity by elementary determinant one matrices and matrices of the form $diag(1, \ldots, u, u^{-1}, \ldots, 1)$ where $u \in \mathcal{U}(\frac{R}{\mathcal{I}})$ a unit if $\mathcal{I}$ satisfies the unital set condition.*

Proof of Claim. To prove this we observe that we can reduce any element to identity using elementary matrices and matrices of the form

$$diag(1, \ldots, u, u^{-1}, \ldots, 1)$$

where $u \in \mathcal{U}(\frac{R}{\mathcal{I}})$ a unit. This reduction can be done because if $(a_1, a_2, \ldots, a_k)$ is a row then there exists an element $i \in \mathcal{I}$ such that $\{a_1, a_2, \ldots, a_k, i\}$ is unital and hence satisifes the unital set condition. So there exists $j \in (a_2, \ldots, a_k, i)$ such that $a_1 + j$ is a unit modulo $\mathcal{I}$. Now the element i can be ignored so that we can bring a unit $\mod I$ in a row by applying only elementary determinant one matrices as column operations. This proves the claim for $SL_k(\frac{R}{\mathcal{I}})$. $\square$

So all matrices are in the image i.e. the reduction map $\sigma : SL_k(R) \longrightarrow SL_k(\frac{R}{\mathcal{I}})$ is onto. This proves the Theorem 2. $\square$

Note 2. *In the proof of the following corollary 1, the Theorem 8 is applied as this can be used to bring a unit modulo the ideal in every row using elementary operations of determinant one.*

Corollary 1. *Let R be a commutative ring with unity. Let $\mathcal{I} \subset R$ be an ideal contained in finitely many maximal ideals. Then the reduction map*

$$SL_k(R) \longrightarrow SL_k(\frac{R}{\mathcal{I}})$$

is onto.

Proof of Corollary. For $k = 1$ there is nothing to prove. For $k > 1$ this corollary follows from the fact that any ideal $\mathcal{I}$ which is contained in finitely many maximal ideals satisfies Unital Set Condition USC using Theorem 8. $\square$

7.1. **A Consequence of Unital Lemma.**

Lemma 5. *Let R be a ring. Let $k > 1$ be a positive integer. Let $(a_1, a_2, \ldots, a_k) \in R^k$ be a vector such that a_i is a unit for some $1 \leq i \leq k$. Then there exists $k-$vectors $\{v_1, v_2, \ldots, v_k\} \subset R^{k-1}$ such that*

$$v_1 \wedge v_2 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_k = a_i$$

Proof. First consider a unital vector $(a_1, a_2, \ldots, a_k)$ with a_1 a unit without loss of generality. Let

$$\begin{aligned} v_1 &= (a_2, -a_1^{-1}a_3, +a_1^{-1}a_4, \ldots, (-1)^i a_1^{-1}a_i, \ldots, (-1)^k a_1^{-1}a_k)^t \\ &= a_2 e_1^{k-1} + \sum_{i=2}^{k-1} (-1)^i a_1^{-1} a_i e_i^{k-1}, v_2 = a_1 e_1^{k-1}, v_3 = e_2^{k-1}, \ldots, v_k = e_{k-1}^{k-1}. \end{aligned}$$

Then we immediately observe that for $1 \leq i \leq k$,

$$v_1 \wedge v_2 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_k = a_i$$

Similarly if any other component a_i is a unit. Hence the lemma follows. $\square$

Lemma 6 (Elementary Row Vector Lemma)**.** *Let $\mathcal{R}$ be a commutative ring with unity. Let $\mathcal{I}$ be an ideal which is contained only in finitely many maximal ideals. Let $k > 1$ be a positive integer. Let $\{a_1, a_2, \ldots, a_k\} \subset R$ be a unital set i.e. $\sum\limits_{i=1}^{k}(a_i) = R$. Then there exists a matrix g in $SL_k(R)$ such that*

$$(a_1, a_2, \ldots, a_k)g \equiv (1, 0, \ldots, 0) \mod \mathcal{I}$$

For $k = 1$ the existence of such a matrix g need not hold.

Proof. We note that if $k = 1$ and a_1 is a unit in R but $a_1 \not\equiv 1 \mod \mathcal{I}$. Then $a_1 g \equiv 1 \mod \mathcal{I}$ does not imply that $g \in SL_1(R)$.

Now assume $k > 1$. Let $(b_1, b_2, \ldots, b_k) \in R^k$ such that $\sum\limits_{i=1}^{k}(-1)^{i-1}a_i b_i = 1$. Now the vector $(b_1, b_2, \ldots, b_k)$ is unital. So from the previous lemma 2 there exists $t_2, t_3, \ldots, t_k \in R$ such that the element $c_1 = b_1 + t_2 b_2 + \ldots + t_k b_k$ is a unit modulo $\mathcal{I}$.
Now consider the vector $(c_1, b_2, \ldots, b_k)$ which has a unit $\mod \mathcal{I}$. Hence using Lemma 5 there exists $k-$vectors $\{v_1, v_2, \ldots, v_k\} \subset R^{k-1}$ such that $v_2 \wedge v_3 \wedge \ldots \wedge v_k \in c_1 + \mathcal{I}$ and

$$v_1 \wedge v_2 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_k \in b_i + \mathcal{I} \text{ if } i > 1$$

Now choose

$$w_1 = v_1, w_2 = v_2 - t_2 v_1, w_3 = v_i + t_3 v_1, \ldots, w_k = v_i + (-1)^{k-1} t_k v_1$$

Then we have for $i \geq 2$

$$w_1 \wedge w_2 \wedge \ldots \wedge \widehat{w_i} \wedge \ldots \wedge w_k \in b_i + \mathcal{I}$$

and $w_2 \wedge w_3 \wedge \ldots \wedge w_k \in b_1 + \mathcal{I}$. So the following matrix has unit determinant modulo $\mathcal{I}$. i.e. treating each w_i is a column $(k-1)-$ vector we have

$$det \begin{pmatrix} a_1 & a_2 & \ldots & a_k \\ w_1 & w_2 & \ldots & w_k \end{pmatrix} \equiv 1 \mod \mathcal{I}$$

So using Theorem 2 there exists a matrix $B \in SL_k(R)$ such that we have

$$B \equiv \begin{pmatrix} a_1 & a_2 & \ldots & a_k \\ w_1 & w_2 & \ldots & w_k \end{pmatrix} \mod \mathcal{I}.$$

We observe that

$$(1, 0, \ldots, 0)B \equiv (a_1, a_2, \ldots, a_k) \mod \mathcal{I}.$$

So we consider $g = B^{-1}$ and this lemma follows. $\square$

8. **Surjectivity Example For a Pair of Maximal Ideals in Arbitrary Commutative Ring With Unity**

Example 2. *Here we describe explicitly the collection of 2×2 determinant one matrices which map onto the product of spaces $\mathbb{PF}^1_{\mathcal{N}} \times \mathbb{PF}^1_{\mathcal{M}}$ for two maximal ideals $\mathcal{N}, \mathcal{M}$ in the ring R.*
Fix any two sections $s_{\mathcal{N}} : \frac{R}{\mathcal{N}} \longrightarrow R$ and $s_{\mathcal{M}} : \frac{R}{\mathcal{M}} \longrightarrow R$ of the quotient maps $\tau_{\mathcal{M}} : R \longrightarrow \frac{R}{\mathcal{M}}, \tau_{\mathcal{N}} : R \longrightarrow \frac{R}{\mathcal{N}}$.
Consider the following set of matrices

$$\mathcal{C}_1 = \left\{ \begin{pmatrix} s & (st-1) \\ 1 & t \end{pmatrix}, s \in image(s_{\mathcal{N}}), t \in image(s_{\mathcal{M}}) \right\}$$

This set of matrices maps into the subset

$$\mathbb{PF}^1_{\mathcal{N}} \times \left(\{[1:t] \in \mathbb{PF}^1_{\mathcal{M}} \mid t \in image(s_{\mathcal{M}})\}\right) \subset \mathbb{PF}^1_{\mathcal{N}} \times \mathbb{PF}^1_{\mathcal{M}}$$

injectively giving rise to distinct elements.

$$\mathcal{C}_1 \hookrightarrow \mathbb{PF}^1_{\mathcal{N}} \times \left(\{[1:t] \in \mathbb{PF}^1_{\mathcal{M}} \mid t \in image(s_{\mathcal{M}})\}\right)$$

There is one more element for each $t \in image(s_{\mathcal{M}})$ with $[1:t]$ as the image corresponding to the second row. It is given as follows. Since $\mathcal{M}, \mathcal{N}$ are comaximal the exists elements $p \in \mathcal{M}$, $q \in \mathcal{N}$ such that the ideals $(p),(q)$ are comaximal i.e. $(p)+(q)=1$. Consider elements $r, q \in R$ such that $rq - kp = 1$ as $(p)+(q)=1$ and for such p,q,r,k, we have that the ideals $(p(1+qr)),(q(1+pk))$ are comaximal. So consider elements l, m such that $lp(1+qr) - mq(1+pk) = 1 - t$ for any given $t \in R$. Now consider 2×2 matrices of determinant 1.

$$\mathcal{C}_2 = \left\{ \begin{pmatrix} (1+rq) & (t+mq) \\ (1+kp) & (t+lp) \end{pmatrix}, t \in image(s_{\mathcal{M}}) \right\}$$

Now the collection $\mathcal{C}_1 \cup \mathcal{C}_2$ maps injectively into the set $\mathbb{PF}^1_{\mathcal{N}} \times \{[1:t] \in \mathbb{PF}^1_{\mathcal{M}} \mid t \in image(s_{\mathcal{M}})\}$. We shall soon observe that this collection actually maps onto this set bijectively. i.e

$$(\mathcal{C}_1 \cup \mathcal{C}_2) \cong \mathbb{PF}^1_{\mathcal{N}} \times \left(\{[1:t] \in \mathbb{PF}^1_{\mathcal{M}} \mid t \in image(s_{\mathcal{M}})\}\right)$$

Now consider the set

$$\mathcal{C}_3 = \left\{ \begin{pmatrix} (1+sp) & s \\ p & 1 \end{pmatrix}, s \in image(s_{\mathcal{N}}) \right\}$$

This set maps injectively into the set $\mathbb{PF}^1_{\mathcal{N}} \times \{[0:1]\}$

$$\mathcal{C}_3 \hookrightarrow \mathbb{PF}^1_{\mathcal{N}} \times \{[0:1]\}$$

We will soon see that the set $\mathcal{C}_3$ misses just one element in the set $\mathbb{PF}^1_{\mathcal{N}} \times \{[0:1]\}$.
Now we describe that one more matrix of determinant one which maps onto the missing element $([p:1],[0:1]) \in \mathbb{PF}^1_{\mathcal{N}} \times \mathbb{PF}^1_{\mathcal{M}}$. Consider elements $x, l \in R$ such that $lq - xp = 1$ as $(p)+(q)=1$. For such integers x,p,l,q we have that the ideals $(p(1+lq)),(q(1+xp))$ are comaximal. So consider elements $y, r \in R$ such that $rq(1+xp) - yp(1+lq) = 1 - p - xp^2$. Then consider 2×2 matrix of determinant 1 given by

$$\begin{pmatrix} (rq+p) & (1+lq) \\ yp & (1+xp) \end{pmatrix}$$

Now we observe that we have a total collection of two by two matrices of determinant one mapping injectively into $\mathbb{PF}^1_{\mathcal{N}} \times \mathbb{PF}^1_{\mathcal{M}}$.
We immediately see that for a fixed $t \in image(s_{\mathcal{M}})$

$$\{[s : st-1] \mid s \in image(s_{\mathcal{N}})\} = \{[1:w] \mid w \in image(s_{\mathcal{N}}), [1:w] \neq [1:t]\} \cup \{[0:1]\}.$$

We also observe that

$$\{[1+sp : s] \mid s \in image(s_{\mathcal{N}})\} = \{[1:w] \mid w \in image(s_{\mathcal{N}}), [1:w] \neq [p:1]\} \cup \{[0:1]\}.$$

Hence the mapping σ_1 is onto and similarly the map σ_2 is also onto. So the intermediate claims of surjectivity of $\mathcal{C}_1 \cup \mathcal{C}_2$ and the set $\mathcal{C}_3$ just missing one element are justified.

