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Abstract

We study a time varying analogue of the Erdős-Rényi graph, which we call the dynamic Erdős-Rényi graph, and concentrate on the topological aspects of its clique complex. Denoting the graph on \( n \) points, with edge connection probability \( p \), and at time \( t \), by \( G(n, p, t) \), the dynamics is determined by each edge of the graph independently evolving as a on/off Markov chain. Our main result is that if \( p = n^{\alpha} \), with \( \alpha \in (-1/k, -1/(k+1)) \), then the time dynamics of the normalized \( k \)–th Betti number of the clique complex associated with the graph converges in distribution to that of the stationary Ornstein-Uhlenbeck process as \( n \to \infty \).

Keywords: Erdős-Rényi graph, Betti numbers, Ornstein-Uhlenbeck process

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1. Introduction

The topological study of random graphs, beyond the classical issues of connectivity and degree, has been the subject of much recent research; see [1, 8, 10, 13, 11, 15, 16] for some of this and [12] for a detailed survey and motivation. All of this literature, however, treats only fixed, or ‘static’ graphs. The aim of this paper is to understand how the topological traits of random
graphs change when they evolve in time. The specific case that we will study is a time varying analogue of the Erdőš-Rényi random graph, which we will refer to as the dynamic Erdőš-Rényi graph, and the topological descriptors that we will use are the Betti numbers of its associated clique complex.

The main result of the paper will be to show convergence to an Ornstein-Uhlenbeck process of a (normalised) random process determined by the Betti numbers. However, before stating this result, it will be useful to review some of the previous results concerning the topology of static Erdőš-Rényi graphs.

1.1. On the topology of static Erdőš-Rényi graphs

Recall that the Erdőš-Rényi graph $G(n,p)$ is a random graph on $n$ vertices where each edge appears with probability $p$, independently of the others. As the pioneering result in random graphs, the Erdőš-Rényi theorem \cite{erdos1959random} established a sharp threshold for connectivity of Erdőš-Rényi graphs.

**Theorem 1.1** (Erdőš and Rényi). Let $\epsilon > 0$ be fixed. If $p \geq (1 + \epsilon) \log(n)/n$, then

$$P\{G(n,p) \text{ is connected}\} \to 1 \text{ as } n \to \infty.$$  

On the other hand, if $p \leq (1 - \epsilon) \log(n)/n$, then

$$P\{G(n,p) \text{ is disconnected}\} \to 1 \text{ as } n \to \infty.$$  

Extending this result, it seems reasonable that if $p$ is chosen significantly larger than $\log(n)/n$, then, instead of merely containing paths between any two vertices, $G(n,p)$ may contain as sub-graphs, interesting patterns made up of cliques of different sizes. A $k$–clique, recall, is a collection of $k$ vertices with all edges present. Figure 1 shows some examples of patterns that may arise.

To formally study these patterns, one typically takes the following two steps. The first step is to build the clique complex $\mathcal{X}(n,p) \equiv \mathcal{X}(G(n,p))$ over $G(n,p)$. Recall that the clique complex $\mathcal{X}(G)$ of an undirected graph $G$ is the abstract simplicial complex formed by the sets of vertices in the cliques of $G$. (Since any subset of a clique is itself a clique, this family of sets meets the requirement of
Figure 1: Patterns made up of 2 and 3 cliques. As explained below, patterns a and b are examples of 1 cycle. Pattern a forms a 2-dimensional hole while Pattern b does not.

Having defined $\mathcal{X}(n,p)$, the second step is to examine its topology, i.e. to look at the various cycles formed using cliques and study the resulting homology groups with a special attention to Betti numbers. The $k$-th Betti number of $\mathcal{X}(n,p)$ is denoted $\beta_{n,k}$. For a formal definition of these terms, see [4, 7].

Intuitively, if a $(k+1)$-clique is thought of as a $k$-dimensional solid object, then a $k$-cycle is a sub-graph made up of only $(k+1)$-cliques the union of which is topologically equivalent to the $k$-sphere $S^k$. A $k$-cycle that cannot be expressed as a boundary of any collection of higher dimensional cliques represents a $(k+1)$-dimensional hole. The $k$-th Homology group of $\mathcal{X}(n,p)$ is the collection of all ‘independent’ $(k+1)$-dimensional holes in $\mathcal{X}(n,p)$ while the $k$-th Betti number is the number of generators of the group; i.e. the number of such holes. The zeroth Betti number is separately defined to be one less than the number of connected components which is as per the convention followed in reduced homology. In the left graph of Figure 1, for example, cycle a represents a 2-dimensional hole while cycle b and the other 3-cliques do not as they can be expressed as the boundary of the triangle(s) they bound. Hence, the first Betti number of left graph is only 1. Similarly, its zeroth Betti number is 21 while the second and higher Betti numbers are zero.

