GEOMETRY OF THE MULTIPLICATIVELY CLOSED SETS GENERATED BY AT MOST TWO ELEMENTS AND ARBITRARILY LARGE GAPS

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Abstract. We prove that the multiplicatively closed subset generated by at most two elements in the set of natural numbers $\mathbb{N}$ has arbitrarily large gaps by explicitly constructing large integer intervals which do not contain any element from the multiplicatively closed set. We also give a criterion by using a geometric correspondence between maximal singly generated multiplicatively closed sets and points of the space $\mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$ as to when a finitely generated multiplicatively closed set gives rise to a doubly multiplicatively closed line.

In the appendix section we discuss another constructive proof for arbitrarily large gap intervals where the prime factorization is not known for the right end-point unlike the constructive proof of the main result of the article in the case of multiplicatively closed set $\{p_1^i p_2^j \mid i,j \in \mathbb{N}\cup\{0\}\}$ with $p_1 < p_2, Log_{p_1}(p_2)$ irrational for which the prime factorization is known for both the end-points of the gap interval via the stabilization sequence of the irrational $\frac{1}{Log_{p_1}(p_2)}$.

1. Introduction

1.1. The Main Result. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of natural numbers. Here in this article we prove the following main result. This result is stated as follows:

Theorem 1. *Let $\mathbb{PP} \neq \{1\}$ be a nonempty set of at most two natural numbers. Let $\mathbb{S} = \{1 < a_1 < a_2 < \ldots < a_n < \ldots\} \subset \mathbb{N}$ be the infinite multiplicatively closed set generated by $\mathbb{PP}$. Then we have*

$$\underset{n\longrightarrow\infty}{limsup}(a_{n+1} - a_n) = \infty.$$

As an application of this Theorem 1 we have the following corollary.

Corollary 1. *Let $\mathbb{PP} = \{p_1 < p_2\}$ be a set of two primes. Consider the multiplicatively closed set $\mathbb{S} = \{1 < a_1 < a_2 < \ldots < a_n < \ldots\} \subset \mathbb{N}$ generated by $\mathbb{PP}$. Then we have*

$$\underset{n\longrightarrow\infty}{limsup}(a_{n+1} - a_n) = \infty.$$

1.2. Structure of the Paper.

In section 2 we associate to every multiplicatively closed set a point in the projective space $\mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$ and conversely to every point, a maximal singly generated multiplicatively closed set in Theorem 2. Then we characterize when two points $P_1, P_2 \in \mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$ give rise to the same point in terms of Log-Rationality in Theorem 3. In Theorem 4 we give a criterion for when a finitely generated multiplicatively closed set is contained in doubly generated multiplicatively closed set and in Theorem 5 we classify doubly multiplicatively closed lines (Refer Definition 4).

In section 3 we first show that for any two relatively prime numbers $1 < p < q$ the gaps between successive approximate inverses of $p \mod q$ is increasing in Theorem 6. In Theorems 7, 8 we prove for a sequence of positive rationals converging to an irrational in $[0, 1]$,

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the sequence of approximate inverses eventually stabilize and the gaps between successive approximate inverses increase.

In section 4 we prove our main Theorem 1. We consider a multiplicatively closed set $\mathbb{S}$ generated by two positive numbers p_1, p_2 representing two distinct points in $\mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$. We apply Theorems 7, 8 to $Log_{p_1}(p_2)$ for a suitable sequence of positive rationals obtained in Lemma 2 and conclude increasing gaps for the stabilized sequence. Then we locate integer intervals in Lemma 3 of arbitrarily large size which has no elements from the multiplicatively closed set $\mathbb{S}$. This finally proves our main Theorem 1.

1.3. **An Open Question.** This article leads to the following open question.

Question 1. *Let $\mathbb{S} = \{1 < a_1 < a_2 < \ldots <\} \subset \mathbb{N}$ be a finitely generated multiplicatively closed infinite set generated by positive integers $d_1, d_2, \ldots, d_n$. How do we construct explicitly arbitrarily large integer intervals which do not contain any elements from the set $\mathbb{S}$ using the positive integer $d_1, d_2, \ldots, d_n$?*

If a multiplicatively closed set $\mathbb{S}$ is generated by $r-$elements and these generators give rise to $s-$distinct points in the projective space $\mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$ (Refer Definition 1) with $s \leq r$ then $\mathbb{S}$ is contained in a multiplicatively closed set $\mathbb{T}$ which is generated by $s-$elements. So the Theorem 1 can be used to answer Question 1 whenever $s \leq 2$ in the affirmative using the same construction(Refer Sec 4). Even otherwise also, if these $s-$points generate a doubly multiplicative closed line (Refer Definition 4 and Theorems 4,5) then the Theorem 1 can be used to answer Question 1 in the affirmative using the same construction(Refer Sec 4).

The Theorem 4 and the Example 1 leads to the following interesting question which is answered in Theorem 5. Before we state the question we need some definitions.

Definition 1. *Let $\mathbb{Q}$ denote the field of rational numbers. Let $\mathbb{Q}_{\geq 0}$ denote the set of non-negative rationals. Define an equivalence relation $\sim_R$ on*

$$\mathbb{Q}^{\infty}_{\geq 0}\backslash\{0\} = \bigoplus_{i=1}^{\infty}\mathbb{Q}_{\geq 0}\backslash\{0\}.$$

We say $(a_1, a_2, \ldots,) \sim_R (b_1, b_2, \ldots,) \in \mathbb{Q}^{\infty}_{\geq 0}\backslash\{0\}$ if there exists $\lambda \in \mathbb{Q}^+$ such that $a_i = \lambda b_i$ for all $i \geq 1$. Let $\mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$ denote the projective space

$$\mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}} = \frac{\mathbb{Q}^{\infty}_{\geq 0}\backslash\{0\}}{\sim_R}$$

Definition 2. *Let $\mathbb{Q}$ denote the field of rational numbers. Define an equivalence relation on*

$$\bigoplus_{i\geq 1}\mathbb{Q}\backslash\{0\}.$$

We say $(a_1, a_2, \ldots, a_n) \sim_R (b_1, b_2, \ldots, b_n)$ if $a_i = \lambda b_i$ for some $\lambda \in \mathbb{Q}^*$. Let $\mathbb{PF}^{\infty}_{\mathbb{Q}}$ denote the space

$$\mathbb{PF}^{\infty}_{\mathbb{Q}} = \frac{\bigoplus_{i\geq 1}\mathbb{Q}\backslash\{0\}}{\sim_R}.$$

Once the inclusions are defined we have $\mathbb{PF}^{\infty}_{\mathbb{Q}} = \bigcup_{n\geq 1}\mathbb{PF}^n_{\mathbb{Q}} = \underset{\overrightarrow{n}}{\lim}\mathbb{PF}^n_{\mathbb{Q}}$. The space $\mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}} \subset \mathbb{PF}^{\infty}_{\mathbb{Q}}$ as the subset of points which have integer representatives. We note that two finite tuples which have positive coordinates are rational multiples of each other then they are positive rational multiples of each other.

Definition 3. *Let $\mathbb{P} = \{p_1 = 2, p_2 = 3, p_3 = 5, \ldots,\} \subset \mathbb{N}$ be the set of primes where p_i denote the $i^{th}-$prime. We say a set $\mathbb{S} \subset \mathbb{N}$ is singly generated multiplicatively closed if $\mathbb{S} = \{1, f, f^2, \ldots,\}$ for some $f \in \mathbb{N}, f \neq 1$. We say $\mathbb{S}$ is a singly generated maximal multiplicatively closed set if $\mathbb{T}$ is any singly generated multiplicatively closed set and $\mathbb{T} \supset \mathbb{S}$ then $\mathbb{T} = \mathbb{S}$.*

Definition 4. *Let L be a line obtained by joining two points $P_1, P_2 \in \mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}} \subset \mathbb{PF}^{\infty}_{\mathbb{Q}}$. We say L is a doubly multiplicatively closed line if we consider only integers and (not elements of $\mathbb{Q}_{\geq 0} \backslash \mathbb{Z}_{\geq 0}$) associated to all tuples whose equivalence classes lie on points of L (Refer to the Proof of Theorem 2) then it gives rise to a doubly generated multiplicatively closed set. In view of the Example 1 not all lines L are doubly multiplicatively closed lines.*
However we note that each point $P \in \mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$ gives rise to a unique maximal singly generated multiplicatively closed set (See Theorem 2).

Now the question is stated as follows:

Question 2. *Classify all lines L obtained by joining two points $P_1, P_2 \in \mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}} \subset \mathbb{PF}^{\infty}_{\mathbb{Q}}$ which are doubly multiplicatively closed lines.*

We answer this question completely in Theorem 5.