9. Unique Factorization Maximal Ideal Monoid of the Ring

In this section we define the Unique Factorization Monoid of maximal ideals of the Ring. We start by proving below a theorem.

Theorem 9 (Unique Factorization Theorem)**.** *Let R be a commutative ring with unity. Suppose for any maximal ideal $\mathcal{M}^i \neq \mathcal{M}^{i+1}$ for all $i \geq 0$. Let $I = \mathcal{M}_1^{t_1}\mathcal{M}_2^{t_2}\ldots\mathcal{M}_k^{t_k} = \mathcal{N}_1^{s_1}\mathcal{N}_2^{s_2}\ldots\mathcal{N}_r^{s_r}$. Then $\{\mathcal{M}_1,\mathcal{M}_2,\ldots,\mathcal{M}_k\} = \{\mathcal{N}_1,\mathcal{N}_2,\ldots,\mathcal{N}_r\}, r = k$ and with a suitable permutation or rearrangement of $\{\mathcal{N}_1,\mathcal{N}_2,\ldots,\mathcal{N}_r\}$ we have $t_i = s_i$ for all $1 \leq i \leq r = k$.*

Proof. If $\mathcal{M} \supset \mathcal{I}$ then $\mathcal{M} = \mathcal{M}_j$ for some $1 \leq j \leq k$. So

$$\{\mathcal{M}_1,\mathcal{M}_2,\ldots,\mathcal{M}_k\} = \{\mathcal{N}_1,\mathcal{N}_2,\ldots,\mathcal{N}_r\}, k = r.$$

Now if the ideal $\mathcal{I}$ is a power of a maximal ideal then the power is uniquely determined because $\mathcal{M}^i \neq \mathcal{M}^{i+1}$ for all $i \geq 0$ and all maximal ideals $\mathcal{M} \subset R$.

Claim 2. *If $\mathcal{M}$ is a maximal ideal and $S = R\backslash\mathcal{M}$. Then we have*

$$S^{-1}\mathcal{M}^i = (S^{-1}\mathcal{M})^i = \{\frac{a}{s} \mid a \in \mathcal{M}^i, s \notin \mathcal{M}\}$$

Conversely if $\frac{b}{t} \in S^{-1}\mathcal{M}^i$ then $b \in \mathcal{M}^i$. Also

$$S^{-1}\mathcal{M}^i \neq S^{-1}\mathcal{M}^{i+1} \textit{ for all } i \geq 0.$$

Proof of Claim. Suppose $\frac{b}{t} \in S^{-1}\mathcal{M}^i$ then there exists $a \in \mathcal{M}^i, s, u \in S$ such that $atu = bsu$. So $b \in \mathcal{M}^i$ as $su \notin \mathcal{M}$. Also we have $S^{-1}\mathcal{M}^i = (S^{-1}\mathcal{M})^i$. Since $\mathcal{M}^i \neq \mathcal{M}^{i+1}$ the other inequality of sets in the claim follows. □

Claim 3. *If $\mathcal{I} = \mathcal{M}_1^{t_1}\mathcal{M}_2^{t_2}\ldots\mathcal{M}_k^{t_k}$ and $S = \mathcal{R}\backslash\mathcal{M}_1$ then $S^{-1}\mathcal{I} = S^{-1}\mathcal{M}_1^{t_1}$.*

Proof of Claim. Let $\frac{b}{t} \in S^{-1}\mathcal{I}$ with $b \in \mathcal{I}, s \in S$. Then $b = \sum\limits_{j=1}^{l} b_j c_j$ with $b_j \in \mathcal{M}_1^{t_1}, c_j \in \mathcal{M}_2^{t_2}\ldots\mathcal{M}_k^{t_k}$. So $\frac{b}{t} \in S^{-1}\mathcal{M}_1^{t_1}$. Conversely if $b \in \mathcal{M}_1^{t_1}$ then pick $s_i \in \mathcal{M}_i\backslash\mathcal{M}_1, 2 \leq i \leq k$ then for any $\frac{b}{s} \in S^{-1}\mathcal{M}_1^{t_1}, \frac{b}{s} = \frac{bs_2^{t_2}s_3^{t_3}\ldots s_k^{t_k}}{ss_2^{t_2}s_3^{t_3}\ldots s_k^{t_k}} \in S^{-1}\mathcal{I}$. So $S^{-1}\mathcal{M}^{t_1} = S^{-1}\mathcal{I}$. This proves the claim. □

Using the previous two claims and upon localization at each $\mathcal{M}_i$ in the factorization of $\mathcal{I}$ we observe that the powers are also uniquely determined and this Lemma 9 follows. □

Definition 8 (A Total Valuation Map V, Valuation $V_{\mathcal{M}}$ at $\mathcal{M}$ on Monoid M)**.** *Let R be a commutative ring with unity. Suppose for any maximal ideal $\mathcal{M}^i \neq \mathcal{M}^{i+1}$ for all $i \geq 0$. Let $\max(spec(R))$ be any finite set. Let $M(\max(spec(R)))$ be the multiplicative monoid of generated by the maximal ideals in $\max(spec(R))$.*
Define two maps

$$V, V_{\mathcal{M}} : M \longrightarrow \mathbb{N} \cup \{0\}$$

as

$$V(\mathcal{J} = \prod_{i=1}^{t}\mathcal{N}_i^{s_i} \in M) = \sum_{i=1}^{t} s_i$$

$$V_{\mathcal{M}}(\mathcal{J} = \prod_{i=1}^{t}\mathcal{N}_i^{s_i} \in M) = s_i \textit{ if } \mathcal{M} = \mathcal{N}_i \textit{ otherwise } 0.$$

This definition of $V, V_{\mathcal{M}}$ is well defined.

Theorem 10 (Non-Emptiness Theorem)**.** *Let R be a commutative ring with identity. Suppose for each maximal ideal $\mathcal{M}, \mathcal{M}^i \neq \mathcal{M}^{i+1}$ and $\bigcap\limits_{i\geq 0} \mathcal{M}^i = (0)$. Let $\mathcal{F} \subset \max(Spec(R))$ be a finite set. Let $M(\mathcal{F})$ be the finitely generated monoid by a finite set $\mathcal{F}$. Let $\mathcal{I} = \mathcal{M}_1^{t_1}\mathcal{M}_2^{t_2}\ldots\mathcal{M}_k^{t_k} \in M(\mathcal{F})$ be a product of maximal ideals. Then the set*

$$\mathcal{I}\backslash\bigg(\bigcup_{\mathcal{J}\in M(\mathcal{F})^*}\mathcal{I}\mathcal{J}\bigg) \neq \emptyset.$$

Proof. We can use the Theorem 7 on ideal avoidance for the ring R. Since the monoid is finitely generated by finitely many maximal ideals in $\mathcal{F}$, we have

$$\mathcal{I}\backslash\bigg(\bigcup_{\mathcal{M}\in\mathcal{F}}\mathcal{I}\mathcal{M}\bigg) = \mathcal{I}\backslash\bigg(\bigcup_{\mathcal{J}\in M(\mathcal{F})^*}\mathcal{I}\mathcal{J}\bigg) \neq \emptyset$$

Here $M(\mathcal{F})$ denote the set $M(\mathcal{F})\backslash\{R\}$. □

Theorem 11 (Determined Valuative Elements)**.** *Let the notation be as in the previous Theorem 10. For every ideal $\mathcal{I} \in M(\mathcal{F})$, let $a_{\mathcal{I}} \in \mathcal{I}\backslash\bigg(\bigcup\limits_{\mathcal{J}\in M(\mathcal{F})^*}\mathcal{I}\mathcal{J}\bigg)$. Let $\mathcal{I}_j \in M(\mathcal{F}) : 1 \leq j \leq r$ are pairwise comaximal. Then*

$$\prod_{i=1}^{r} a_{\mathcal{I}_i} \in \prod_{i=1}^{r}\mathcal{I}_i\backslash\bigg(\bigcup_{\mathcal{J}\in M(\mathcal{F})^*}\mathcal{J}\prod_{i=1}^{r}\mathcal{I}_i\bigg)$$

Proof.

Claim 4. *If $a \in R$ and $s \notin \mathcal{M}$ then*

$$a \in \mathcal{M}^i\backslash\mathcal{M}^{i+1} \Leftrightarrow as \in \mathcal{M}^i\backslash\mathcal{M}^{i+1}.$$

Proof of Claim. If $a \in \mathcal{M}^i$ then $as \in \mathcal{M}^i$. If $as \in \mathcal{M}^{i+1}$ then since $s \notin \mathcal{M}, a \in \mathcal{M}^{i+1}$. So one way implication follows. Now the other way implication also follows similarly. This proves the claim. □

In the lemma above since the ideals $\mathcal{I}_i : 1 \leq i \leq r$ are comaximal the valuations with respect to any maximal ideal in $\mathcal{F}$ gets exactly determined for the product $\prod\limits_{i=1}^{r} a_{\mathcal{I}_i}$ and the Theorem follows using the previous Claim. □

Definition 9. *Let R be a commutative ring with unity. Suppose for each maximal ideal $\mathcal{M}$ we have $\mathcal{M}^i \neq \mathcal{M}^{i+1}$ and $\bigcap \mathcal{M}^i = (0)$. Let $\mathcal{F} \subset \max(Spec(R))$ be a finite set. Let $M(\mathcal{F})$ be the finitely generated monoid by a finite set $\mathcal{F}$. Let $\mathcal{I} \in M(\mathcal{F})$. Define the set*

$$\mathcal{S}_{\mathcal{I}} \overset{\text{def}}{=\!=} \mathcal{I}\backslash\bigg(\bigcup_{\mathcal{J}\in M(\mathcal{F})^*}\mathcal{I}\mathcal{J}\bigg).$$

By Non-Emptiness Theorem 10 this set $\mathcal{S}_{\mathcal{I}}$ is non-empty.

Note 3. *Let R be a commutative ring with unity. If two sets $S_1, S_2 \subset R$ satisfy the property that their sum of the ideals $(S_1) + (S_2) = R$. It does not imply that there exists $s_1 \in S_1, s_2 \in S_2$ such that $ideal(s_1) + ideal(s_2) = R$. However it does imply that there exists finite set of elements $s_{i1}, s_{i2}, \ldots, s_{it_i} \in S_i$ such that the sum of the ideals*

$$(s_{11}, s_{12}, \ldots, s_{1t_1}) + (s_{21}, s_{22}, \ldots, s_{2t_2}) = R.$$

Theorem 12 (Comaximality of the Ideals of the Sets Theorem)**.** *Let R be a commutative ring with unity. Suppose for each maximal ideal $\mathcal{M} \subset R$ we have*

- $\mathcal{M}^i \neq \mathcal{M}^{i+1}$.
- $\bigcap\limits_{i\geq 0} \mathcal{M}^i = (0)$.