Fix $k \geq 1$. With small and large temporarily being qualifiers that depend
on $k$, the above interpretation suggests that both small values as well as large values of $p$ are bad for $\beta_{n,k}$. A small $p$ inhibits the formation of $k$--cycles, while with a large $p$, a $k$--cycle more often than not borders higher dimensional cliques. For $\beta_{n,k}$ to be significant, $p$ needs to lie somewhere between small and large. The following result from [11, Theorem 1.1] (see also the discussion below [14, (1)]) gives a rigorous version of this intuition.

**Theorem 1.2** (Kahle). Fix $k \geq 1$. Let $p = n^\alpha$, $\alpha \in \left(-\frac{1}{k}, -\frac{1}{k+3}\right)$. Then, for any $M > 0$,

$$\lim_{n \to \infty} \mathbb{P}\{\beta_{n,k} \neq 0, \beta_{n,j} = 0, \forall j \neq k\} = 1 - o(n^{-M}).$$

Having settled the issue of the non-triviality of Betti numbers, one turns to questions on the limiting distribution of $\beta_{n,k}$. The following result from [13, Theorem 2.4] and [14, Theorem 1.1] gives a central limit theorem for Betti numbers.

**Theorem 1.3** (Kahle and Meckes). Fix $k \geq 1$, and let $p$ be as in Theorem 1.2. Then, as $n \to \infty$,

$$\frac{\beta_{n,k} - \mathbb{E}[\beta_{n,k}]}{\sqrt{\text{Var}[\beta_{n,k}]}} \Rightarrow \mathcal{N}(0, 1),$$

where $\mathcal{N}(0, 1)$ is standard Gaussian and $\Rightarrow$ denotes convergence in distribution.

With the brief overview of static Erdős-Rényi graphs done, we now turn to the main scenario of this paper, the dynamic case.

1.2. Dynamic Erdős-Rényi topology

We begin with the definition of dynamic Erdős-Rényi graph.

**Definition 1.1.** Fix $n \in \mathbb{N}$, $p \in [0, 1]$, and $\lambda > 0$. The dynamic Erdős-Rényi graph $\{G(n,p,t) : t \geq 0\}$ is a time-varying graph on $n$ vertices with the following properties.

- For $t \geq 0$, each edge independently evolves as a continuous time on/off Markov chain; ’on’ denotes edge is present while ’off’ denotes edge is absent. The waiting time in the states ’off’ and ’on’ is exponential respectively with parameters $\lambda p$ and $\lambda(1 - p)$. 


• The initial configuration of each edge at $t = 0$ is determined independently by a Bernoulli random variable which takes the ‘on’ state with probability $p$ and ‘off’ state with probability $1 - p$.

Consider an arbitrary edge $e$ of the dynamic Erdős-Rényi graph and let $e(t)$ denote its state at time $t$. Then, using Definition [1.1], it is straightforward to check that for any non-negative times $t_1$ and $t_2$,

$$P\{e(t_2) = \text{on} \mid e(t_1) = \text{on}\} = p + (1 - p)e^{-\lambda|t_2 - t_1|},$$

(1.1)

and

$$P\{e(t_2) = \text{off} \mid e(t_1) = \text{off}\} = (1 - p) + pe^{-\lambda|t_2 - t_1|}.$$ 

(1.2)

From this, it follows that for any $t \geq 0$,

$$P\{e(t) = \text{on}\} = p.$$ 

(1.3)

Definition [1.1] and the above facts put together show that $\{G(n, p, t) : t \geq 0\}$ is a stationary reversible Markov process and, for each $t \geq 0$, it is the usual Erdős-Rényi graph on $n$ vertices with edge probability $p$.

For each $t \geq 0$, let $\mathcal{X}(n, p, t) \equiv \mathcal{X}(G(n, p, t))$ denote the clique complex on $G(n, p, t)$ and let $\beta_{n,k}(t)$ be the $k$-th Betti number of $\mathcal{X}(n, p, t)$. Let $p$ be as in Theorem 1.2. Then, it follows from Theorem 1.3 that the normalized Betti number

$$\bar{\beta}_{n,k}(t) := \beta_{n,k}(t) - E[\beta_{n,k}(t)] \sqrt{\text{Var}[\beta_{n,k}(t)]}$$

(1.4)

is asymptotically Gaussian for each $t$. Combining this with stationarity of the dynamic Erdős-Rényi graph, one might expect that the process $\{\bar{\beta}_{n,k}(t) : t \geq 0\}$ would converge to some stationary Gaussian process. This turns to be indeed true and is the key contribution of this paper.