1.4. **History.** Historically the distribution of integers with exactly $k-$distinct prime factors has been studied by many authors. It was first shown by Landau [3] that for a fixed $k \geq 1$, the function defined by

$$\pi(x, k) = \sum_{n \leq x} f_k(n)$$

where $f_k(n) = 1$ if n has exactly k-prime factors and 0 otherwise satisfies

$$\pi(x, k) = \left(\frac{x}{log\ x}\right) \frac{(log\ log\ x)^{k-1}}{(k-1)!}(1 + o(1)). \tag{1}$$

Among the other authors who have obtained similar or better asymptotic expressions are Sathe, L.G. [4], Selberg [6],Hensely [1],Hildebrand and Tenenbaum [2].
Let $\{p_1, p_2, \ldots, p_k\}$ be any set of $k-$distinct primes. Let $\mathbb{S}_{\{p_1,p_2,\ldots,p_k\}}$ be the multiplicatively closed set generated by 1 and numbers which have exactly and all the factors from $\{p_1, p_2, \ldots, p_k\}$. Let $\mathcal{C}$ be the collection of all $k-$subsets of prime numbers. Consider the set

$$\mathbb{S}_k = \bigcup_{c \in \mathcal{C}} \mathbb{S}_c$$

Using any of the results say the result by Landau [3] about asymptotics of $\pi(x, k)$ we conclude that there are arbitrarily large gaps in $\mathbb{S}$. We observe here that using Equation 1 we have

$$\lim_{x \longrightarrow \infty} \frac{\pi(x, k)}{x} = 0$$

If the gaps were bounded then we have that

$$\liminf_{x \longrightarrow \infty} \frac{\pi(x, k)}{x} > 0$$

would be a non-zero constant. Hence the gaps must be arbitrarily large in the set $\mathbb{S}_k$.
With an additional bit of effort on the result of Landau [3] we can extend and conclude arbitrary large gaps for the set

$$\bigcup_{i=1}^{k} \mathbb{S}_i.$$

Now choose a base say $b = 2$. If we use asymptotics for a multiplicatively closed set $\mathbb{T}$ generated by primes $\{p_1, p_2, \ldots, p_k\}$ then we get for large x the following inequality

$$\lceil \frac{log_b\ x}{\sum\limits_{i=1}^{k} log_b\ p_i} \rceil \leq \#(\mathbb{T} \cap [1, x]) \leq \prod_{i=1}^{k} \lceil \frac{log_b\ x}{log_b\ p_i} \rceil$$

Hence again we have

$$\lim_{x \longrightarrow \infty} \frac{\#(\mathbb{T} \cap [1, x])}{x} = 0$$

from which we will be able to conclude that there are arbitrarily large gaps in $\mathbb{T}$.
However here in this article we give a more constructive proof for multiplicatively closed sets which are contained in doubly generated multipicatively closed sets. First we consider multiplicatively closed sets generated by two primes or two positive integers (> 1) which are not Log-Rational to each other. We note here that the multiplicatively closed set can contain numbers with single prime factor unlike the set which is considered in the result by Landau [3]. Using the technique of rational approximation and stabilization of the sequences of approximate inverses and increasing gaps between two such successive ones we explicitly construct by locating large intervals of natural numbers which do not contain any element in the given multiplicatively closed set there by proving Theorem 1 given below.

2. Geometry of Singly and Doubly Generated Multiplicatively Closed sets

We start the section with a correspondence theorem.

Theorem 2. *Let $\mathcal{S} = \{\mathbb{S} \subset \mathbb{N} \mid$ such that $\mathbb{S}$ is maximal singly generated.$\}$ There is a bijective correspondence between*

$$\mathcal{S} \longleftrightarrow \mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$$

i.e. between maximal singly generated multiplicatively closed sets and the points of the space $\mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$.

Proof. The bijection is given as follows. Let

$$\mathbb{S} = \{1, f, f^2, \ldots\}$$

be any singly generated multiplicatively closed set. Let

$$f = \prod_{j=1}^{k} p_{i_j}^{r_{i_j}} \text{ with } p_{i_1}, p_{i_2}, \ldots, p_{i_k} \in \mathbb{P}, r_{i_1}, \ldots, r_{i_k} \in \mathbb{N}$$

To this multiplicatively closed set we associate the point

$$P = [\ldots : r_{i_1} : \ldots : r_{i_2} : \ldots : \ldots : \ldots : r_{i_k} : \ldots] \in \mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}.$$

The condition that $\mathbb{S}$ is maximal is equivalent to the condition

$$gcd(r_{i_1}, r_{i_2}, \ldots, r_{i_k}) = 1.$$

Also given any point P in $\mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$ there is a unique integer coordinate representative of P with *gcd* of the coordinates equal to one which gives rise to the integer $f \in \mathbb{N}$ with $f \neq 1$. This establishes the bijection and hence the Theorem 2 follows. □

Theorem 3 (Log-Rationality)**.** *Let P_1, P_2 be two points (possibly the same point) in $\mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$. Let*

$$g_1 = \prod_{j=1}^{t} p_{i_j}^{r_{i_j}}, g_2 = \prod_{j=1}^{u} q_{i_j}^{s_{i_j}}$$

two positive integers (>1) *with their unique prime factorizations such that*

$$P_1 = [\ldots : r_{i_1} : \ldots : r_{i_2} : \ldots : \ldots : \ldots : r_{i_t} : \ldots]$$
$$P_2 = [\ldots : s_{i_1} : \ldots : s_{i_2} : \ldots : \ldots : \ldots : s_{i_u} : \ldots]$$

Let f_1, f_2 *be the corresponding positive integers* (>1) *under the bijection given in the Theorem 2 then the following are equivalent.*

1. *(Log-Rationality:)* $Log_{g_1}(g_2)$ *is rational.*
2. $P_1 = P_2$
3. $f_1 = f_2$.
4. *The multiplicatively closed set* $\mathbb{T} = \{g_1^i g_2^j \mid i, j \geq 0\}$ *is contained in a singly generated maximal multiplicatively closed set.*

Proof. Suppose $Log_{g_1}(g_2) = \frac{m}{n}$ is rational. Then we have $g_2^n = g_1^m$. So the distinct prime factors of g_1, g_2 agree and we also have that their exponents are projectively equivalent. Hence we get $P_1 = P_2$. So this implies $f_1 = f_2 = f$ say. Then we get that $\mathbb{T} \subset \{1, f, f^2, \ldots, \}$. For the converse if $\mathbb{T} \subset \{1, f, f^2, \ldots, \}$ for some $1 \neq f \in \mathbb{N}$ then $g_1 = f^n$, $g_2 = f^m$ and we have $g_2^n = g_1^m$. Hence $Log_{g_1}(g_2) = \frac{m}{n}$ is rational. This completes the equivalence of the statements $(1), (2), (3), (4)$ and also proves the Theorem 2. □

Now we have the following corollary.

Corollary 2. *A multiplicatively closed set* $\mathbb{T} = \{g_1^i g_2^j \mid i, j \geq 0, g_1, g_2 \in \mathbb{N} \backslash \{1\}\}$ *is not contained in a singly generated multiplicatively closed set if and only if* $Log_{g_1}(g_2), Log_{g_2}(g_1)$ *are both irrational if and only if* g_1, g_2 *represent two distinct points in the Projective Space* $\mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$.

In the theorem that follows we give a criterion as to when a multiplicatively closed set is contained in a doubly generated multiplicatively closed set.

Theorem 4. *Let* $\mathbb{S} = \{g_1^{i_1} g_2^{i_2} \ldots g_r^{i_r} \mid i_1, i_2, \ldots, i_r \in \mathbb{N} \cup \{0\}\}$ *be a multiplicatively closed set generated by* $r-$*elements. Suppose corresponding to these positive integers* $g_i : 1 \leq i \leq r$ *the points* $[g_i] : 1 \leq i \leq r \in \mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}} \subset \mathbb{PF}^{\infty}_{\mathbb{Q}}$ *lie on a projective line* L *obtained by joining two points of* $\mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$ *whose corresponding integers are relatively prime. Then* $\mathbb{S}$ *is contained in a doubly generated multiplicatively closed set.*

Proof. Let $P_1, P_2 \in \mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$ be any two distinct points which gives rise to the projective line L. Let p_1, p_2 be the positive integers which represent these points P_1, P_2 with $gcd(p_1, p_2) = 1$. Then the hypothesis that the points $[g_i]$ lie on the projective line $P_1 P_2$ implies that there exists integers $a_i, b_i, c_i \geq 0$ such that $p_1^{a_i} p_2^{b_i} = g_i^{c_i}$ for $1 \leq i \leq r$. Consider the unique prime factorization of

$$p_1 = q_1^{s_1} q_2^{s_2} \ldots q_l^{s_l}, p_2 = q_1^{t_1} q_2^{t_2} \ldots q_l^{t_l}.$$

where we assume without loss of generality that $gcd(s_1, s_2, \ldots, s_l) = 1, gcd(t_1, t_2, \ldots, t_l) = 1$. If in addition we have $gcd(p_1, p_2) = 1$ then we have $s_j t_j = 0 : 1 \leq j \leq l$ but one of s_j and t_j is non-zero for each j. In all cases we conclude that $c_i \mid s_j a_i, c_i \mid t_j b_i$ for $1 \leq j \leq l$. So $c_i \mid a_i, c_i \mid b_i : 1 \leq i \leq r$ as $gcd(s_1, s_2, \ldots, s_l) = 1, gcd(t_1, t_2, \ldots, t_l) = 1$. Hence the set $\mathbb{T} = \{p_1^i p_2^j \mid i, j \geq 0\} \supset \mathbb{S}$ and this proves Theorem 4. □

Example 1. *Let* $g_1 = 45, g_2 = 20, g_3 = 30$. *Then we have* $g_1 g_2 = g_3^2$. *So the doubly generated multiplicatively closed set generated by* g_1, g_2 *contains* g_3^2 *but not* g_3. *However there is no doubly generated multiplicatively closed set containing all* g_1, g_2, g_3 *because there are no two distinct non-trivial common factors of* g_1, g_2, g_3 *as* $gcd(g_1, g_2, g_3) = 5$ *which is prime. Now the corresponding exponents satisfy*

$$[0 : 2 : 1 : 0 : \ldots] + [2 : 0 : 1 : 0 : \ldots] = 2.[1 : 1 : 1 : 0 : \ldots].$$

since $g_1g_2 = g_3^2$ and the exponent vectors lie on a Projective Line $L \subset \mathbb{PF}^{\infty}_{\mathbb{Q}}$.