Let $\mathcal{F} \subset \max(Spec(R))$ be a finite set. Suppose every non-zero element $r \in R$ is contained in finitely many maximal ideals. Let $M(\mathcal{F})$ be the finitely generated monoid by a finite set $\mathcal{F}$. There exists a nowhere zero choice multiplicative monoid map $\Sigma : M(\mathcal{F}) \longrightarrow R$ such that

(1) (Unit Condition): $\Sigma(R) = 1$.
(2) (Choice Set Condition): $\Sigma(\mathcal{I}) \in \mathcal{S}_{\mathcal{I}}$ for all $\mathcal{I} \in M(\mathcal{F})$.
(3) (Multiplicativity Condition): If $\mathcal{I}, \mathcal{J} \in M(\mathcal{F})$ are comaximal then $\Sigma(\mathcal{I}\mathcal{J}) = \Sigma(\mathcal{I})\Sigma(\mathcal{J})$.
(4) (Comaximality Condition): For ideals $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_r \in M(\mathcal{F})$

$$\text{If } \mathcal{I}_1 + \mathcal{I}_2 + \ldots + \mathcal{I}_r = 1 \text{ then } (\Sigma(\mathcal{I}_1)) + (\Sigma(\mathcal{I}_2)) + \ldots + (\Sigma(\mathcal{I}_r)) = 1.$$

Proof. We prove this theorem as follows.

Claim 5. *If $\mathcal{I}, \mathcal{J} \in M(\mathcal{F})$ are comaximal then we have $(\mathcal{S}_{\mathcal{I}}) + (\mathcal{S}_{\mathcal{J}}) = 1$ i.e. the ideals of the sets are comaximal and may not be the sets themselves.*

Proof of Claim. Let $\mathcal{M}$ be a maximal ideal containing the set $\mathcal{S}_{\mathcal{I}}$ then $\mathcal{M}$ occurs in the unique factorization of $\mathcal{I} \in M(\mathcal{F})$. Suppose not then Ideal Avoidance Theorem 7 does not hold as $\mathcal{I} = \mathcal{I}\mathcal{M} \bigcup\limits_{\mathcal{N}\in\mathcal{F}} \mathcal{I}\mathcal{N}$. Since there are no common maximal ideals occuring in the unique factorization of $\mathcal{I}, \mathcal{J}$ the claim follows. □

Define $\Sigma(R) = 1$. Let $\mathcal{F} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k\}$. Since every non-zero element is contained in finitely many maximal ideals we find the points $\Sigma(\mathcal{M}_i^{t_i}) \in S_{\mathcal{M}_i^{t_i}}$ inductively as follows.

First we choose any $\Sigma(\mathcal{M}_1) \in S_{\mathcal{M}_1}$. Now this element is contained in finitely many maximal ideals. Choose $\Sigma(\mathcal{M}_2) \in S_{\mathcal{M}_2}$ avoiding these finitely many maximal ideals and continue this process till we find a configuration of elements $\#(\mathcal{F}) = k$-elements $m_i \in \mathcal{S}_{\mathcal{M}_i}$ inductively for $1 \leq i \leq k$ which are pairwise comaximal again using the Theorem 7 on Ideal Avoidance in every inductive step.
Note that it may so happen that $\Sigma(\mathcal{M}_1)^2 = 0$ and hence it belongs to all ideals. So we just cannot raise these values to higher powers. Instead now we find $\Sigma(\mathcal{M}_1^2) \in S_{\mathcal{M}_1^2}$ which is comaximal to all the previously found elements corresponding to other maximal ideals using the Theorem 7 on Ideal Avoidance and also comaximal to maximal ideals other than $\mathcal{M}_1$ containing $\Sigma(\mathcal{M}_1)$. So continuing this way we have defined Σ for all powers of maximal ideals in $\mathcal{F}$. Now extend Σ multiplicatively to the entire monoid. We use Theorem 11 to conclude $\Sigma(\mathcal{I}) \in \mathcal{S}_{\mathcal{I}}$.
Now if $\mathcal{I}_1 + \mathcal{I}_2 + \ldots + \mathcal{I}_r = 1$. Let $\mathcal{M}$ be any maximal ideal. If $\mathcal{M}$ contains all the elements $\Sigma(\mathcal{I}_1), \Sigma(\mathcal{I}_2), \ldots, \Sigma(\mathcal{I}_r)$ then $\mathcal{M}$ contains $\Sigma(\mathcal{M}_i^{l_i})$ and $\Sigma(\mathcal{M}_j^{l_j})$ for two distinct maximal ideals $\mathcal{M}_i \neq \mathcal{M}_j$ in $\mathcal{F}$. So comaximality condition follows.
Now the fact that $\Sigma(\mathcal{I}) \in S_{\mathcal{I}}$ implies that Σ is nowhere zero. Now the Theorem 12 follows. □

Observation 1. *In Theorem 12 while defining the map $\Sigma_{\mathcal{F}}$ it satisfies the following property automatically. If $\mathcal{A} = \mathcal{M}_1^{t_1}\mathcal{M}_2^{t_2}\ldots\mathcal{M}_l^{t_l}, \mathcal{B} = \mathcal{M}_1^{s_1}\mathcal{M}_2^{s_2}\ldots\mathcal{M}_l^{s_l}$ with $\mathcal{M}_1, \ldots, \mathcal{M}_l \in \mathcal{F}$ then we not only have*

$$\Sigma_{\mathcal{F}}(\mathcal{A}) = \Sigma_{\mathcal{F}}(\mathcal{M}_1^{t_1})\ldots\Sigma_{\mathcal{F}}(\mathcal{M}_l^{t_l}), \Sigma_{\mathcal{F}}(\mathcal{B}) = \Sigma_{\mathcal{F}}(\mathcal{M}_1^{s_1})\ldots\Sigma_{\mathcal{F}}(\mathcal{M}_l^{s_l})$$

If $t_i \neq s_i$ for all $1 \leq i \leq l$, we have for each $1 \leq i, j \leq l$, the set of maximal ideals containing $\Sigma(\mathcal{M}_i^{t_i})$ other $\mathcal{M}_i$ is distinct from the set of maximal ideals containing $\Sigma(\mathcal{M}_j^{s_j})$ other than $\mathcal{M}_j$.

Example 3.

- *Let $R = \mathbb{Z}$. Here Σ can be defined for the entire monoid $M(R)$. The map $\Sigma : M(R) \longrightarrow R$ given by $\Sigma((p_1^{t_1} p_2^{t_2} \ldots p_k^{t_k})) = p_1^{t_1} p_2^{t_2} \ldots p_k^{t_k}$ where $p_i : 1 \leq i \leq k$ are $k-$distinct primes.*
- *Let R be a dedekind domain with finitely many maximal ideals. It is a principal ideal domain. Any element in $\pi_i \in \mathcal{P}_i \backslash \left(\bigcup_{j \neq i} \mathcal{P}_j \cup \mathcal{P}_i^2 \right)$ is a generator as its ideal factorization in R is given by $(\pi_i) = \mathcal{P}_i$. Here the monoid $M(R)$ is finitely generated. Then define $\Sigma(\prod_{i=1}^{k} \mathcal{P}_i^{t_i}) = \prod_{i=1}^{k} \pi_i^{t_i}$.*
- *A dedekind domain R is a principal ideal domain if and only if for every maximal ideal $\mathcal{M}$, the set*
$$\mathcal{M} \backslash \left(\Big(\bigcup_{\mathcal{N} \in Spec(R), \mathcal{N} \neq \mathcal{M}} \mathcal{N} \Big) \bigcup \mathcal{M}^2 \right) \neq \emptyset.$$
Then we could define the map Σ similar to the ring of integers explicitly.

10. Surjectivity of the Map $SL_2(R) \longrightarrow \mathbb{PF}^1_{\mathcal{I}} \times \mathbb{PF}^1_{\mathcal{J}}$

Question 5. *Let R be a commutative ring with unity. Let $\mathcal{I}, \mathcal{J} \in \mathcal{RADINF}(R)^*$ be two comaximal ideals. Then when is the map*
$$SL_2(R) \longrightarrow \mathbb{PF}^1_{\mathcal{I}} \times \mathbb{PF}^1_{\mathcal{J}}$$
given by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow [a : b], [c : d]$$
surjective?

In this section we attempt to answer this question. We know from Section 8 that if $\mathcal{I}, \mathcal{J}$ are two distinct maximal ideals then the map is surjective.

10.1. Representation of Elements in One Dimensional Projective Space associated to Ideals.

Lemma 7 (A Representation Lemma)**.** *Let R be a ring with unity. Let $\mathcal{M}$ be a maximal ideal. Suppose $dim_{\frac{R}{\mathcal{M}}}(\frac{\mathcal{M}^t}{\mathcal{M}^{t+1}}) = 1$ for $0 \leq t \leq (k-1)$. Let $p_t \in \mathcal{M}^t \backslash \mathcal{M}^{t+1}$ represent a basis modulo $\mathcal{M}^{t+1}$ for the $\frac{R}{\mathcal{M}}-$vector space $\frac{\mathcal{M}^t}{\mathcal{M}^{t+1}}, 0 \leq t \leq (k-1)$. Then the projective space*
$$\mathbb{PF}^1_{\mathcal{M}^k} = \{[1 : p_t u] \mid \bar{u} \in \mathcal{U}(\frac{R}{\mathcal{M}^{k-t}}), 0 \leq t \leq (k-1)\}$$
$$\bigcup \{[p_t u : 1] \mid \bar{u} \in \mathcal{U}(\frac{R}{\mathcal{M}^{k-t}}), 0 \leq t \leq (k-1)\} \bigcup \{[1 : 0], [0 : 1]\}$$

Proof. Clearly if $[a : b] \in \mathbb{PF}^1_{\mathcal{M}^k}$ then either $a \notin \mathcal{M}$ or $b \notin \mathcal{M}$. So without loss of generality we can assume either $a = 1$ or $b = 1$. So assume $a = 1$. Then $[1 : b_1] = [1 : b_2]$ if and only if $b_1 - b_2 \in \mathcal{M}^k$. Moreover for each $i = 1, 2$ either for some $0 \leq t < k, b_i \in (\mathcal{M}^t \backslash \mathcal{M}^{t+1})$ or $b_i \in \mathcal{M}^k$. Also for any $0 \leq t < k$
$$b_1 \in \mathcal{M}^t \backslash \mathcal{M}^{t+1} \Leftrightarrow b_2 \in \mathcal{M}^t \backslash \mathcal{M}^{t+1}$$
and for $t = k$
$$b_1 \in \mathcal{M}^k \Leftrightarrow b_2 \in \mathcal{M}^k.$$
Now let $b \in \mathcal{M}^t \backslash \mathcal{M}^{t+1}$ and let $b = p_t u + \mathcal{M}^{t+1}$ and here u actually can be varied in a coset of $\mathcal{M}$. Because if
$$p_t u + \mathcal{M}^{t+1} = p_t u' + \mathcal{M}^{t+1}$$

Then by the basis condition $u - u^{'} \in \mathcal{M}$.
However we need to answer the question of representing an element $b \in \mathcal{M}^t \backslash \mathcal{M}^{t+1}$ in the required form. If $t+1 = k$ we are through. Now we will answer the question of representing the element of projective space if $k > t+1$.
Here first we observe that

$$b - p_t u = \sum x_l y_l \text{ with } x_l \in \mathcal{M}^t, y_l \in \mathcal{M}.$$

Now again expressing each x_l in terms of the basis $\{p_t\}$ modulo $\mathcal{M}^{t+1}$ and repeating this process and pushing the powers to $y's$ from $x's$ till we reach $\mathcal{M}^k$ we can actually assume that

$$b = p_t v + \mathcal{M}^k$$

for possibly some other $v \notin \mathcal{M}$.
This representation yields surjectivity and also as now if $k > t+1$ then we can actually vary v in the coset of $\mathcal{M}^{k-t}$ without changing the projective element $[1 : b]$.
This proves the lemma 7. $\square$