For $\lambda > 0$, let $\{U_\lambda(t) : t \geq 0\}$ denote the stationary Ornstein-Uhlenbeck process satisfying $E[U_\lambda(t)] \equiv 0$ and $\text{Cov}[U_\lambda(t_1), U_\lambda(t_2)] = e^{-\lambda|t_1 - t_2|}$.

**Theorem 1.4.** Fix $k \geq 1$ and let $\lambda$ be as in Definition 1.1. Let $p = n^\alpha$, $\alpha \in \left(-\frac{1}{k}, -\frac{1}{k+1}\right)$. Then, as $n \to \infty$,

$$\{\bar{\beta}_{n,k}(t) : t \geq 0\} \Rightarrow \{U_\lambda(t) : t \geq 0\}. $$
1.3. Essence of Theorem 1.4

It is well known that \( \{ U_\lambda(t) : t \geq 0 \} \), in addition to being stationary and Gaussian, is also Markov. Combining this with Theorem 1.4, it follows that as \( n \to \infty \) the distribution of the stationary process \( \{ \beta_{n,k}(t) : t \geq 0 \} \) is also approximately Gaussian and Markov. While the asymptotic Gaussianity is to expected, the Markov property is not obvious since \( \{ \beta_{n,k}(t) : t \geq 0 \} \), for finite \( n \) is not Markov. We now provide an example to support this last claim.

Recall that the dynamic Erdős-Rényi graph \( \{ G(n,p,t) : t \geq 0 \} \) is a stationary reversible continuous time Markov chain. Hence \( \{ \beta_{n,k}(t) : t \geq 0 \} \) can be interpreted as a function of the Markov chain \( \{ G(n,p,t) : t \geq 0 \} \). The following result from [3, Theorem 4] gives necessary and sufficient conditions for a function of Markov chain to be Markov.

**Theorem 1.5** (Burke and Rosenblatt). Let \( \{ X(t) : t \geq 0 \} \) be a Markov chain with a finite number of states, \( i = 1, \ldots, m \), and stationary transition probability function \( P(t) = (p_{ij}(t)) \), where

\[
p_{ij}(t) = \mathbb{P}\{X(r+t) = j | X(r) = i\}
\]

is continuous in \( t \). Assume that \( \lim_{t \to 0} P(t) = I \), the identity matrix. Let \( \psi \) be a given function on the state space \( 1, \ldots, m \) and let \( Y(t) = \psi(X(t)) \). The states \( i \) of the original process \( \{ X(t) : t \geq 0 \} \) where \( \psi \) equals some fixed constant are collapsed into a single state of the new process \( \{ Y(t) : t \geq 0 \} \). Call these collapsed sets of states \( S_j, j = 1, \ldots, r, r \leq m \). Then \( \{ Y(t) : t \geq 0 \} \) is Markovian, whatever be the initial distribution of \( \{ X(t) : t \geq 0 \} \), if and only if, for each \( j = 1, \ldots, r \), either

1. \( p_{i,S_j} = 0 \) for all \( i \notin S_j \), or
2. \( p_{i,S_j} = C_{S_j,S_j} \) for every \( i \in S_j \), for \( j' = 1, \ldots, r \), where \( C_{S_j,S_j} \) is a constant that depends only on \( S_j \) and \( S_j \).

The required counter example is the following. Consider the dynamic Erdős-Rényi graph with \( n = 4 \) and arbitrary \( p \in (0,1) \). At any given time \( t \), each of its
Figure 2: $\beta_{4,1}(t) = 1$ if $G(4, p, t)$ is in one of the above configurations. Note that there is no vertex at the intersection of the cross edges.

6 edges, say $e_1, \ldots, e_6$, can be either in an ‘on’ or ‘off’ state. Thus, $G(4, p, t)$ has $m = 64$ possible configurations. On the other hand, the process $\{\beta_{4,1}(t) : t \geq 0\}$ can only take $r = 2$ values, i.e. zero or one. Further, $\beta_{4,1}(t) = 1$ if and only if $G(4, p, t)$ is in one of the configurations given in Figure 2. Using (1.1) and (1.2), observe that

$$P\{\beta_{4,1}(t+r) = 1 | e_1(r) = \cdots = e_6(r) = \text{off}\} = 3p^4(1 - e^{-t})^4((1 - p) + pe^{-t})^2$$

while

$$P\{\beta_{4,1}(t+r) = 1 | e_1(r) = \cdots = e_6(r) = \text{on}\} = 3(p+(1-p)e^{-t})^4(1-p)^2(1-e^{-t})^2.$$ 

Clearly, for a generic $p$ and $t$, the above two equations are unequal. But $\beta_{4,2}(r) = 0$ when either $e_1(r) = \cdots = e_6(r) = \text{off}$, or $e_1(r) = \cdots = e_6(r) = \text{on}$. This fact along with Theorem 1.5 shows that the process $\{\beta_{4,1}(t) : t \geq 0\}$ is not Markov.