Note 1. *The Theorem 4 can be generalized as follows. Let $\mathbb{S} = \{g_1^{i_1}g_2^{i_2}\ldots g_r^{i_r} \mid i_1, i_2, \ldots, i_r \in \mathbb{N} \cup \{0\}\}$ be a multiplicatively closed set generated by $r-$elements. Fix a prime $p = 2$. Suppose there exists two positive integers p_1, p_2 such that the monoid $\{aLog_p(p_1) + bLog_p(p_2) \mid a, b \in \mathbb{Z}_{\geq 0}\}$ contains the set $\{Log_p(g_i) : 1 \leq i \leq r\}$ then the set $\mathbb{S} \subset \mathbb{T} = \{p_1^ip_2^j \mid i, j \geq 0\}$.*

Note 2. *In Example 1, for all integer representatives $f \in \mathbb{N}$ such that $[f] \in L$ we have $5 \mid f$. This is the only prime with this property for the line L which is not a doubly multiplicatively closed line. So does there exist a doubly multiplicatively closed line with such a prime? Definitely not when there are only two primes involved with the line L.*
In the Example 1, we have the following properties holding true.

- *For all integer representatives $f \in \mathbb{N}$ such that $[f] \in L$ we have $5 \mid f$ and this is the only such prime. Neither of the primes $2, 3$ satisfy this property.*
- *There exists numbers $g_1 = 45, g_2 = 20$ whose points lie on L and two primes $2, 3$ such that*
$$3 \mid 45, 3 \nmid 20, 2 \mid 20, 2 \nmid 45.$$
- *The lattice M corresponding to L is a two dimensional lattice which does NOT possess a basis $\{x, y\}$ such that $M \cap \mathbb{Z}^r_{\geq 0}$ satisfies the monoid addition property. i.e.*
$$\boxed{ax + by \in M \cap \mathbb{Z}^r_{\geq 0} \Leftrightarrow a \geq 0, b \geq 0}$$

Theorem 5. *A line L joining two points $P_1, P_2 \in \mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}} \subset \mathbb{PF}^{\infty}_{\mathbb{Q}}$ is a doubly multiplicatively closed line if and only if there exists two points $Q_1 = [q_1], Q_2 = [q_2] \in \mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}} \subset \mathbb{PF}^{\infty}_{\mathbb{Q}}$ with positive integers*
$$q_1 = p_1^{a_1}p_2^{a_2}\ldots p_r^{a_r}, q_2 = p_1^{b_1}p_2^{b_2}\ldots p_r^{b_r}.$$
with the following properties.

(1) Trivial Index Property(Alternative): The gcd of two by two minors of $\begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}$ is one.
(2) Monoid Addition Property(Alternative): There exists two subscripts i, j such that $a_ib_i = 0 = a_jb_j$ and either $a_ib_j \neq 0$ or $a_jb_i \neq 0$.

In particular if there exists two points $Q_1, Q_2 \in L \cap \mathbb{PF}^{\infty}_{\mathbb{Q}_{\geq 0}}$ such that their corresponding integer representatives are relatively prime then L is a doubly multiplicatively closed line.

Proof. First we prove the latter assertion. Suppose there exists such points Q_1, Q_2 on L and let q_1, q_2 be the corresponding integers. Let
$$q_1 = p_1^{s_1}p_2^{s_2}\ldots p_l^{s_l}, q_2 = p_{l+1}^{s_{l+1}}p_{l+2}^{s_{l+2}}\ldots p_n^{s_n}.$$
be their unique prime factorizations with $s_i \in \mathbb{N} : 1 \leq i \leq n, gcd(q_1, q_2) = 1$. Now we choose q_1, q_2 such that $gcd(s_1, s_2, \ldots, s_l) = 1 = gcd(s_{l+1}, s_{l+2}, \ldots, s_n)$.
If $g \in \mathbb{N}$ such that $[g] \in L$ then there exist $a, b, c \in \mathbb{N}$ such that $q_1^aq_2^b = g^c$. This implies
$$c \mid as_i : 1 \leq i \leq l, c \mid bs_i : l + 1 \leq i \leq n \Rightarrow c \mid a, c \mid b$$
So $g \in \mathbb{T} = \{q_1^iq_2^j \mid i, j \in \mathbb{N} \cup \{0\}\}$. In this particular case we also have in the matrix
$$\begin{pmatrix} s_1 & \ldots & s_l & 0 & \ldots & 0 \\ 0 & \ldots & 0 & s_{l+1} & \ldots & s_n \end{pmatrix}$$
has the property that its gcd of two minors equal $gcd(s_is_j : 1 \leq i \leq l, l + 1 \leq j \leq n) = 1$. This proves the latter assertion.

Now we prove the first assertion. If L has such points Q_1, Q_2. Let $V = \mathbb{Q} - span$ of the vectors $\{a = (a_1, \ldots, a_r), (b_1, \ldots, b_r)\}$. Let $M = \mathbb{Z} - span$ of the vectors $\{a, b\}$. Then we have the following properties.

- Trivial Index Property:
$$V \cap \mathbb{Z}^r = M.$$
This follows because of the gcd of the 2×2 minors is one. i.e. M has a trivial index in $V \cap \mathbb{Z}^r$. Using the theorem for sublattices we get that for the tower of sublattices
$$M \subset V \cap \mathbb{Z}^r \subset \mathbb{Z}^r$$
that there exists a basis of $\mathbb{Z}^r$ given by $\{u_1, u_2, \ldots, u_r\}$ and positive integers d_1, d_2 such that $\{u_1, u_2\}$ is a basis of $V \cap \mathbb{Z}^r$ and $\{d_1u_1, d_2u_2\}$ is a basis of M which has therefore index d_1d_2 which is also gcd of the 2×2 minors of $\{d_1u_1, d_2u_2\}$. Since $\{d_1u_1, d_2u_2\}$ differ from the basis $\{a, b\}$ of M by an $SL_2(\mathbb{Z})$ matrix we have $d_1d_2 = 1$.
- Monoid Addition Property:
$$\alpha a + \beta b \in M \cap \mathbb{Z}^r_{\geq 0} \Leftrightarrow \alpha \geq 0, \beta \geq 0.$$
This follows because there exists two subscripts i, j such that $a_ib_i = 0 = a_jb_j$ and either $a_ib_j \neq 0$ or $a_jb_i \neq 0$ and the coordinate entries of both a, b are non-negative.

So we get that if $g \in \mathbb{N}$ such that $[g] \in L$ then $g = q_1^i q_2^j$ for some $i, j \in \mathbb{N} \cup \{0\}$. So the required multiplicatively closed set representing the line is $\mathbb{T} = \{q_1^i q_2^j \mid i, j \in \mathbb{N} \cup \{0\}\}$.
Now we prove the converse. Suppose L is a multiplicatively closed line with the multiplicatively closed set being $\mathbb{T} = \{q_1^i q_2^j \mid i, j \in \mathbb{N} \cup \{0\}\}$. So if $g \in \mathbb{N}$ such that $[g] \in L$ then $g \in \mathbb{T}$. Let the prime exponent vectors of q_1, q_2 be s, t with $s = (s_1, s_2, \ldots, s_r), t = (t_1, t_2, \ldots, t_r)$. This proof is a bit long. We prove both the Trivial Index Property and the Monoid Addition Property for $\{s, t\}$.