Lemma 8 (A Fundamental Observation between the Addition and Multiplication in the Ring)**.** *Let R be a commutative ring with unity. Let $\mathcal{I} = \mathcal{M}^k$ where $\mathcal{M} \subset R$ be a maximal ideal. Suppose $dim_{\frac{R}{\mathcal{M}}}(\frac{\mathcal{M}^t}{\mathcal{M}^{t+1}}) = 1$. Suppose $p_i, \tilde{p}_i \in \mathcal{M}^i \backslash \mathcal{M}^{i+1}$ for $0 \leq i \leq k$. For any $u \in R\backslash\mathcal{M}$ there exists $v \in R\backslash\mathcal{M}$ such that $[1 : p_i u] = [1 : \tilde{p}_i v] \in \mathbb{PF}^1_{\mathcal{I}}$. i.e. $p_i u - \tilde{p}_i v \in \mathcal{M}^k$. Moreover u and v can be varied in their respective cosets $\mod \mathcal{P}^{k-i}$ without changing the element in the projective space $\mathbb{PF}^1_{\mathcal{M}^k}$.*

Proof. Since we have exhibited representing elements in case when the ideal $\mathcal{I} = \mathcal{M}^k$ a power of a maximal ideal for any fixed set of representatives $p_i \in \mathcal{M}^i \backslash \mathcal{M}^{i+1}$ for $0 \leq i \leq (k-1)$ and $p_k = 0$ in the previous Lemma 7 this Lemma 8 follows. $\square$

Theorem 13. *Let R be a commutative ring with unity. Suppose R is a Dedekind Domain (refer Definition 1). Let $M(R)$ be the monoid generated by maximal ideals in R. Let*

$$\mathcal{I} = \mathcal{M}_1^{k_1}\mathcal{M}_2^{k_2}\ldots\mathcal{M}_r^{k_r} \in M(R)$$

be an ideal. Let $\mathcal{F}$ be any finite set of maximal ideals containing $V(\mathcal{I})$. Then the projective space

$$\mathbb{PF}^1_{\mathcal{I}} = \{[\Sigma(\mathcal{I}_1) : \Sigma(\mathcal{I}_2)u] \Big| u \in R\backslash\left(\bigcup_{\mathcal{M}\in\mathcal{F}}\mathcal{M}\right) + \mathcal{I}_3 \text{ where for } i = 1,2,3 \ \ \mathcal{I} \subset \mathcal{I}_i \in M(R)$$
$$\text{with } \mathcal{I}_1, \mathcal{I}_2 \text{ are co-maximal and } \mathcal{I}_1\mathcal{I}_2\mathcal{I}_3 = \mathcal{I}\}$$

Here the map Σ is the no where zero choice monoid multiplicative map for the monoid $M(\mathcal{F})$ from Theorem 12.

Proof. Consider an element $e = (e_1, e_2, \ldots, e_r) \in \prod_{i=1}^{r}\mathbb{PF}^1_{\mathcal{M}_i^{k_i}}$. Let $A \sqcup B$ be a partition of the set $\{1, 2, \ldots, r\}$ such that if $i \in A$ then $e_i = [1 : \Sigma(\mathcal{M}_i^{j_i})u_i]$ for some $u_i \notin \mathcal{M}_i$ and if $i \in B$ then $e_i = [\Sigma(\mathcal{M}_i^{j_i})v_i : 1]$ for some $v_i \notin \mathcal{M}_i$. Here $0 \leq j_i \leq k_i$. This representation holds for e using the representation Lemma 7. Using the Chinese Remainder Reduction Isomorphism in Theorem 1 there exists an element $[a : b] \in \mathbb{PF}^1_{\mathcal{I}}$ such that $[a : b] \equiv e_i \mod \mathcal{M}_i^{k_i}$. Let $\mathcal{I}_1 = \prod_{i\in B}\mathcal{M}_i^{j_i}, \mathcal{I}_2 = \prod_{i\in A}\mathcal{M}_i^{j_i}$. Let $\mathcal{I}_3$ be the unique ideal which is a product of maximal ideals and $\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3 = \mathcal{I}$. We observe that $\mathcal{I}_1, \mathcal{I}_2$ are co-maximal as A, B are disjoint. Now we factor $\Sigma(\mathcal{I}_1), \Sigma(\mathcal{I}_2)$ from a, b respectively using congruences especially using Lemma 8. Let $i \in A$. Then $a \equiv 1 \mod \mathcal{M}_i^{k_i}, b \equiv \Sigma(\mathcal{M}_i^{j_i})u_i \mod \mathcal{M}_i^{k_i}$. Let $t\Sigma(\mathcal{I}_1) \equiv 1 \mod \mathcal{M}_i^{k_i}$. We observe both $b, \Sigma(\mathcal{I}_2)t \in \mathcal{M}_i^{j_i} \backslash \mathcal{M}_i^{j_i+1}$ unless $j_i = k_i$ in which case both $b, \Sigma(\mathcal{I}_2)t \in \mathcal{M}_i^{k_i}$. Now we use Lemma 8 to conclude that that there exists $x_i \in R \backslash \mathcal{M}_i$ such that $b - \Sigma(\mathcal{I}_2)tx_i \in \mathcal{M}_i^{k_i}$. This proves that

$$[a : b] = [1 : \Sigma(\mathcal{I}_2)tx_i] = [\Sigma(\mathcal{I}_1) : \Sigma(\mathcal{I}_2)x_i] \in \mathbb{PF}^1_{\mathcal{M}_i^{k_i}}.$$

We can do similarly if $i \in B$. So we have factored $\Sigma(\mathcal{I}_1), \Sigma(\mathcal{I}_2)$ from a, b for all $1 \leq i \leq r$ respectively obtaining suitable elements $x_i \in R \backslash \mathcal{M}_i$ for $1 \leq i \leq r$. So we get that

$$[a : b] = (e_1, e_2, \ldots, e_r) = ([\Sigma(\mathcal{I}_1) : \Sigma(\mathcal{I}_2)x_1], [\Sigma(\mathcal{I}_1) : \Sigma(\mathcal{I}_2)x_2], \ldots, [\Sigma(\mathcal{I}_1) : \Sigma(\mathcal{I}_2)x_r])$$

Now we obtain the element $u \in R \backslash \left(\bigcup\limits_{\mathcal{M} \in \mathcal{F}} \mathcal{M} \right)$ as follows. We solve three sets of congruences simultaneously.

The first set of congruences is

$$u \equiv x_i \mod \mathcal{M}_i^k$$

The second set of congruences is as follows. For $\mathcal{M} \in \mathcal{F} \backslash \{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k\}$,

$$u \equiv 1 \mod \mathcal{M}$$

The third set of congruences is as follows. Since any element $r \in R$ is in finitely many maximal ideals, let $\mathcal{G}$ be the finite set of maximal ideals which contain $\Sigma(\mathcal{I}_1), \Sigma(\mathcal{I}_2)$. Then we solve for $\mathcal{M} \in \mathcal{G} \backslash \mathcal{F}$

$$u \equiv 1 \mod \mathcal{M}$$

So by solving these congruences we not only obtain $u \in R \backslash \left(\bigcup\limits_{\mathcal{M} \in \mathcal{F}} \mathcal{M} \right)$ we also have that there is no common maximal ideal containing $u, \Sigma(\mathcal{I}_1), \Sigma(\mathcal{I}_2)$. So $[\Sigma(\mathcal{I}_1) : \Sigma(\mathcal{I}_2)u] \in \mathbb{PF}^1_{\mathcal{I}}$ is not only a well defined element but also the required element.

Now the fact that we can modify u to another $\tilde{u} \in u + \mathcal{I}_3$ provided $[\Sigma(\mathcal{I}_1) : \Sigma(\mathcal{I}_2)\tilde{u}] \in \mathbb{PF}^1_{\mathcal{I}}$ is well defined is a easy consequence.

This proves the Theorem 13. $\square$

Theorem 14. *Let R be a commutative ring with unity. Suppose R is a Dedekind Domain (refer Definition 1). Let $M(R)$ be the monoid generated by maximal ideals in R. Let $\mathcal{I}, \mathcal{J} \in M(R)$ be two comaximal ideals. Then the map*

$$SL_2(R) \longrightarrow \mathbb{PF}^1_{\mathcal{I}} \times \mathbb{PF}^1_{\mathcal{J}}$$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow ([a : b], [c : d])$$

is surjective.

Proof. Consider the two co-maximal ideals

$$\mathcal{I} = \prod_{i=1}^{r} \mathcal{M}_i^{k_i}, \mathcal{J} = \prod_{i=1}^{s} \mathcal{N}_i^{l_i} \in M(R).$$

Let

$$\mathcal{F} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_r, \mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_s\}.$$

Let Σ be the choice monoid multiplicative map for the monoid $M(\mathcal{F})$ from Theorem 12. Using the previous Theorem 13 consider an element

$$([\Sigma(\mathcal{I}_1) : \Sigma(\mathcal{I}_2)u], [\Sigma(\mathcal{J}_1) : \Sigma(\mathcal{J}_2)v]) \in \mathbb{PF}^1_{\mathcal{I}_1} \times \mathbb{PF}^1_{\mathcal{I}_2}$$

where $\mathcal{I}_1,\mathcal{I}_2,\mathcal{J}_1,\mathcal{J}_2 \in M(R)$ with $\mathcal{I} \subset \mathcal{I}_1,\mathcal{I}_2$ which are co-maximal, and $\mathcal{J} \subset \mathcal{J}_1,\mathcal{J}_2$ which are co-maximal where $u,v \in R\backslash\bigg(\bigcup\limits_{\mathcal{M}\in\mathcal{F}}\mathcal{M}\bigg)$. Let $\mathcal{I}_3,\mathcal{J}_3 \in M(R)$ be the unique ideals such that $\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3 = \mathcal{I}, \mathcal{J}_1\mathcal{J}_2\mathcal{J}_3 = \mathcal{J}$. Let

$$x \in R\backslash\bigg(\bigcup_{i=1}^{r}\mathcal{M}_i\bigg), y \in R\backslash\bigg(\bigcup_{i=1}^{s}\mathcal{N}_i\bigg), i_3 \in \mathcal{I}_3, j_3 \in \mathcal{J}_3$$

and consider the following matrix

$$\begin{pmatrix} \Sigma(\mathcal{I}_1)x & \Sigma(\mathcal{I}_2)(xu+i_3) \\ \Sigma(\mathcal{J}_1)y & \Sigma(\mathcal{J}_2)(yv+j_3) \end{pmatrix}.$$

Now we solve for x,y,i_3,j_3 such that the above matrix has determinant one. For this purpose let $\alpha,\beta \in R, I_3 \in \mathcal{I}_3, J_3 \in \mathcal{J}_3, i_3 = I_3\beta\Sigma(\mathcal{I}_1), j_3 = J_3\alpha\Sigma(\mathcal{J}_1)$ and consider the equation

(1)
$$\Sigma(\mathcal{I}_1)\Sigma(\mathcal{J}_2)x(J_3\alpha\Sigma(\mathcal{J}_1)) - \Sigma(\mathcal{I}_2)\Sigma(\mathcal{J}_1)y(I_3\beta\Sigma(\mathcal{I}_1)) = 1 + (\Sigma(\mathcal{I}_2)\Sigma(\mathcal{J}_1)u - \Sigma(\mathcal{I}_1)\Sigma(\mathcal{J}_2)v)xy$$

Consider the comaximal ideals

$$\mathcal{K}_1 = (\Sigma(\mathcal{I}_1)), \mathcal{K}_2 = (\Sigma(\mathcal{J}_1))$$

Now we solve the following congruences for $A \in R$ given by

$$1 + \Sigma(\mathcal{I}_2)\Sigma(\mathcal{J}_1)uA \in \mathcal{K}_1, 1 + \Sigma(\mathcal{I}_1)\Sigma(\mathcal{J}_2)vA \in \mathcal{K}_2$$