Claim 1. *Let V be a two dimensional $\mathbb{Q}-$vector space spanned by $s = (s_1, s_2, \ldots, s_r), t = (t_1, t_2, \ldots, t_r)$. Let $M = V \cap \mathbb{Z}^r$. Then there exists a basis $\{s, w\}$ for M where the coordinate entries of w are all non-negative.*

Proof of Claim. We observe that V is the corresponding affine space defined by the projective line L and also that $M = V \cap \mathbb{Z}^r$ is a two dimensional lattice. Now by a theorem on sublattices of $\mathbb{Z}^r$ it follows that there exists a basis of $\mathbb{Z}^r$ say
$$\{u = (u_1, u_2, \ldots, u_r), v = (v_1, v_2, \ldots, v_r), w^1, w^2, \ldots, w^{r-2}\}$$
and positive integers d_1, d_2 such that $\{d_1u, d_2v\}$ is a basis of M with $d_1 \mid d_2$. Since M contains a gcd one vector either s or t we have $d_1 = 1$. If $\alpha u + \beta v = s$ then $gcd(\alpha, \beta) = 1$ because $(\alpha) + (\beta) = (\alpha u_1, \alpha u_2, \ldots, \alpha u_r) + (\beta v_1, \beta v_2, \ldots, \beta v_r) \supset (\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2, \ldots, \alpha u_r + \beta v_r) = (s_1, s_2, \ldots, s_r) = \mathbb{Z}$. Hence there exists $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ such that
$$\begin{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & 0_{2\times(r-2)} \\ 0_{(r-2)\times 2} & I_{(r-2)\times(r-2)} \end{pmatrix} \begin{pmatrix} u \\ v \\ w^1 \\ \vdots \\ w^{r-2} \end{pmatrix} = \begin{pmatrix} s \\ w \\ w^1 \\ \vdots \\ w^{r-2} \end{pmatrix}$$
where $w = \gamma u + \delta v$. Apriori w need not have non-negative entries. However we note that the $\mathbb{Q}-Span$ of $\{u, d_2v\}, \{u, v\}, \{s, w\}$ are all equal to the two dimensional $\mathbb{Q}-$ vector space V defined by the projective line L. Using a unipotent lower triangular matrix over $\mathbb{Z}$ we need to consider only those entries of w whose corresponding entries in s are zero. Now by a similar procedure we conclude that $\mathbb{Q} - Span$ of $\{u, d_2v\}, \{u, v\}, \{t, x\}$ are the same for

some unital $x \in \mathbb{Z}^r$. Now the vector t which has non-negative entries lies in the span of s, w. i.e $t = \epsilon s + \mu w$ with $\epsilon \in \mathbb{Q}, \mu \in \mathbb{Q}^*$. Now if $s_i = 0, w_i \neq 0$ then $sign(w_i) = sign(\mu)$. If this sign is negative then we consider $-w$ instead of w. Then we get that the w_i has non-negative sign whenever s_i is zero. Now again using unipotent lower triangular matrix over $\mathbb{Z}$ we make the sign of the remaining entries of w non-negative. Hence we arrive at a basis $\{s, w\}$ such that both have non-negative integer entries and whose $\mathbb{Z}-$span is the same as $\mathbb{Z}-$apan of $\{u, v\}$ and whose $\mathbb{Q}-$span is exactly V. We obtain $\{s, w\}$ from $\{u, v\}$ by an $\tilde{S}L_2(\mathbb{Z})$ transformation with determinant ± 1. Now we also have the $\mathbb{Z}-$span M of $\{u, d_2 v\}$ is contained in the $\mathbb{Z}-$span of $\{s, w\}$. Here we observe that on the other hand $V \cap \mathbb{Z}^r$ contains $\mathbb{Z}-$span of $\{s, w\}$ as this span has integer entries. Hence we have $d_2 = 1$ and

$$\mathbb{Z} - \text{Span of}\{u, v\} = \mathbb{Z} - \text{Span of}\{s, w\} = M = V \cap \mathbb{Z}^r.$$

So we have obtained a basis $\{s, w\}$ of $M = V \cap \mathbb{Z}^r$ with non-negative entries. This proves the Claim 1 □

Claim 2 (Trivial Index Property). *Let $\{s, w\}$ be the basis of M obtained from the Claim 1. Let*

$$q_1 = p_1^{s_1} p_2^{s_2} \ldots p_r^{s_r}, e = p_1^{w_1} q_2^{w_2} \ldots p_r^{w_r}.$$

Since $M \cap \mathbb{Z}^r_{\geq 0}$ corresponds to a doubly multiplicatively closed set $\mathbb{T} = \{q_1^i q_2^j \mid i, j \in \mathbb{N} \cup \{0\}\}$ we have

- $$\{p_1^i e^j \mid i, j \in \mathbb{Z}\} = \{q_1^i q_2^j \mid i, j \in \mathbb{Z}\}.$$
- *The $\mathbb{Z} -$ span of $\{s, t\}$ is the same as $\mathbb{Z} -$ span of $\{s, w\}$.*
- *If for some $\alpha, \beta \in \mathbb{Q}, \alpha s + \beta t \in \mathbb{Z}^r$ then $\alpha, \beta \in \mathbb{Z}$.*

Proof of Claim. Since p_1, e corresponds to points in $M \cap \mathbb{Z}^r_{\geq 0}$ we have $p_1 = f_1^{l_1} f_2^{l_2}$ and $e = f_1^{m_1} f_2^{m_2}$. So we have $\{p_1^i e^j \mid i, j \in \mathbb{Z}\} \subset \{f_1^i f_2^j \mid i, j \in \mathbb{Z}\}$. The other way containment is immediate. Now the rest of the claim for exponents follows as $\{s, w\}$ is a $\mathbb{Z}-$basis for $M = V \cap \mathbb{Z}^r$ and $\mathbb{Q}-$basis for V. This proves the trivial index property for $\{s, t\}$. □

Claim 3 (Monoid Addition Property). *The basis $\{s, t\}$ has monoid addition property.*

Proof of Claim. Now suppose if all the coordinate entries of s is positive. Then for some large $m \in \mathbb{N}$ we have $ms - t$ has non-negative entries which is a contradiction. Hence there exist a subscript i such that $s_i = 0, t_i \neq 0$. Similarly there exist a subscript j such that $t_j = 0, s_i \neq 0$. This proves the monoid addition property that

$$\alpha s + \beta t \in M \cap \mathbb{Z}^r_{\geq 0} \Leftrightarrow \alpha \geq 0, \beta \geq 0.$$

□

This completes the proof of Theorem 5. □

Example 2. *Let $g_1 = 10$,$g_2 = 15$. Then the line joining the points $[g_1], [g_2]$ is a multiplicatively closed line using Theorem 5. where as the line in Example 1 joining $[\tilde{g}_1 = 20], [\tilde{g}_2 = 45]$ is not multiplicatively closed. Now we could also use Theorem 5 to prove this fact in another way.*

3. Irrationals and Behaviour of Rational Approximations, Arithmetic Progressions,Stabilization

We start this section by proving a theorem below on increasing gaps for the successive approximate inverses.

Theorem 6 (Increasing Gaps between Successive Approximate Inverses)**.** *Let p,q be two positive integers with $gcd(p,q)=1, p<q$. Consider the arithmetic progressions $p\mathbb{Z}^+$ and $q\mathbb{Z}^+$. Consider the sequence $(p\mathbb{Z}^+\cup q\mathbb{Z}^+\cup\{0\})\cap\{0,1,2,\ldots,qp\}$ in the set $\{0,1,2,\ldots,qp\}$.*

$$\begin{aligned}&l_0=0,p,2p,3p,\ldots,l_1p,q\\&(l_1+1)p,(l_1+2)p,\ldots,l_2p,2q,\\&(l_2+1)p,\ldots,l_ip,iq\\&(l_i+1)p,\ldots,(q-1)p,qp\end{aligned}$$

Now consider the sequence of numbers

$$\begin{aligned}&\{l_0=0\}\cup\{l_j\mid q\geq j\geq 1,(l_j+1)p-jq\mid (l_j+1)p-jq<\underset{0\leq i<j}{min}\{(l_i+1)p-iq\}\}\\&=\{l_{j_1},l_{j_2},\ldots,l_{j_r}\}\end{aligned}$$

$$\begin{cases}=\{0=l_0=l_{j_1}=p^{-1}-1 \mod q\} \text{ if } p=1\\=\{0=l_{j_1}=l_0<l_{j_2}=l_1<l_{j_3}<\ldots<(l_{j_r}=p^{-1}-1 \mod q)\} \text{ if } p\neq 1\end{cases}$$

Then the gaps $l_{j_{i+1}}-l_{j_i}$ in the above sequence is increasing.