Such solutions exist because the pairs of ideals $((\Sigma(\mathcal{I}_2)\Sigma(\mathcal{J}_1)u),\mathcal{K}_1),((\Sigma(\mathcal{I}_1)\Sigma(\mathcal{J}_2)v),\mathcal{K}_2)$ are also comaximal. i.e.

$$(\Sigma(\mathcal{I}_2)\Sigma(\mathcal{J}_1)u) + (\Sigma(\mathcal{I}_1)) = R$$
$$(\Sigma(\mathcal{I}_1)\Sigma(\mathcal{J}_2)v) + (\Sigma(\mathcal{J}_1)) = R.$$

If A_0 is one common solution then the set of common solutions is given by

$$A_0 + \mathcal{K}_1\mathcal{K}_2 = \{A_0 + a \mid a \in \mathcal{K}_1\mathcal{K}_2\}$$

because $\frac{R}{\mathcal{K}_1\mathcal{K}_2} \cong \frac{R}{\mathcal{K}_1} \oplus \frac{R}{\mathcal{K}_2}$. Moreover we have the sum of the ideals $(A_0) + \mathcal{K}_1\mathcal{K}_2 = R$. So let $(A_0) + (B_0) = R$ for some $B_0 \in \mathcal{K}_1\mathcal{K}_2$. Here in the Theorem 6 we choose the set

$$E = V(\mathcal{I}) \cup V(\mathcal{J}) \cup V(\Sigma(\mathcal{I}_2)) \cup V(\Sigma(\mathcal{J}_2)) \text{ a finite set.}$$

Because each set in the union is a finite set. Here choice multiplicative monoid map Σ never takes a zero value. Now we note that $\Sigma(\mathcal{I}_2)\Sigma(\mathcal{J}_2) = \Sigma(\mathcal{I}_2\mathcal{J}_2) \neq 0$ by multiplicativity and So using the Theorem 6 which is the Fundamental Lemma on Arithmetic Progressions for Schemes there exists an element of the form $C_0 = A_0 + nB_0$ for some $n \in R$ such that

$$(C_0) + \mathcal{I}\mathcal{J}(\Sigma(\mathcal{I}_2)\Sigma(\mathcal{J}_2)) = R.$$

Now choose $x = 1, y = C_0$ in their respective sets such that their associated principal ideals are obviously co-maximal and also comaximal to each ideal $\mathcal{I},\mathcal{J}$. We observe that

$$1 + (\Sigma(\mathcal{I}_2)\Sigma(\mathcal{J}_1)u - \Sigma(\mathcal{I}_1)\Sigma(\mathcal{J}_2)v)xy \in \mathcal{K}_1 \cap \mathcal{K}_2 = \mathcal{K}_1\mathcal{K}_2 = (\Sigma(\mathcal{I}_1)\Sigma(\mathcal{J}_1) = \Sigma(\mathcal{I}_1\mathcal{J}_1)).$$

Now let $1+(\Sigma(\mathcal{I}_2)\Sigma(\mathcal{J}_1)u-\Sigma(\mathcal{I}_1)\Sigma(\mathcal{J}_2)v)xy = \Sigma(\mathcal{I}_1\mathcal{J}_1)t$. We solve for I_3, J_3 in the following equation which is obtained from Equation 1.

$$\Sigma(\mathcal{J}_2)xJ_3\alpha - \Sigma(\mathcal{I}_2)yI_3\beta = t$$

Now consider the two ideals $\Sigma(\mathcal{J}_2)x\mathcal{J}_3, \Sigma(\mathcal{I}_2)C_0\mathcal{I}_3$. They are comaximal because $\Sigma(\mathcal{J}_2)x\mathcal{J}_3 = \Sigma(\mathcal{J}_2)\mathcal{J}_3$. Also the ideals $(\Sigma(\mathcal{I}_2)),\mathcal{I}_3$ are comaximal with ideals $(\Sigma(\mathcal{J}_2)),\mathcal{J}_3$ and the ideal (C_0) is comaximal with $(\Sigma(\mathcal{J}_2))$ and $\mathcal{J}$ itself hence $\mathcal{J}_3$ also. So solving for $I_3\beta \in \mathcal{I}_3, J_3\alpha \in \mathcal{J}_3$ is possible in the above equation.

This proves the Theorem 14. ∎

10.2. **Unique Factorization of a non-zero element with respect to a Finitely Generated Monoid Generated by Maximal Ideals.**

Definition 10. *Let R be a commutative ring with unity. The ring R satisfies the following properties.*

(1) *For each maximal ideal $\mathcal{M}$ we have $\mathcal{M}^i \neq \mathcal{M}^{i+1}$ for all $i \geq 0$.*
(2) $\bigcap\limits_{n\geq 0} \mathcal{M}^i = (0)$.
(3) *Every non-zero element $r \in R$ is contained in finitely many maximal ideals.*

Let $M(R)$ be the monoid generated by maximal ideals in R. Let $\mathcal{F}$ be a finite set of maximal ideals. Then for any $0 \neq x \in R$ we can define a valuation $V_{\mathcal{F}}$ with respect to the monoid $\mathcal{F}$. Since $x \neq 0$ for each maximal ideal $\mathcal{M}$ there exists a largest integer $i = i_{\mathcal{M}} \geq 0$ such that $x \in \mathcal{M}^i \backslash \mathcal{M}^{i+1}$. The maps

$$V_{\mathcal{F}} : R^* \longrightarrow M(\mathcal{F}), V_{\mathcal{M}} : R^* \longrightarrow \mathbb{N}$$

are defined as $V_{\mathcal{F}}(x) = \prod\limits_{\mathcal{M}\in\mathcal{F}} \mathcal{M}^{i_{\mathcal{M}}}$ and $V_{\mathcal{M}}(x) = i_{\mathcal{M}}$. Clearly $x \in V_{\mathcal{F}}(x)$ and $V_{\mathcal{F}}(x)$ is the unique factorization of the element x with respect to the monoid $M(\mathcal{F})$.

10.3. **Representation of Elements in Higher Dimensional Projective Space associated to Ideals.**

Theorem 15. *Let R be a commutative ring with unity. Suppose R is a Dedekind Domain (refer Definition 1). Let $M(R)$ be the monoid generated by maximal ideals in R. Let $\mathcal{I} \in M(R)$ be a product of maximal ideals. Let $k \geq 2$ be a positive integer. Let $\mathcal{F}$ be any finite set of maximal ideals containing $V(\mathcal{I})$. Let Σ be the nowhere zero choice monoid multiplicative map for the monoid $M(\mathcal{F})$ from Theorem 12. Then the description of the $k-$dimensional projective space is given by*

$$\mathbb{PF}^k_{\mathcal{I}} = \{[\Sigma(\mathcal{J}_0)v_0 : \Sigma(\mathcal{J}_1)v_1 : \ldots : \Sigma(\mathcal{J}_k)v_k] \Big| \mathcal{J}_i \supset \mathcal{I}, \sum_{i=0}^{k} \mathcal{J}_i = R \Rightarrow \sum_{i=0}^{k} \Sigma(\mathcal{J}_i) = R$$
$$v_0, v_1, \ldots, v_k \in R \backslash \bigcup_{\mathcal{M}\in\mathcal{F}} \mathcal{M}, \sum_{i=0}^{k} (\Sigma(\mathcal{J}_i)v_i) = R\}.$$

Proof. Let $\mathcal{I} = \mathcal{M}_1^{t_1}\mathcal{M}_2^{t_2}\ldots\mathcal{M}_l^{t_l} \in M(R)$. Let $[x_0 : x_1 : \ldots : x_k] \in \mathbb{PF}^k_{\mathcal{I}}$. Assume each x_i is non-zero by replacing the element by a non-zero element of $\mathcal{I}$. This also does not alter the condition $\sum\limits_{i=0}^{k}(x_i) = R$. We define the ideal $\mathcal{J}_i$ as follows. Let $\mathcal{G} = \{\mathcal{M}_1, \ldots, \mathcal{M}_l\} = V(\mathcal{I})$. Consider the unique factorizations of x_i with respect to the monoid $M(\mathcal{G})$. Define the ideal

$$\text{for } 0 \leq i \leq k, \mathcal{J}_i = \prod_{j=1}^{l} \mathcal{M}_j^{min(t_j, V_{\mathcal{M}_j}(x_i))} \Rightarrow \mathcal{J}_i \supset V_{\mathcal{G}}(x_i) \supset \{x_i\}, \mathcal{J}_i \supset \mathcal{I}.$$

So $\sum\limits_{i=0}^{k}(\mathcal{J}_i) = R$. Hence we also have $\sum\limits_{i=0}^{k}\Sigma(\mathcal{J}_i) = R$ for $\Sigma : M(\mathcal{F}) \longrightarrow R$ where $\mathcal{F} \supset \mathcal{G}$. Now we factor $\Sigma(\mathcal{J}_i)$ from x_i for $0 \leq i \leq k$ using congruences. First for a fixed $1 \leq j \leq l$, using the Lemma 8 we conclude that there exists $v_{ij} \in R\backslash\mathcal{M}_j$ such that $x_i - \Sigma(\mathcal{J}_i)v_{ij} \in \mathcal{M}_j^{t_j}$. Note if $V_{\mathcal{M}_j}(x_i) \geq t_j$ then we could choose $v_{ij} = 1$. By chinese remainder theorem for a fixed i we lift v_{ij} to an element $v_i \in R\backslash \bigcup\limits_{\mathcal{M}\in\mathcal{F}} \mathcal{M}$ by solving congruences.

$$v_i \equiv v_{ij} \mod \mathcal{M}_j^{t_j}$$

We may need to solve some additional finitely many congruences of the type

$$v_i \equiv 1 \mod \mathcal{N}$$

to avoid a maximal ideal $\mathcal{N}$ and also to ensure the condition that

$$\sum_{i=0}^{k}(\Sigma(\mathcal{J}_i)v_i) = R$$

which can be done as every non-zero element is contained in finitely many maximal ideals. Hence the Theorem 15 follows. □

Now we prove the third main Theorem 3 of our article.

Proof. We prove this theorem by proving the following three claims.

Claim 6 (Well definedness of the map $r_{g^{-1}}$)**.** *Let $g \in SL_k(R)$. Let*

$$([a_{11} : a_{12} : \ldots : a_{1k}], [a_{21} : a_{22} : \ldots : a_{2k}], \ldots, [a_{k1} : a_{k2} : \ldots : a_{kk}]) \in \prod_{i=1}^{k} \mathbb{PF}^{k-1}_{\mathcal{I}_i}.$$

Consider the matrix $A = [a_{ij}]_{k\times k}$. Let $Ag^{-1} = [b_{ij}]_{k\times k}$. Then the map

$$r_{g^{-1}} : \prod_{i=1}^{k} \mathbb{PF}^{k-1}_{\mathcal{I}_i} \longrightarrow \prod_{i=1}^{k} \mathbb{PF}^{k-1}_{\mathcal{I}_i}$$

given by

$$([a_{11} : a_{12} : \ldots : a_{1k}], [a_{21} : a_{22} : \ldots : a_{2k}], \ldots, [a_{k1} : a_{k2} : \ldots : a_{kk}]) \underset{r_{g^{-1}}}{\longrightarrow}$$
$$([b_{11} : b_{12} : \ldots : b_{1k}], [b_{21} : b_{22} : \ldots : b_{2k}], \ldots, [b_{k1} : b_{k2} : \ldots : b_{kk}])$$

is well defined. This gives a left action of $SL_k(R)$ on the space $\prod\limits_{i=1}^{k} \mathbb{PF}^{k-1}_{\mathcal{I}_i}$.