Proof. If $p=1$ then there is nothing to prove. So assume $p>1$. First we observe that p is a unit in $\mathbb{Z}/q\mathbb{Z}=\{0,1,2\ldots,q-1\}$. The values (l_i+1) tend to the inverse of p because the least possible value for $(l_i+1)p-iq$ is one. If we consider the sequence of multiples $\{(l_{j_1}+1)p \ mod \ q,(l_{j_2}+1)p \ mod \ q,\ldots,(l_{j_r}+1)p \ mod \ q\}$ then the values are distinct and decrease to 1 as multiples of p given by $0,p,2p,\ldots,(q-1)p$ gives rise to all residue classes modulo q. Now suppose we consider three consecutive elements in the sequence $l_{j_i},l_{j_{i+1}},l_{j_{i+2}}$ then we have

$$\begin{aligned}(l_{j_i}+1)p&=k_{j_i}q+x_{j_i}\\(l_{j_{i+1}}+1)p&=k_{j_{i+1}}q+x_{j_{i+1}}\\(l_{j_{i+2}}+1)p&=k_{j_{i+2}}q+x_{j_{i+2}}\end{aligned}$$

$$\text{and the residue classes satisfy } x_{j_i}>x_{j_{i+1}}>x_{j_{i+2}}$$

and moreover for any $t<l_{j_{i+1}}-l_{j_i}$ we have

$$\text{if } (l_{j_i}+1+t)p=kq+x \text{ then } x>x_{j_i}$$

because of the minimality condition on $(l_{j_i}+1)p-j_iq$ as the lesser than $(l_{j_i}+1)$ multiples of p are not as close to multiples of q where we compare multiples of p to numbers which are smaller and multiples of q. So we have

$$(l_{j_{i+1}}+1+t)p=(l_{j_{i+1}}-l_{j_i})p+(l_{j_i}+1+t)p=(k_{j_{i+1}}-k_{j_i}+k)q+x_{j_{i+1}}-x_{j_i}+x$$

Now note in the right hand side we have the following inequalities for the residue classes *mod* q.

$$\begin{aligned}&0<x_{j_i}<q\\&0<x_{j_{i+1}}<q\\&0<x_{j_i}-x_{j_{i+1}}<q\\&0<x_{j_{i+1}}<x_{j_{i+1}}-x_{j_i}+x<x<q\end{aligned}$$

This is a subtle argument about the residue classes. Hence we have $l_{j_{i+2}} > l_{j_{i+1}} + t$ for all $t < l_{j_{i+1}} - l_{j_i}$ and for $t = l_{j_{i+1}} - l_{j_i}$ we have $x = x_{j_{i+1}}$ so a candidate for the residue class is $(2x_{j_{i+1}} - x_{j_i})$ and

$$(l_{j_{i+1}} + 1 + t)p = (2k_{j_{i+1}} - k_{j_i})q + (2x_{j_{i+1}} - x_{j_i})$$

So we have if $0 < (2x_{j_{i+1}} - x_{j_i})$ then the residue class is $(2x_{j_{i+1}} - x_{j_i})$ and

$$0 < (2x_{j_{i+1}} - x_{j_i}) = x_{j_{i+1}} + x_{j_{i+1}} - x_{j_i} < x_{j_{i+1}} < q$$

So $l_{j_{i+2}} = 2l_{j_{i+1}} - l_{j_i}$ or $l_{j_{i+2}} - l_{j_{i+1}} = l_{j_{i+1}} - l_{j_i}$. Otherwise if $0 < x_{j_{i+1}} < 2x_{j_{i+1}} < x_{j_i} < q$ then the residue class is given by $q + 2x_{j_{i+1}} - x_{j_i}$ and we observe that

$$q > q + 2x_{j_{i+1}} - x_{j_i} > x_{j_{i+1}} \text{ because } q > q + x_{j_{i+1}} - x_{j_i} > 0$$

we conclude that $l_{j_{i+2}} > 2l_{j_{i+1}} - l_{j_i}$ or $l_{j_{i+2}} - l_{j_{i+1}} > l_{j_{i+1}} - l_{j_i}$. It is also clear that the residue classes decrease to one. Now the Theorem 6 follows. The sequence

$$\{1 = l_0 + 1 = l_{j_1} + 1, l_{j_2} + 1, \ldots, l_{j_r} + 1\}$$

is the sequence of approximate inverses of $p \mod q$. □

Theorem 7 (Stabilization and Eventual Invariance)**.** *Let p_n, q_n be a sequence of positive integers with $gcd(p_n, q_n) = 1$ and suppose $\frac{p_n}{q_n}$ is a cauchy sequence converging to an irrational number $0 < \alpha < 1$. Define as in the previous lemma the sequence $l_i(n)$ and consider the set*

$$\{0 = l_{j_1(n)}(n) < l_{j_2(n)}(n) = l_1(n) < l_{j_3(n)}(n) < \ldots < l_{j_{r_n}(n)}(n) = p_n^{-1} - 1 \ mod\ q_n\}.$$

The values $j_i(n)$ stabilize and also $l_{j_i(n)}(n)$ is eventually a constant as $n \longrightarrow \infty$ for a stabilized j_i.

Proof. We can assume that $p_n < q_n$ and $p_n \neq 1$. If $p_n = 1$ for infinitely many positive integer $n > 0$ then $\frac{p_n}{q_n} \longrightarrow 0$ which is a contradiction. We observe that $l_i(n) = \lfloor \frac{iq_n}{p_n} \rfloor$ and for fixed $i, l_i(n)$ is eventually $\lfloor \frac{i}{\alpha} \rfloor$ as $n \longrightarrow \infty$. Also we have the sequence $j_i(n)$ stabilizes as $n \longrightarrow \infty$ because in the inductive definition, we have $j_i(n)$ satisfies the property that

$$(l_{j_i(n)}(n) + 1)p_n - j_i(n)q_n < \underset{0 \leq i < j_i(n)}{min} \{(l_i(n) + 1)p_n - iq_n\}$$

or equivalently that

$$(l_{j_i(n)}(n) + 1)\frac{p_n}{q_n} - j_i(n) < \underset{0 \leq i < j_i(n)}{min} \{(l_i(n) + 1)\frac{p_n}{q_n} - i\}.$$

Now if $n \longrightarrow \infty$ then we get that $(l_i(n) + 1)\frac{p_n}{q_n} - i \longrightarrow \left(\lfloor \frac{i}{\alpha} \rfloor + 1\right)\alpha - i$ which is independent of n. Now the independence of n here implies the stabilization of $j_i(n)$ follows as $n \longrightarrow \infty$. This completes the proof of this Theorem 7. □

Theorem 8. *Let p_n, q_n be a sequence of positive integers with $gcd(p_n, q_n) = 1$ with $p_n < q_n$ and suppose $\frac{p_n}{q_n}$ is a cauchy sequence converging to an irrational number $0 < \alpha < 1$. Define as in the previous lemma the sequence $l_i(n)$ and consider the set*

$$\{0 = l_{j_1(n)}(n) < l_{j_2(n)}(n) = l_1(n) < l_{j_3(n)}(n) < \ldots < l_{j_{r_n}(n)}(n) = p_n^{-1} - 1 \ mod\ q_n\}.$$

Using previous lemma let $j_i = \underset{n \longrightarrow \infty}{lim} j_i(n), l_i = \underset{n \longrightarrow \infty}{lim} l_i(n)$. Then we have

$$\underset{i \longrightarrow \infty}{lim} l_{j_{i+1}} - l_{j_i} = \infty$$

Proof. We can assume that $p_n \neq 1$ eventually. We observe that using the previous Theorem7 we have for every

$$i \in \mathbb{N}, l_{j_{i+2}} - l_{j_{i+1}} \geq l_{j_{i+1}} - l_{j_i}.$$

If the above limit is not infinity (say equal to d) then eventually l_{j_i} form an arithmetic progression with common difference d. Then $(l_{j_i}+1)=\lceil\frac{j_i}{\alpha}\rceil$ is in arithmetic progression with common difference d. On the one hand the sequence

$$\lceil\frac{j_i}{\alpha}\rceil\alpha - j_i \searrow 0$$

On the other hand the sequence has a distribution if l_{j_i} are in arithemetic progression. Because if $l_{j_i}=l_{j_{i_0}}+kd$ with $k\in\mathbb{N}$ and fractional parts z_{j_i} are such that $\frac{j_i}{\alpha}+z_{j_i}=\lceil\frac{j_i}{\alpha}\rceil=l_{j_i}+1$. Then we get $(l_{j_{i_0}}+kd+1)\alpha-j_i=z_{j_i}\alpha\searrow 0$. However the fractional parts $\{(l_{j_{i_0}}+kd+1)\alpha-j_i\}=\{(l_{j_{i_0}}+kd+1)\alpha\}$ are distributed in the unit interval uniformly as $k\in\mathbb{N}$ by Weyl's Criterion. So this is a contradiction and the lemma follows. □

We mention Weyl's Equidistributive Criterion here (See also [7].)

Theorem 9. *Let α be a positive irrational. Let $0\le a\le b\le 1$. For $x\in\mathbb{R}^+$, let $\{x\}$ denote the fractional part of x. Then we have*

$$\frac{\#\{n\mid a\le\{n\alpha\}\le b, 1\le n\le N\}}{N}\longrightarrow (b-a) \text{ as } N\longrightarrow\infty$$

4. The Main Theorem and Construction of Arbitrarily Large Gaps

Before we prove the main Theorem 1 we prove the following three lemmas.