Proof of Claim. Suppose $([\tilde{a}_{11} : \tilde{a}_{12} : \ldots : \tilde{a}_{1k}], [\tilde{a}_{21} : \tilde{a}_{22} : \ldots : \tilde{a}_{2k}], \ldots, [\tilde{a}_{k1} : \tilde{a}_{k2} : \ldots : \tilde{a}_{kk}]) = ([a_{11} : a_{12} : \ldots : a_{1k}], [a_{21} : a_{22} : \ldots : a_{2k}], \ldots, [a_{k1} : a_{k2} : \ldots : a_{kk}]) \in \prod\limits_{i=1}^{k} \mathbb{PF}^{k-1}_{\mathcal{I}_i}$. Then we have for every $1 \leq \alpha, \beta, \gamma \leq k$, $a_{\alpha\beta}\tilde{a}_{\alpha\gamma} - \tilde{a}_{\alpha\beta}a_{\alpha\gamma} \in \mathcal{I}_\alpha$. Now we observe that if

$$\begin{pmatrix} a_{\alpha 1} & a_{\alpha 2} & \ldots & a_{\alpha k} \\ \tilde{a}_{\alpha 1} & \tilde{a}_{\alpha 2} & \ldots & \tilde{a}_{\alpha k} \end{pmatrix} g^{-1} = \begin{pmatrix} b_{\alpha 1} & b_{\alpha 2} & \ldots & b_{\alpha k} \\ \tilde{b}_{\alpha 1} & \tilde{b}_{\alpha 2} & \ldots & \tilde{b}_{\alpha k} \end{pmatrix}$$

Then we have for any fixed $1 \leq \alpha \leq k$ and for every $1 \leq \mu < \delta \leq k$

$$b_{\alpha\mu}\tilde{b}_{\alpha\delta} - \tilde{b}_{\alpha\mu}b_{\alpha\delta} \in ideal(a_{\alpha\beta}\tilde{a}_{\alpha\gamma} - \tilde{a}_{\alpha\beta}a_{\alpha\gamma} : 1 \leq \beta < \gamma \leq k) \subset \mathcal{I}_\alpha.$$

and conversely because g is invertible. Moreover

$$(a_{\alpha i} : 1 \leq i \leq k) = (\tilde{a}_{\alpha i} : 1 \leq i \leq k) = R \Leftrightarrow (b_{\alpha i} : 1 \leq i \leq k) = (\tilde{b}_{\alpha i} : 1 \leq i \leq k) = R$$

This proves the claim. □

Claim 7 (Invariance of the Image)**.** *The image of the map σ_1 is $SL_k(R)$ invariant.*

Proof of Claim. We observe that $g.\sigma_1(A) = \sigma_1(Ag^{-1})$. Each row of Ag^{-1} is unital if and only if each row of A is unital. So the claim follows. □

Claim 8. *The image of σ_1 equals $\prod\limits_{i=1}^{k} \mathbb{PF}^{k-1}_{\mathcal{I}_i}$.*

Proof of Claim. Let $([a_{11}:a_{12}:\ldots:a_{1k}],[a_{21}:a_{22}:\ldots:a_{2k}],\ldots,[a_{k1}:a_{k2}:\ldots:a_{kk}])\in \prod\limits_{i=1}^{k}\mathbb{PF}^{k-1}_{\mathcal{I}_i}$. Let $A=[a_{ij}]_{k\times k}\in M_{k\times k}(R)$. Now we reduce the matrix A to an element in $SL_k(R)$ to prove the claim in a step by step manner.

Since each row generates a unit ideal using the Lemma 2 we can right multiply A by an $SL_k(R)-$matrix so that a_{11} element is a unit modulo $\mathcal{I}_1$. Now replace the first row by an equivalent row where $a_{11}=1$. Then we can transform the first row to $e^k_1=(1,0,0,\ldots,0)$ using another $SL_k(R)-$matrix. Now we use the previous Theorem 15 to represent appropriately the elements of the projective spaces by choosing the map Σ on the finitely generated monoid $M(\mathcal{F})$ where

$$\mathcal{F}=V(\mathcal{I}_1)\cup V(\mathcal{I}_2)\cup\ldots\cup V(\mathcal{I}_k).$$

Let the second row be

$$[\Sigma(\mathcal{I}_{21})v_{21}:\Sigma(\mathcal{I}_{22})v_{22}:\ldots:\Sigma(\mathcal{I}_{2k})v_{2k}]$$

We have

$$\sum_{i=1}^{k}(\Sigma(\mathcal{I}_{2i})v_{2i})=R,\ \text{and}\ \mathcal{I}_1+\sum_{i=2}^{k}(\Sigma(\mathcal{I}_{2i})v_{2i})=R$$

by the choice of the monoid. Hence we get

$$(\Sigma(\mathcal{I}_{21})v_{21})\mathcal{I}_1+\sum_{i=2}^{k}(\Sigma(\mathcal{I}_{2i})v_{2i})=R$$

So there exists $i_1\in\mathcal{I}_1$ such that the vector

$$(\Sigma(\mathcal{I}_{21})v_{21}i_1,\Sigma(\mathcal{I}_{22})v_{22},\ldots,\Sigma(\mathcal{I}_{2k})v_{2k})$$

is unital in R. Now $\mathcal{I}_2$ satisfies unital set condition USC. So by the Theorem 8 there exists $s_1,s_3,\ldots,s_k\in R$ such that the element

$$\Sigma(\mathcal{I}_{22})v_{22}+\Sigma(\mathcal{I}_{21})v_{21}i_1s_1+\sum_{i=3}^{k}\Sigma(\mathcal{I}_{2i})v_{2i}s_i$$

is a unit modulo $\mathcal{I}_2$. The second summand in the above expression is in the ideal $\mathcal{I}_1$. Now we use a suitable column operation on A to transform a_{22} to the above expression. This does not alter the first row because it replaces the element a_{12} by an element of $\mathcal{I}_1$. Hence we could replace the first row of A back by e^k_1. Now we have obtained a_{22} a unit mod $\mathcal{I}_2$. We can make this element $a_{22}=1$ exactly by replacing the second row with another equivalent projective space element representative in $\mathbb{PF}^k_{\mathcal{I}_2}$ however in the same equivalence class. Now by applying suitable column operations we can transform the second row to $e^k_2=(0,1,\ldots,0)$.

Inductively suppose we arrive at the $j^{th}-$row for $j\leq k$. Let the j^{th} row be given by

$$[\Sigma(\mathcal{I}_{j1})v_{j1}:\Sigma(\mathcal{I}_{j2})v_{j2}:\ldots:\Sigma(\mathcal{I}_{jk})v_{jk}]$$

using again the Theorem 15 with respect to the same monoid map Σ.

We have

$$\sum_{i=1}^{k}(\Sigma(\mathcal{I}_{ji})v_{ji})=R,\ \text{and}\ \mathcal{I}_1\mathcal{I}_2\ldots\mathcal{I}_{j-1}+\sum_{i=j}^{k}(\Sigma(\mathcal{I}_{ji})v_{ji})=R$$

by the choice of the monoid. Hence we get

$$\sum_{i=1}^{j-1}(\Sigma(\mathcal{I}_{ji})v_{ji})\mathcal{I}_1\mathcal{I}_2\ldots\mathcal{I}_{j-1}+\sum_{i=j}^{k}(\Sigma(\mathcal{I}_{ji})v_{ji})=R$$

So there exists $t_1, t_2, \ldots, t_{j-1} \in \prod_{i=1}^{j-1} \mathcal{I}_i$ such that the vector

$$(\Sigma(\mathcal{I}_{j1})v_{j1}t_1, \Sigma(\mathcal{I}_{j2})v_{j2}t_2, \ldots, \Sigma(\mathcal{I}_{j(j-1)})v_{j(j-1)}t_{j-1}, \Sigma(\mathcal{I}_{jj})v_{jj}, \ldots, \Sigma(\mathcal{I}_{jk})v_{jk})$$

is unital in R. Now $\mathcal{I}_j$ satisfies unital set condition USC. So by the Theorem 8 we make a_{jj} element an unit $\mod \mathcal{I}_j$ without actually changing the previous $(j-1)$-rows as projective space elements because $t_1, t_2, \ldots, t_{j-1} \in \bigcap_{i=1}^{j-1} \mathcal{I}_i$. Now we make the $a_{jj} = 1$ exactly and then by applying an $SL_{k+1}(R)$ matrix make the $j^{th}-$row equal to $e^k_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$. We continue this procedure till $j = k$. We arrive at the identity matrix. Hence the map σ_1 is surjective and the Claim 8 follows. □

Similarly the map σ_2 is also surjective and the Theorem 3 also follows. □

10.4. **A Consequence of Sujectivity.**

Theorem 16. *Let R be a commutative ring with unity.*

(1) Let R be a Dedekind domain (refer Definition 1).

(2) R has infinitely many maximal ideals.

Suppose Let $M(R)$ be the monoid generated by maximal ideals in R. Let $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_r \in M(R)$ be $r-$ pairwise co-maximal ideals. Let $k \geq 2$ be a positive integer. Consider for $r \leq k$

$$G_{r,k}(R) = \{A = [a_{ij}]_{r\times k} \in M_{r\times k}(R) \mid \text{ such that the } r \times r \text{ minors generate unit ideal}\}.$$

Then the map

$$\tau : G_{r,k}(R) \longrightarrow \mathbb{PF}^{k-1}_{\mathcal{I}_1} \times \mathbb{PF}^{k-1}_{\mathcal{I}_2} \times \ldots \times \mathbb{PF}^{k-1}_{\mathcal{I}_r}$$

given by

$$\tau : (A) = ([a_{11} : a_{12} : \ldots : a_{1k}], [a_{21} : a_{22} : \ldots : a_{2k}], \ldots, [a_{r1} : a_{r2} : \ldots : a_{rk}])$$

are surjective.

Proof. Since the dedekind domain has infinitely many maximal ideals by hypothesis, let $\mathcal{I}_{r+1}, \ldots, \mathcal{I}_k \in M(R)$ be pairwise comaximal which are also comaximal to each of $\mathcal{I}_1, \ldots, \mathcal{I}_r$. Such ideals exist. Now using the main Theorem 3 we conclude surjectivity of this map τ. Hence this Theorem 16 also follows. □

11. Two Examples for Subgroups of $SL_k(R)$

Example 4. *Let $\mathbb{K}$ be an algebraically closed field. Let $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$. Consider the standard action of $SL_2(R)$ on R^2. Let $G(R)$ be the stabilizer subgroup of the element $(1, 1)^{tr} \in R^2$ i.e. $G_2(R) = \{A \in SL_2(R) \mid A.(1, 1)^{tr} = (1, 1)^{tr}\}$. Let $\mathcal{M}, \mathcal{N}$ be two maximal ideals in R. Then the map*

$$G_2(R) \longrightarrow \mathbb{PF}^1_{\mathcal{M}} \times \mathbb{PF}^1_{\mathcal{N}}$$

is not surjective.

We observe that $G_2(R)$ is also given as follows.

$$G_2(R) = \{\begin{pmatrix} 1+b & -b \\ b & 1-b \end{pmatrix} \mid b \in R\}$$

So the image of $G_2(R)$ is exactly $\{([1 + b : -b], [b : 1 - b]) \mid b \in R\} \subset \mathbb{PF}^1_{\mathcal{M}} \times \mathbb{PF}^1_{\mathcal{N}} = \mathbb{PF}^1_{\mathbb{K}} \times \mathbb{PF}^1_{\mathbb{K}} \subset \mathbb{PF}^3_{\mathbb{K}}$. The image is precisely

$$([x_1 : y_1], [x_2 : y_2]) \in \mathbb{PF}^1_{\mathbb{K}} \times \mathbb{PF}^1_{\mathbb{K}} \text{ where } (x_1 + y_1)(x_2 + y_2) \neq 0.$$

In fact the image does not contain any element from the set

$$\left(\{[1:-1]\} \times \mathbb{PF}^1_{\mathbb{K}}\right) \bigcup \left(\mathbb{PF}^1_{\mathbb{K}} \times \{[1:-1]\}\right)$$

which is a union of two projective lines meeting at the point $([1:-1],[1:-1])$.

Example 5. *In Theorem 2 we have proved that if an ideal $\mathcal{I}$ satisfies the Unital Set Condition in 7 then the map*

$$SL_k(R) \longrightarrow SL_k(\frac{R}{\mathcal{I}})$$

is surjective for all $k > 0$. Any ideal $\mathcal{I}$ which is a product of maximal ideals in R where R be a commutative ring with unity satisfies USC. So for the group $SL_k(R)$ surjectivity onto $SL_k(\frac{R}{\mathcal{I}})$ always follows. Now here we consider a smaller subgroup $H_k(R) \subset SL_k(R)$.
Let R be a commutative ring with unity. Suppose R is a Dedekind Domain (refer Definition 1). Let $M(R)$ be the monoid generated by maximal ideals in R. Let $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_k \in M(R)$ be $k-$ pairwise co-maximal ideals. Let $k \geq 2$ be a positive integer. Let $H_k(R) \subset SL_k(R)$ be a subgroup. Suppose

$$H_k(R) \supset \{A = [a_{ij}]_{k\times k} \mid a_{ii} \in 1 + \mathcal{I}_1^t, a_{ij} \in \mathcal{I}_1^t \text{ if } i < j \text{ for some fixed large integer } t\}.$$

i.e. $H_k(R)$ contains matrices in $SL_k(R)$ which are lower triangular modulo $\mathcal{I}_1^t$ for some large positive integer $t > 0$. Then the maps

$$\sigma_1, \sigma_2 : H_k(R) \longrightarrow \mathbb{PF}^{k-1}_{\mathcal{I}_1} \times \mathbb{PF}^{k-1}_{\mathcal{I}_2} \times \ldots \times \mathbb{PF}^{k-1}_{\mathcal{I}_k}$$

given by

$$\sigma_1 : (A) = ([a_{11} : a_{12} : \ldots : a_{1k}], [a_{21} : a_{22} : \ldots : a_{2k}], \ldots, [a_{k1} : a_{k2} : \ldots : a_{kk}]),$$
$$\sigma_2 : (A) = ([a_{11} : a_{21} : \ldots : a_{k1}], [a_{12} : a_{22} : \ldots : a_{k2}], \ldots, [a_{1k} : a_{2k} : \ldots : a_{kk}])$$

are surjective.
This can be proved similarly. In the proof of Theorem 3 we make the first row equal to e_1^k and then it is enough that $H_k(R)$ contains matrices in $SL_k(R)$ which are lower triangular modulo $\mathcal{I}_1^t$ for some large positive integer $t > 0$ for the remaining rows in order to transform the matrix to a identity matrix in the rest of the procedure as we can use $\mathcal{I}_1^t$ instead of $\mathcal{I}_1$.

12. A Surjectivity Theorem for the Sum-Product Equation

In this section we prove a surjectivity theorem for the Sum-Product Equation.

Remark 2. *Let R be a commutative ring with unity. Let $k > 0$ be a positive integer. Let $(a_1, a_2, \ldots, a_{k+1})$ be a unital set in R. Suppose $a_1x_1 + a_2x_2 + \ldots + a_kx_k + a_{k+1}x_{k+1} = 1$ and $\{x_1, x_2, \ldots, x_k\}$ is also a unital set. i.e. $b_1x_1 + b_2x_2 + \ldots + b_kx_k = 1$ then we have*

$$(a_1 + a_{k+1}x_{k+1}b_1)x_1 + (a_2 + a_{k+1}x_{k+1}b_2)x_2 + \ldots + (a_k + a_{k+1}x_{k+1}b_k)x_k = 1$$

i.e. there exists $t_1, t_2, \ldots, t_k \in (a_{k+1})$ such that the set $\{a_1 + t_1, a_2 + t_2, \ldots, a_k + t_k\}$ is unital in R.

Before we state the main theorem in this section we prove the following two important Lemmas 9, 10.

Lemma 9. *Let R be a commutative ring with unity in which every non-zero element is contained in finitely many maximal ideals. Let $\mathcal{I} \subset R$ be an ideal. Let $x, y \in R$. Suppose $(x)+(y)+\mathcal{I} = R$. Then there exists $a, b \in R$ such that $ax+by \equiv 1 \mod \mathcal{I}$ and $(a)+(b) = R$.*

Proof. Suppose $a_1x+b_1y+i=1$ for some $i\in\mathcal{I}$. Either $b_1\neq 0$ or $a_1\neq 0$. Suppose $b_1\neq 0$. Then by Fundamental Lemma on Arithmetic Progressions for Schemes 6 we have that there exists $t\in R$ such that $(a_1-t(b_1y+i))+(b_1)=R$. So we have

$$(a_1-t(b_1y+i))x+(1+tx)(b_1y+i)=1$$
$$(a_1-t(b_1y+i))+(b_1)=R$$
$$(a_1-t(b_1y+i))x+(1+tx)b_1y+(1+tx)i=1$$

So choosing $a=(a_1-t(b_1y+i)), b=(1+tx)b_1$ we have $(a)+(b)=R$ and $ax+by\equiv 1 \mod \mathcal{I}$. Now the lemma follows. $\square$

Lemma 10. *Let R be a commutative ring with unity. Suppose R is a Dedekind domain (refer Definition 1). Let $\mathcal{I}\subset R$ be an ideal. Let $r>1$ be a positive integer. Suppose $(a_1,a_2,\ldots,a_r)$ is a unital set modulo $\mathcal{I}$. Then there exists $t_1,t_2,\ldots,t_r\in\mathcal{I}$ such that the set $\{a_1+t_1,\ldots,a_r+t_r\}$ is unital in R.*

Proof. Let $a_1x_1+\ldots+a_rx_r+i=1$ for $i\in\mathcal{I}$. If $i=0$ then there is nothing to prove. So assume $i\neq 0$.
Suppose if two of the $x_j's$ are non-zero. Say $x_1\neq 0, x_2\neq 0$. Let $e=a_2x_2+\ldots+a_rx_r+i$. Then $a_1x_1+e=1$. By using Fundamental Lemma for Arithmetic Progressions for Dedekind Domains 5 there exists $t\in R$ such that $(x_1-te)+(x_2)=R$. We also have $(x_1-te)a_1+(1+ta_1)e=1$. So we get

$$a_1(x_1-te)+a_2x_2(1+ta_1)+a_3x_3(1+ta_1)+\ldots+a_rx_r(1+ta_1)+i(1+ta_1)=1$$

Now we have both $ideal(x_1-te)+ideal(x_2)=R, ideal(x_1-te)+ideal(1+ta_1)=R$ so $ideal(x_1-te)+ideal(x_2(1+ta_1))=R$. There exists $s_1,s_2\in R$ such that

$$(x_1-te)s_1+x_2(1+ta_1)s_2=1\Rightarrow (x_1-te)s_1i(1+ta_1)+x_2(1+ta_1)s_2i(1+ta_1)=i(1+ta_1)$$

Hence we get

$$(a_1+s_1(1+ta_1)i)(x_1-te)+(a_2+s_2(1+ta_1)i)x_2(1+ta_1)+a_3x_3(1+ta_1)+\ldots+a_rx_r(1+ta_1)=1$$

So choosing $t_1=is_1(1+ta_1), t_2=is_2(1+ta_2)\in\mathcal{I}, t_3=t_4=\ldots=0$ we get $\{a_i+t_i:1\leq i\leq r\}$ is a unital set.
Suppose all but one of the x_i is zero. Say $x_1\neq 0$ and $x_2,x_3,\ldots,x_r=0$. Then $a_1x_1+i=1$ and suppose $a_j=0$ for some $j\geq 2$. Then choose $t_j=i, t_l=0$ for $l\neq j$ and we have the set $\{a_1,a_2,\ldots,a_{j-1},a_j+t_j,a_{j+1},\ldots,a_r\}$ is unital.
Now if $x_1\neq 0, x_2=x_3=\ldots=x_r=0, a_2,a_3,\ldots,a_r\neq 0$ and $r\geq 3$ then we could choose $x_2=a_3, x_3=-a_2$ and we have atleast two of the $x_j's$ non-zero which is considered before.
Now consider the possibility where $r=2$. Let $(a_1)+(a_2)+\mathcal{I}=R$. Now using the previous Lemma 9 we have that there exists x_1,x_2 such that $(x_1)+(x_2)=R$ and $a_1x_1+a_2x_2+i=1$ for some $i\in\mathcal{I}$. So if $x_1y_1+x_2y_2=1$ then $x_1y_1i+x_2y_2i=i$. So we get $\{a_1+y_1i,a_2+y_2i\}$ is a unital set.
This completes the proof of this Lemma 10. $\square$

Now we prove the main result of this section on surjectivity theorem for the Sum-Product Equation.

Theorem 17. *Let R be a commutative ring with unity. Suppose R is a Dedekind domain (refer Definition 1). Let $M(R)$ be the monoid generated by maximal ideals in R. Let $\mathcal{I}_1,\mathcal{I}_2,\ldots,\mathcal{I}_r\in M(R)$ be $r-$ pairwise co-maximal ideals. Let $r\geq 2, k\geq 2$ be two positive integer. Consider*

$$M(r,k)(R)=\{A=[a_{ij}]_{r\times k}\mid \sum_{j=1}^{k}\prod_{i=1}^{r}a_{ij}=1\}.$$

Then the map

$$\lambda : M(r,k)(R) \longrightarrow \mathbb{PF}^{k-1}_{\mathcal{I}_1} \times \mathbb{PF}^{k-1}_{\mathcal{I}_2} \times \ldots \times \mathbb{PF}^{k-1}_{\mathcal{I}_r}$$

given by

$$\lambda : (A) = ([a_{11} : a_{12} : \ldots : a_{1k}], [a_{21} : a_{22} : \ldots : a_{2k}], \ldots, [a_{r1} : a_{r2} : \ldots : a_{rk}])$$

is surjective.

Proof. For $r = 1$ the theorem is not true. Choose $R = \mathbb{Z}$. $\mathcal{I}_1 = p\mathbb{Z}$. The point

$$[1 : -1] \in \mathbb{PF}^1_{\mathcal{I}_1} = \mathbb{PF}^1_p$$

is not in the image of $M(1,2)(\mathbb{Z})$.
Assume $r \geq 2$. Let us prove this by induction on r. First we prove for $r = 2$. Let

$$([x_1 : \ldots : x_k], [y_1 : \ldots : y_k]) \in \mathbb{PF}^{k-1}_{\mathcal{I}_1} \times \mathbb{PF}^{k-1}_{\mathcal{I}_2}.$$

Suppose there exists

$$([x^0_1 : \ldots : x^0_k], [y^0_1 : \ldots : y^0_k]) = ([x_1 : \ldots : x_k], [y_1 : \ldots : y_k]) \in \mathbb{PF}^{k-1}_{\mathcal{I}_1} \times \mathbb{PF}^{k-1}_{\mathcal{I}_2}$$

such that $\sum_{j=1}^{k} x^0_j y^0_j = 1 + i_2$ where $i_2 \in \mathcal{I}_2$. Let $\sum_{j=1}^{k} x^0_j z^0_j = 1$ because we have $\sum_{j=1}^{k} (x^0_j) = R$. By choosing $u_j = x^0_j, v_j = y^0_j - z^0_j i_2$ we have $\sum_{j=1}^{k} u_j v_j = 1$ and

$$([u_1 : \ldots : u_k], [v_1 : \ldots : v_k]) = ([x_1 : \ldots : x_k], [y_1 : \ldots : y_k]) \in \mathbb{PF}^{k-1}_{\mathcal{I}_1} \times \mathbb{PF}^{k-1}_{\mathcal{I}_2}$$

So it is enough to prove that there exists $([x^0_1 : \ldots : x^0_k], [y^0_1 : \ldots : y^0_k]) = ([x_1 : \ldots : x_k], [y_1 : \ldots : y_k]) \in \mathbb{PF}^{k-1}_{\mathcal{I}_1} \times \mathbb{PF}^{k-1}_{\mathcal{I}_2}$ such that $\sum_{j=1}^{k} x^0_j y^0_j \equiv 1 \mod \mathcal{I}_2$.