Lemma 1. *Let $p_1<p_2$ be two natural numbers such that $gcd(p_1,p_2)=1$. Then*
- *Either $p_1=1$.*
- *Or $Log_{p_1}(p_2), Log_{p_2}(p_1)$ are both simultaneously irrationals.*

Proof. If $p_1=1$ then there is nothing to prove. Suppose $Log_{p_1}(p_2)=\frac{m}{n}$ for some positive integers $m,n>0$. Then we have $p_2^n=p_1^m$ a contradiction to unique factorization into primes. So $Log_{p_1}(p_2)$ is irrational. □

Definition 5. *We say a pair $(p_1,p_2)\in\mathbb{N}^2$ is an irrational pair if $p_1\neq 1$ and $p_2\neq 1$ and both $Log_{p_1}(p_2), Log_{p_2}(p_1)$ are irrationals. For example a $GCD-$one pair $(p_1,p_2)\in\mathbb{N}^2$ where $p_1\neq 1\neq p_2$ is an irrational pair.*

Lemma 2. *Let $(p_1,p_2)\in\mathbb{N}^2$ be such that $p_1<p_2$ and is an irrational pair. Let $\alpha=Log_{p_2}(p_1)<1$. Let $x_2(i)=\lceil\frac{i}{\alpha}\rceil$. For every positive integer i let*

$$z_i=-i+x_2(i)\alpha.$$

Define a subsequence with the property that $z_{k_j}<z_{k_{j-1}}=min\{z_1,z_2,\ldots,z_{k_j-1}\}$. Then
1. *$z_{k_j}\searrow 0$.*
2. *k_j-k_{j-1} is increasing.*
3. *$\lim\limits_{j\longrightarrow\infty}(k_j-k_{j-1})=\infty$.*

Proof. First we define a sequence of number parts $0<y_i<1$ defined by the equation

$$y_i+\frac{i}{\alpha}=\lceil\frac{i}{\alpha}\rceil=x_2(i).$$

Define a subsequence with the property that $y_{k_j}<y_{k_{j-1}}=min\{y_1,y_2,\ldots,y_{k_j-1}\}$. Since the number parts of $\{\frac{i}{\alpha}\mid i\in\mathbb{N}\}$ is also dense in $[0,1]$ we have that $y_{k_j}\searrow 0$.
We also have for every i, $z_i=y_i\alpha$. So z_{k_j} also satisfies the property that

$$z_{k_j}<z_{k_{j-1}}=min\{z_1,z_2,\ldots,z_{k_j-1}\}$$

Now we have $x_2(k_j) = \frac{k_j}{\alpha} + y_{k_j}$ and $y_{k_j} \searrow 0$. Since $y_{k_j}\alpha < 1$ then

$$\lfloor x_2(k_j)\alpha \rfloor = k_j.$$

Now we apply the previous Theorems 7, 8 as follows. The sequence

$$\frac{k_j}{x_2(k_j)} = \alpha - \frac{z_{k_j}}{x_2(k_j)} \longrightarrow \alpha \text{ as } j \longrightarrow \infty$$

In Theorems 7, 8 we choose α which is an irrational satisfying the property that $0 < \alpha < 1$ and the sequence of rationals $\frac{k_j}{x_2(k_j)} = \frac{p_j}{q_j} \longrightarrow \alpha$ as $j \longrightarrow \infty$ where $gcd(p_j, q_j) = 1$. Now by the very definition of z_{k_j} and using the properties of stabilization and eventual invariance we have

- $x_2(k_j) - x_2(k_{j-1})$ is increasing.
- $\underset{j\longrightarrow\infty}{lim} (x_2(k_j) - x_2(k_{j-1})) = \infty$.

This implies we also have

- $k_j - k_{j-1}$ is increasing.
- $\underset{j\longrightarrow\infty}{lim} (k_j - k_{j-1}) = \infty$.

This proves the Lemma 2. □

Now we prove the following Lemma 3

Lemma 3. *Let $p_1 < p_2$ be two integers such that (p_1, p_2) is an irrational pair. Using the notations of the previous Lemma 2, we have for any integer $0 \leq t < k_{j+1} - k_j$ there are no numbers of the form $p_1^b p_2^a$ in the integer interval excluding the end-points.*

$$(p_2^{k_j+t}, \ldots, p_1^{x_2(k_j)} p_2^t).$$

Proof. Let $\alpha = Log_{p_2}(p_1) < 1$. Here we use the following fact. We have $\lfloor x_2(k_j)\alpha \rfloor = k_j$. Suppose if there exists such a number $p_2^{k_j+t} < p_1^b p_2^a < p_1^{x_2(k_j)} p_2^t$ then we have

$$\begin{aligned} & k_j + t < a + b\alpha < t + x_2(k_j)\alpha < t + k_j + 1 \\ \rightarrow & k_j < -t + a + b\alpha < x_2(k_j)\alpha < k_j + 1 \\ \rightarrow & k_j + t - a < b\alpha < k_j + t - a + 1 \end{aligned}$$

So we have that $b \neq 0$. Similarly $b \neq x_2(k_j)$. If $b = x_2(k_j)$ then we get that $k_j = k_j + t - a$ which implies $t = a$. Hence $p_1^b p_2^a$ is an end-point which is not considered.
Let $b\alpha = k_j + t - a + z$. Consider the case $k_j + t - a < k_{j+1}$. Then by definition of $z_{k_j}, z_{k_{j+1}}$ and since $b \neq x_2(k_j)$ we have $z \geq z_{k_j} > z_{k_{j+1}}$. Hence

$$k_j < k_j + z = -t + a + b\alpha < x_2(k_j)\alpha = k_j + z_{k_j} < k_j + 1.$$

Hence we get $z < z_{k_j}$ which is a contradiction. Hence we must have $k_j + t - a \geq k_{j+1}$ which implies $t \geq k_{j+1} - k_j + a \geq k_{j+1} - k_j$ which is again a contradiction to the hypothesis $0 \leq t < k_{j+1} - k_j$. This proves the Lemma 3. □

Using these three Lemmas 1, 2, 3 we prove our main Theorem 1 of this article and its Corollary 1.

Proof. Suppose $\mathbb{S} = \{1, f, f^2, \ldots\}$ a singly generated multiplicatively closed set then we immediately have $\underset{j\longrightarrow\infty}{lim} (f^{j+1} - f^j) = \infty$.

Now suppose $\mathbb{S} = \{g_1^i g_2^j \mid i, j \geq 0\}$ and $Log_{g_1}(g_2)$ is rational then $\mathbb{S}$ is contained in a singly generated multiplicatively closed set $\mathbb{T}$ using the Theorem 3. So there exists arbitrarily large gaps in $\mathbb{S}$ as well.

Now suppose $\mathbb{S} = \{p_1^i p_2^j \mid i,j \geq 0\}$ and $Log_{p_2}(p_1)$ is irrational. Then in Lemma 3 we substitute $t = k_{j+1} - k_j - 1$ and we obtain a gap of size

$$0 < p_1^{x_2(k_j)} p_2^t - p_2^{k_j+t} = p_2^t(p_1^{x_2(k_j)} - p_2^{k_j}) \geq p_2^t = p_2^{k_{j+1}-k_j-1}$$

Hence the limit superior of the gaps tend to infinity in the multiplicatively closed set $\mathbb{S}$ using Lemma 2. Now the Theorem 1 follows. □

Note 3. *Via the sequence k_j we know the prime factorization of the end points of the intervals $(p_2^{k_j+t}, p_1^{x_2(k_j)} p_2^t)$ for $0 < t < k_{j+1} - k_j$ which are all gap intervals.*

To prove Corollary 1 we can use Theorem 1 by observing that using Lemma 1 the pair (p_1, p_2) is an irrational pair if both p_1, p_2 are primes which also implies that both $Log_{p_1}(p_2), Log_{p_2}(p_1)$ are irrational.

Here we give an example.

Example 3 (Main Example)**.** *Consider the irrational $\frac{1}{Log_2(3)}$. The first few terms of the sequence k_j which is defined by the fractional parts*

$$z_{k_j} < z_{k_{j-1}} = min\{z_1, z_2, \ldots, z_{k_j-1}\}$$

is given by

$$\{1, 3, 5, 17, 29, 41, 94, 147, 200, 253, 306, 971, 1636, 2301, 2966, 3631, 4296, 4961, 5626, 6291, 6956, 7621, 8286, 8951, 9616, 10281, 10946, 11611, 12276, 12941, 13606, 14271, 14936, 15601, 47468, 79335, 190537\}$$

The corresponding first few terms of the sequence $x_2(k_j)$ is given by

$$\{2, 5, 8, 27, 46, 65, 149, 233, 317, 401, 485, 1539, 2593, 3647, 4701, 5755, 6809, 7863, 8917, 9971, 11025, 12079, 13133, 14187, 15241, 16295, 17349, 18403, 19457, 20511, 21565, 22619, 23673, 24727, 75235, 125743, 301994\}$$