Sicne $\mathcal{I}_1 + \mathcal{I}_2 = R$, let $a \in \mathcal{I}_1, b \in \mathcal{I}_2$ such that $a + b = 1 - \sum_{j=1}^{k} x_i y_i$. Now there exists $w_i \in R$ such that $\sum_{j=1}^{k} w_i y_i = 1$ because $\sum_{j=1}^{k} (y_i) = R$. Hence $\sum_{j=1}^{k} (x_i + a w_i) y_i = 1 - b \equiv 1 \mod \mathcal{I}_2$. Now apriori we do not have $\sum_{j=1}^{k} (x_j + a w_j) = R$. Instead we have

$$\sum_{j=1}^{k} (x_j + a w_j) + \mathcal{I}_2 = R, \sum_{j=1}^{k} (x_j + a w_j) + \mathcal{I}_1 = R.$$

Hence $\sum_{j=1}^{k} (x_j + a w_j) + \mathcal{I}_1 \mathcal{I}_2 = R$. So using Lemma 10 we conclude that there exists $t_1, t_2, \ldots, t_k \in \mathcal{I}_1 \mathcal{I}_2$ such that $\sum_{j=1}^{k} (x_j + a w_j + t_j) = R$ and

$$\sum_{j=1}^{k} (x_j + a w_j + t_j) y_j = 1 - b + \sum_{j=1}^{k} t_j y_j \equiv 1 \mod \mathcal{I}_2$$

So chooisng $x^0_j = x_j + a w_j + t_j, y^0_j = y_j$ we have proved this Theorem 17 for the case when $r = 2$.
Now we prove for any positive integer $r > 2$. Let

$$\mathcal{F} = V(\mathcal{I}_1) \cup V(\mathcal{I}_2) \cup \ldots \cup V(\mathcal{I}_r).$$

Let $\Sigma = \Sigma_{\mathcal{F}} : M(\mathcal{F}) \longrightarrow R$ be the no where zero choice multiplicative monoid map using Theorem 12. Let

$$([\Sigma(\mathcal{J}_{11})v_{11} : \Sigma(\mathcal{J}_{12})v_{12} : \ldots : \Sigma(\mathcal{J}_{1k})v_{1k}], [\Sigma(\mathcal{J}_{21})v_{21} : \Sigma(\mathcal{J}_{22})v_{22} : \ldots : \Sigma(\mathcal{J}_{2k})v_{2k}], \ldots,$$
$$[\Sigma(\mathcal{J}_{r1})v_{r1} : \Sigma(\mathcal{J}_{r2})v_{r2} : \ldots : \Sigma(\mathcal{J}_{rk})v_{rk}]) \in \prod_{i=1}^{r} \mathbb{PF}^{k-1}_{\mathcal{I}_i}.$$

Let $\mathcal{I} = \prod_{i=1}^{r} \mathcal{I}_i$. We note that $(v_{ij}) + \mathcal{I} = R$ for every $(i,j) \in \{1,2,\ldots,r\} \times \{1,2,\ldots,k\}$. We replace v_{ij} by $w_{ij} \in v_{ij} + \mathcal{I}$ such that the following two property holds.

- For every $(i,j) \neq (e,f) \in \{1,2,\ldots,r\} \times \{1,2,\ldots,k\}$ the sets of maximal ideals containing w_{ij} and w_{ef} are disjoint i.e. $V(w_{ij}) \cap V(w_{ef}) = \emptyset$.
- For every $(i,j),(e,f) \in \{1,2,\ldots,r\} \times \{1,2,\ldots,k\}$ the sets of maximal ideals containing w_{ij} and $\Sigma(\mathcal{J}_{ef})$ are disjoint i.e. $V(w_{ij}) \cap V(\Sigma(\mathcal{J}_{ef})) = \emptyset$.

This can be done using the Theorem 5 on Arithmetic Progressions for Dedekind Domains. This immediately implies that for each i we have a well defined element representing the same element

$$[\Sigma(\mathcal{J}_{i1})w_{i1} : \Sigma(\mathcal{J}_{i2})w_{i2} : \ldots : \Sigma(\mathcal{J}_{ik})w_{ik}] = [\Sigma(\mathcal{J}_{i1})v_{i1} : \Sigma(\mathcal{J}_{i2})v_{i2} : \ldots : \Sigma(\mathcal{J}_{ik})v_{ik}] \in \mathbb{PF}^{k-1}_{\mathcal{I}_i}$$

We observe that any maximal ideal containing the coordinates $\Sigma(\mathcal{J}_{ij})w_{ij}, 1 \leq j \leq k$ contains all $\Sigma(\mathcal{J}_{ij})$ for $1 \leq j \leq k$ and hence has to be a unit ideal which is a contradiction.

Now for a fixed $1 \leq j \leq k$ we observe that the maximal ideals containing $\Sigma(\mathcal{J}_{ij})$ outside $V(\mathcal{I}_i)$ distinct for $1 \leq i \leq r$. For this purpose we use the Observation 1 and we have $\mathcal{J}_{ij} \supset \mathcal{I}_i$ for $1 \leq i \leq r$ with $\mathcal{I}_i$ being mutually comaximal.

We have for $1 \leq j \leq k$

$$\prod_{i=2}^{r} \Sigma(\mathcal{J}_{ij}) = \Sigma(\prod_{i=2}^{r} \mathcal{J}_{ij}).$$

Now consider a maximal ideal $\mathcal{M}$ containing the set $\{\prod_{i=2}^{r} \Sigma(\mathcal{J}_{ij}) \mid j = 1,\ldots,k\}$. Then $\mathcal{M}$ contains one of the factors $\Sigma(\mathcal{J}_{ij})$ for some $2 \leq i \leq r, 1 \leq j \leq k$.

Again for $1 \leq j \leq k$, in the factorization of $\prod_{i=2}^{r} \mathcal{J}_{ij}$, for any maximal ideal $\mathcal{M}_i \in \mathcal{F}$, the maximal ideal $\mathcal{M}_i$ does not occur to the same power for all $1 \leq j \leq k$. So if $\mathcal{M} \notin \mathcal{F}$ then it is a maximal ideal containing $\Sigma(\mathcal{M}_i^{t_i}), \Sigma(\mathcal{M}_j^{s_j})$ for some $i \neq j$ or if $i = j$ then $t_i \neq s_i = s_j$ which contradicts the Observation 1.

Now suppose $\mathcal{M} \in \mathcal{F}$, then it immediately follows that $\mathcal{M}$ contains $\{\Sigma(\mathcal{J}_{ij}) \mid j = 1,\ldots,k\}$ for a fixed subscript $1 \leq i \leq r$ which implies that $\mathcal{M}$ is a unit ideal which is a contradiction. So the the set

$$\{\prod_{i=2}^{r} \Sigma(\mathcal{J}_{ij}) \mid j = 1,\ldots,k\}$$

is unital.

Similarly now the set

$$\{\prod_{i=2}^{r} \Sigma(\mathcal{J}_{ij})w_{ij} = \Sigma(\prod_{i=2}^{r} \mathcal{J}_{ij})w_{ij} \mid j = 1,\ldots,k\}.$$

is also unital.

Now consider the element

$$[\prod_{i=2}^{r}\Sigma(\mathcal{J}_{i1})w_{i1} : \prod_{i=2}^{r}\Sigma(\mathcal{J}_{i2})w_{i2} : \ldots : \prod_{i=2}^{r}\Sigma(\mathcal{J}_{ik})w_{ik}] \in \mathbb{PF}^k_{\left(\left(\prod_{i=2,j=1}^{r,k}\Sigma(\mathcal{J}_{ij})w_{ij}\right)\mathcal{I}_2\mathcal{I}_3\ldots\mathcal{I}_k\right)}.$$

Now we reduce to the case when $r = 2$ and apply this Theorem 17 for the above element in

$$\mathbb{PF}^k_{\left(\left(\prod_{i=2,j=1}^{r,k}\Sigma(\mathcal{J}_{ij})w_{ij}\right)\mathcal{I}_2\mathcal{I}_3\ldots\mathcal{I}_k\right)}$$

and the element $[\Sigma(\mathcal{J}_{11})w_{11} : \Sigma(\mathcal{J}_{12})w_{12} : \ldots : \Sigma(\mathcal{J}_{1k})w_{1k}] \in \mathbb{PF}^k_{\mathcal{I}_1}$. We note that the two ideals

$$\left(\left(\prod_{i=2,j=1}^{r,k}\Sigma(\mathcal{J}_{ij})w_{ij}\right)\mathcal{I}_2\mathcal{I}_3\ldots\mathcal{I}_k\right), \mathcal{I}_1$$

are comaximal.

Now there exists elements $b_{1j} : 1 \leq j \leq k$ with $b_{1j} \equiv \Sigma(\mathcal{J}_{1j})w_{1j} \mod \mathcal{I}_1$ and

$$[b_{11} : b_{12} : \ldots : b_{1k}] = [\Sigma(\mathcal{J}_{11})w_{11} : \Sigma(\mathcal{J}_{12})w_{12} \ldots : \Sigma(\mathcal{J}_{1k})w_{1k}] \in \mathbb{PF}^k_{\mathcal{I}_1}$$

and there exists $t_1, t_2, \ldots, t_k \in \mathcal{I}_2\mathcal{I}_3\ldots\mathcal{I}_k$ such that

$$\sum_{j=1}^{k} b_{1j}\left(\prod_{i=2}^{r}\Sigma(\mathcal{J}_{ij})w_{ij} + t_j\prod_{i=2,j=1}^{r,k}\Sigma(\mathcal{J}_{ij})w_{ij}\right) = 1.$$

Now consider the same element with these representatives

$([b_{11} : b_{12} : \ldots : b_{1k}], [b_{21} : b_{22} : \ldots : b_{2k}], \ldots, [b_{r1} : b_{r2} : \ldots : b_{rk}]) \in \mathbb{PF}^k_{\mathcal{I}_1} \times \mathbb{PF}^k_{\mathcal{I}_2} \times \ldots \times \mathbb{PF}^k_{\mathcal{I}_r}$ where for $r > i > 1$ we have $b_{ij} = \Sigma(\mathcal{J}_{ij})w_{ij}$ and for $i = r$ we have

$$b_{rj} = \Sigma(\mathcal{J}_{rj})\left(w_{rj} + t_j\frac{\prod_{i=2,j=1}^{r,k}\Sigma(\mathcal{J}_{ij})w_{ij}}{\prod_{i=2}^{r-1}\Sigma(\mathcal{J}_{ij})w_{ij}}\right).$$

Then we observe that

$$\sum_{j=1}^{k}\prod_{i=1}^{r} b_{ij} = 1$$

The map λ is surjective and the Theorem 17 follows for any $r > 2$. □

13. **Acknowledgements**

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