The first few terms of the rational approximation seqence to α is given by

$$\{\frac{1}{2}, \frac{3}{5}, \frac{5}{8}, \frac{17}{27}, \frac{29}{46}, \frac{41}{65}, \frac{94}{149}, \frac{147}{233}, \frac{200}{317}, \frac{253}{401}, \frac{306}{485}, \frac{971}{1539}, \frac{1636}{2593}, \frac{2301}{3647}, \frac{2966}{4701}, \frac{3631}{5755}, \frac{4296}{6809}, \frac{4961}{7863}, \frac{5626}{8917}, \frac{6291}{9971}, \frac{6956}{11025}, \frac{7621}{12079}, \frac{8286}{13133}, \frac{8951}{14187}, \frac{9616}{15241}, \frac{10281}{16295}, \frac{10946}{17349}, \frac{11611}{18403}, \frac{12276}{19457}, \frac{12941}{20511}, \frac{13606}{21565}, \frac{14271}{22619}, \frac{14936}{23673}, \frac{15601}{24727}, \frac{47468}{75235}, \frac{79335}{125743}, \frac{190537}{301994}\}$$

This stabilized seqence for approximate inverses for the fraction $\frac{190537}{301994}$ is given by

$$\{1, 2, 5, 8, 27, 46, 65, 149, 233, 317, 401, 485, 1539, 2593, 3647, 4701, 5755, 6809, 7863, 8917, 9971, 11025, 12079, 13133, 14187, 15241, 16295, 17349, 18403, 19457, 20511, 21565, 22619, 23673, 24727, 75235, 125743, 301994\}$$

We note that it matches with $x_2(k_j)$. Actually this can be obtained for any suitable rational approximation sequence for α. The first few gaps of intervals with the prime factorization of end-points of the gap intervals of the form

$$(p_2^{k_{j+1}-1} \ldots p_1^{x_2(k_j)} p_2^{k_{j+1}-k_j-1})$$

using this method is given by

$(3^2\dots 2^2 3^1), (3^4\dots 2^5 3^1), (3^{16}\dots 2^8 3^{11}), (3^{28}\dots 2^{27} 3^{11}), (3^{40}\dots 2^{46} 3^{11}), (3^{93}\dots 2^{65} 3^{52}),$
$(3^{146}\dots 2^{149} 3^{52}), (3^{199}\dots 2^{233} 3^{52}), (3^{252}\dots 2^{317} 3^{52})(3^{305}\dots 2^{401} 3^{52}), (3^{970}\dots 2^{485} 3^{664}),$
$(3^{1635}\dots 2^{1539} 3^{664}), (3^{2300}\dots 2^{2593} 3^{664}), (3^{2965}\dots 2^{3647} 3^{664}), (3^{3630}\dots 2^{4701} 3^{664}),$
$(3^{4295}\dots 2^{5755} 3^{664}), (3^{4960}\dots 2^{6809} 3^{664}), (3^{5625}\dots 2^{7863} 3^{664}), (3^{6290}\dots 2^{8917} 3^{664}),$
$(3^{6955}\dots 2^{9971} 3^{664}), (3^{7620}\dots 2^{11025} 3^{664}), (3^{8285}\dots 2^{12079} 3^{664}), (3^{8950}\dots 2^{13133} 3^{664}),$
$(3^{9615}\dots 2^{14187} 3^{664}), (3^{10280}\dots 2^{15241} 3^{664}), (3^{10945}\dots 2^{16295} 3^{664}), (3^{11610}\dots 2^{17349} 3^{664}),$
$(3^{12275}\dots 2^{18403} 3^{664}), (3^{12940}\dots 2^{19457} 3^{664}), (3^{13605}\dots 2^{20511} 3^{664}), (3^{14270}\dots 2^{21565} 3^{664}),$
$(3^{14935}\dots 2^{22619} 3^{664}), (3^{15600}\dots 2^{23673} 3^{664}), (3^{47467}\dots 2^{24727} 3^{31866}),$
$(3^{79334}\dots 2^{75235} 3^{31866}), (3^{190536}\dots 2^{125743} 3^{111201}).$

5. Appendix

In this appendix section we prove some interesting lemmas about gaps, also present some motivating examples and give another constructive proof and discuss advantages and disadvantages with respect to the above given constructive proof.
We begin with a lemma.

Lemma 4. *(1) Let $S \subset \mathbb{N}$ be an infinite set. If*

$$\underset{n\longrightarrow\infty}{liminf}\frac{\#(S\cap[1,\dots,n])}{n}=0$$

there are arbitrarily large gaps in S.

(2) Let $S_i \subset \mathbb{N} : 1 \leq i \leq k$ be $k-$infinite subsets. If for each $1 \leq i \leq k$

$$\underset{n\longrightarrow\infty}{lim}\frac{\#(S_i\cap[1,\dots,n])}{n}=0$$

there are arbitrarily large gaps in $S = \bigcup_{i=1}^{k} S_i$.

Proof. To prove (1) we observe that if the gaps we bounded then $\underset{n\longrightarrow\infty}{liminf}\frac{\#(S\cap[1,\dots,n])}{n} > 0$.
To prove (2) we have

$$0 \leq \underset{n\longrightarrow\infty}{lim}\frac{\#(S\cap[1,\dots,n])}{n} \leq \underset{n\longrightarrow\infty}{lim}\sum_{i=1}^{k}\frac{\#(S_i\cap[1,\dots,n])}{n}=0.$$

Hence using (1) the gaps in S is unbounded. □

Example 4. *The following sets have arbitrarily large gaps.*

- *A multiplicatively closed set generated by finitely many positive integers > 1.*
- *The set of all integers which have exactly $k-$prime factors.*
- *The set of all integers which have atmost $k-$prime factors.*

Theorem 10. *Let $\mathbb{S}_1, \mathbb{S}_2$ be two infinite subsets of $\mathbb{N}$. Let $\mathbb{S}_3 = \mathbb{S}_1 \cup \mathbb{S}_2, \mathbb{S}_4 = \mathbb{S}_1\mathbb{S}_2 = \{s_1 s_2 \mid s_i \in \mathbb{S}_i, i = 1, 2\}$. Let $\mathbb{S}_i = \{1 < a_{i1} < a_{i2} < \dots\}$ for $i = 1, 2, 3, 4$. Then*

(1) $\underset{j\longrightarrow\infty}{limsup}(a_{i(j+1)} - a_{ij}) = \infty$ for $i = 1, 2 \not\Rightarrow \underset{j\longrightarrow\infty}{limsup}(a_{i(j+1)} - a_{ij}) = \infty$ for $i = 3, 4$.

(2) $\underset{j\longrightarrow\infty}{lim}(a_{1(j+1)} - a_{1j}) = \infty, \underset{j\longrightarrow\infty}{limsup}(a_{2(j+1)} - a_{2j}) = \infty$ then $\underset{j\longrightarrow\infty}{limsup}(a_{3(j+1)} - a_{3j}) = \infty$ and does not imply $\underset{j\longrightarrow\infty}{limsup}(a_{4(j+1)} - a_{4j}) = \infty$.

(3) $\underset{j\longrightarrow\infty}{lim}(a_{i(j+1)}-a_{ij})=\infty$ *for* $i=1,2\Rightarrow \underset{j\longrightarrow\infty}{limsup}(a_{4(j+1)}-a_{4j})=\infty$.

Proof. Let us prove (1) by giving a counter example.

- Consider the set of natural numbers $\mathbb{N}$. Decompose $\mathbb{N}$ into two sets $\mathbb{S}_1,\mathbb{S}_2$ as follows. Keep the first element of $\mathbb{N}$ in $\mathbb{S}_1$. The next two elements in $\mathbb{S}_2$. The next three elements in $\mathbb{S}_1$ and so on i.e.
$$\mathbb{S}_1=\bigcup_{i\geq 0}\{(2i+1)(i+1)-2i,\ldots,(2i+1)(i+1)\}$$
$$\mathbb{S}_2=\bigcup_{i\geq 1}\{i(2i+1)-2i+1,\ldots,i(2i+1)\}$$
Then $\mathbb{S}_1\cup\mathbb{S}_2=\mathbb{N}$.
- Partion the set of primes $\mathbb{P}$ into two infinite subsets of primes $\mathbb{PP}_1,\mathbb{PP}_2$. Let $\mathbb{S}_i$ be the multiplicatively closed set generated by $\mathbb{PP}_i$ for $i=1,2$. Then $\mathbb{S}_1\mathbb{S}_2=\mathbb{N}$ and $\underset{j\longrightarrow\infty}{limsup}(a_{i(j+1)}-a_{ij})=\infty$ for $i=1,2$ by an application of chinese remainder theorem.

Let us prove (2). Given any $N>0$ there exists M such that $a_{1(k+1)}-a_{1k}>N$ for all $k>M$ and there exists infintely many $l>M$ such that $a_{2(l+1)}-a_{2l}>N$. Also choose large enough $l=l_0>M$ such that if $a_{1k_0}>a_{2l_0}$ then $k_0>M$. If $a_{2l_0}<a_{2(l_0+1)}$ are consecutive in $\mathbb{S}_1\cup\mathbb{S}_2$ then we have produced a gap more than N. If $a_{2l_0}<a_{1k_0}$ are consecutive then

- We have either $a_{2l_0}<a_{1k_0}<a_{1(k_0+1)}$ as consecutive integers in $\mathbb{S}_1\cup\mathbb{S}_2$.
- Or $a_{2l_0}<a_{1k_0}<a_{2(l_0+1)}$ as consecutive integers in $\mathbb{S}_1\cup\mathbb{S}_2$.

In the first case we are done again. In the second case we have either $a_{1k_0}-a_{2l_0}>\frac{N}{2},a_{2(l_0+1)}-a_{1k_0}>\frac{N}{2}$. Hence we have produced a gap more than $\frac{N}{2}$. Moreover these gaps can be produced arbitrary number of times by choosing M larger and larger for any positive integer N. So we have $\underset{j\longrightarrow\infty}{limsup}(a_{3(j+1)}-a_{3j})=\infty$.

Now for second part of (2) we give a counter example. Let $\mathbb{S}_1=\{n^2\mid n\in\mathbb{N}\}$. Let $\mathbb{S}_2=\{n\in\mathbb{N}\mid n \text{ is square free}\}$. Then $\mathbb{S}_1\mathbb{S}_2=\mathbb{N}$. We have $\underset{j\longrightarrow\infty}{lim}(a_{1(j+1)}-a_{1j})=\infty$. Also by an application of chinese remainder theorem we have $\underset{j\longrightarrow\infty}{limsup}(a_{2(j+1)}-a_{2j})=\infty$.

Let us prove (3). Fix a large integer K. Let $\mathbb{T}_1=\{1<a_{11}<a_{12}<\ldots<a_{1N}\},\mathbb{T}_2=\{1<a_{21}<a_{22}<\ldots<a_{2M}\}$. Suppose $a_{1(t+1)}-a_{1t}\geq K$ for all $t\geq N-1$ and $a_{2(t+1)}-a_{2t}\geq K$ for all $t\geq M-1$. Let $a_{1N}a_{2M},a_{1\tilde{N}}a_{2\tilde{M}}$ be two successive numbers in the set $\mathbb{S}_1\mathbb{S}_2$. Then we have either $\tilde{N}>N$ or $\tilde{M}>M$. We note that for $\tilde{N}>N$ we have
$$a_{1\tilde{N}}a_{2\tilde{M}}-a_{1N}a_{2M}\geq(a_{1\tilde{N}}-a_{1N})a_{2\tilde{M}}\geq K \text{ if } \tilde{M}\geq M.$$
For
$$a_{1\tilde{N}}a_{2\tilde{M}}-a_{1N}b_{2M}\geq(a_{1\tilde{N}}-a_{1N})a_{2M}\geq K \text{ if } M>\tilde{M}.$$
The argument is similar if $\tilde{M}>M$. This holds for any large K. So $\underset{j\longrightarrow\infty}{limsup}(a_{4(j+1)}-a_{4j})=\infty$

Hence we have completed the proof of this theorem. □

Theorem 11. *Let* $\mathbb{S}_i:1\leq i\leq n$ *be finitely many infinite subsets of* $\mathbb{N}$. *Let* $\mathbb{S}_{n+1}=\bigcup_{i=1}^{n}\mathbb{S}_i,\mathbb{S}_{n+2}=\prod_{i=1}^{n}\mathbb{S}_i=\{s_1s_2\ldots s_n\mid s_i\in\mathbb{S}_i,1\leq i\leq n\}$. *Let* $\mathbb{S}_i=\{1<a_{i1}<a_{i2}<\ldots\}:1\leq i\leq n+2$. *If* $\underset{j\longrightarrow\infty}{lim}(a_{i(j+1)}-a_{ij})=\infty$ *for* $i=1,\ldots,n$ *then* $\underset{j\longrightarrow\infty}{limsup}(a_{i(j+1)}-a_{ij})=\infty$ *for* $i=n+1,n+2$.

Proof. The proof of this theorem is left to the interested reader. □

Corollary 3. *(1) The set of natural numbers $\mathbb{N}$ cannot be written as a finite product of sets $\mathbb{S}_1\mathbb{S}_2\ldots\mathbb{S}_n$ where the gaps in $\mathbb{S}_i$ diverges to ∞ for $1 \leq i \leq n$.*

(2) The set of natural numbers $\mathbb{N}$ cannot be written as a finite union of sets $\mathbb{S}_1 \cup \mathbb{S}_2 \cup \ldots \cup \mathbb{S}_n$ where the gaps in $\mathbb{S}_i$ diverges to ∞ for $1 \leq i \leq n$.

(3) The multiplicatively closed subset $\mathbb{S}$ of $\mathbb{N}$ generated by finitely many positive integers > 1 has arbitrarily large gaps.

Theorem 12 (Another Constructive Proof)**.** *The multiplicatively closed subset of $\mathbb{N}$ generated by finitely many positive integers $\mathbb{S}$ has arbitrarily large gaps.*

Proof. We give here another constructive proof in this Theorem. Let K be an arbitrary positive integer. Let $n_1, n_2, \ldots, n_k$ be the generators of the multiplicatively closed set.
Define $\lceil Log_{n_i}(K)\rceil = a_i$. Then we have for all

$$t_i \geq a_i, t_i \in \mathbb{N}, n_i^{t_i+1} - n_i^{t_i} = n_i^{t_i}(n_i - 1) \geq n_i^{t_i} \geq K.$$

The gap between $n_1^{t_1}n_2^{t_2}\ldots n_k^{t_k}$ and the next number l in the set $\mathbb{S}_1\mathbb{S}_2\ldots\mathbb{S}_k$ is at least K. Let $l = n_1^{s_1}n_2^{s_2}\ldots n_k^{s_k}$ be the next number. Then there is at least one $i = i_0$ such that $s_i > t_i$.
So we get that $n_1^{s_1}n_2^{s_2}\ldots n_k^{s_k} - n_1^{t_1}n_2^{t_2}\ldots n_k^{t_k} \geq n_i^{s_i}a - n_i^{t_i}b \geq n_i^{t_i}(n_i^{s_i-t_i}a - b) \geq n_i^{t_i} \geq K.$ □

Note 4. *The difference between this constructive proof and the other constructive proof is that we do not exactly know the right end point l of this Gap-Interval as we do not know its prime factorization exactly. However we were able to locate a point $n_1^{a_1}n_2^{a_2}\ldots n_k^{a_k}$ and a gap interval of size at least K with this integer as the left end point for every positive integer $K > 0$.*
In the proof of the Main Theorem 1 we know the prime factorizations of both the end points of the gap interval via the stabilization sequence. Sometimes knowing factorizations is helpful.

In an attempt to answer Question 1 we prove a lemma which says that the same technique may or may not be extendable for more than two generators.

Lemma 5. *Let $G = \{p_1 < p_2 < \ldots < p_l\}$ be a finite set of primes. Let k be any positive integer. Consider the monoid $T = \{\sum_{i=1}^{l-1} x_i Log_{p_l} p_i \mid x_i \in \mathbb{N} \cup \{0\}\}$. Consider the set $T_k = T \cap (k, k+1)$. Let $z_k = min(T_k - k)$. Let z_{k_j} be a monotone decreasing sequence converging to zero constructed from z_k defined by the property that*

$$z_{k_j} < z_{k_{j-1}} = min\{z_1, z_2, \ldots, z_{k_j-1}\}$$

then the sequence of integers $\{k_{j+1} - k_j : j \in \mathbb{N}\}$ need not be increasing.

Proof. Consider the following example. Let $\{p_1 = 2 < p_2 = 3 < p_3 = 5\}$. By calculating the logarithm of numbers to the base 5 in the sequence $\{2^i 3^j \mid 0 \leq i, j \leq 50\}$ or by actually showing inequalities we obtain

- $k_0 = 0, z_{k_0} = z_0 = Log_5(2) - 0.$
- $k_1 = 1, z_{k_1} = z_1 = Log_5(2.3) - 1.$
- $k_2 = 2, z_{k_2} = z_2 = Log_5(3^3) - 2.$
- $k_3 = 3, z_{k_3} = z_3 = Log_5(2^7) - 3.$
- $k_4 = 7, z_{k_4} = z_7 = Log_5(2^2.3^9) - 7.$
- $k_5 = 8, z_{k_5} = z_8 = Log_5(2^{17}.3) - 8.$
- $k_6 = 13, z_{k_6} = z_{13} = Log_5(2^8.3^{14}) - 13.$

- $k_7 = 14, z_{k_7} = z_{14} = Log_5(2^{23}.3^6) - 14.$

We can show the inequalities $z_{k_0} > z_{k_1} > z_{k_2} > z_{k_3} > z_{k_4} > z_{k_5} > z_{k_6} > z_{k_7}$ and

$$k_1 - k_0 = k_2 - k_1 = k_3 - k_2 = 1 < k_4 - k_3 = 4 > k_5 - k_4 = 1 < k_6 - k_5 = 5 > k_7 - k_6 = 1$$

which is not increasing. This proves the lemma. However we mention that it is possible that $\underset{j \longrightarrow \infty}{limsup}(k_{j+1} - k_j) = \infty$ which additionally requires a proof. □

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