1.4. On the proof of Theorem 1.4

Since working directly with Betti numbers is not easy, we adopt an approach developed in [9, 13]. To describe this, we require some definitions. Let $f_{n,k}(t)$ denote the number of $k$-dimensional faces, or, equivalently $(k+1)$-cliques in the clique complex $X(n,p,t)$. Let

$$\chi_n(t) := \sum_{j=0}^{n-1} (-1)^j f_{n,j}(t) \quad (1.5)$$

denote the Euler-Poincaré characteristic of $X(n,p,t)$. Alternatively, and equivalently, (e.g. [4][p101]) we have that

$$\chi_n(t) := \sum_{j=0}^{n-1} (-1)^j \beta_{n,j}(t). \quad (1.6)$$
Analogously to (1.4), let
\[ \tilde{f}_{n,k}(t) := f_{n,k}(t) - \frac{E[f_{n,k}(t)]}{\sqrt{\text{Var}[f_{n,k}(t)]}} \]  
(1.7)
and
\[ \tilde{\chi}_n(t) := \frac{\chi_n(t) - E[\chi_n(t)]}{\sqrt{\text{Var}[\chi_n(t)]}}. \]  
(1.8)

To establish the weak convergence for Betti numbers, we first show that a corresponding result holds true for the process \( \{\tilde{f}_{n,k}(t) : t \geq 0\} \). Using (1.5), we then establish weak convergence for \( \{\tilde{\chi}_n(t) : t \geq 0\} \). Finally, Theorem 1.4 is proven using (1.6) and Theorem 1.2.

1.5. Structure of the paper

In Section 2, we collect a number of standard results needed for the proofs. In Section 3, we discuss some preliminary results concerning the mean and variance of \( f_{n,k}(t), \chi_n(t), \) and \( \beta_{n,k}(t) \). The covariance functions of the processes \( \{\tilde{f}_{n,k}(t) : t \geq 0\}, \{\tilde{\chi}_n(t) : t \geq 0\}, \) and \( \{\beta_{n,k}(t) : t \geq 0\} \) are derived in Section 4. Knowing these covariances is key for the convergence of the finite dimensional distributions of the process \( \{\tilde{\beta}_{n,k}(t) : t \geq 0\} \), which we prove in Section 5. In Section 6, we establish tightness for the process \( \{\tilde{\beta}_{n,k}(t) : t \geq 0\} \), and complete the proof of Theorem 1.4.

2. Mathematical Background

We first describe the normal approximation theorem from [2] useful for establishing central limit theorems for a sequence of dissociated or partially dependent real valued random variables. We then recall from [6] sufficient conditions for a sequence of stochastic processes to converge to the stationary Ornstein-Uhlenbeck process.

2.1. Normal Approximation Theorem

We first define the notion of the \( L_1 \)-Wasserstein metric for real valued random variables.
Definition 2.1. The $L_1$-Wasserstein metric between two real valued random variables $Y_1$ and $Y_2$ is

$$d_1(Y_1, Y_2) = \sup_{\psi} |E[\psi(Y_1)] - E[\psi(Y_2)]|,$$

where the sup is over all functions $\psi : \mathbb{R} \rightarrow \mathbb{R}$ with $\sup_{y_1 \neq y_2} \left| \frac{\psi(y_1) - \psi(y_2)}{|y_1 - y_2|} \right| \leq 1$. For real valued random variables $\{Y_n\}$, if $\lim_{n \to \infty} d_1(Y_n, Y) = 0$, then $Y_n \Rightarrow Y$.

The normal approximation theorem from [2] (see Theorem 1 and (2.7)) is stated next.

Theorem 2.1 (Barbour, Karoński and Ruciński). Let $\{Y_i\}$ be a sequence of dissociated or partially dependent real valued random variables. That is, for each $i$, there exists $\mathcal{N}(i)$ such that, for each $j \notin \mathcal{N}(i)$, $Y_j$ and $Y_i$ are independent. For each $n$, let $\mathcal{W}_n = \sum_{i \in \mathcal{J}_n} Y_i$. If $E[Y_i] \equiv 0$ and $\text{Var}[\mathcal{W}_n] = 1$ for each $n$, then there exists a universal constant $\gamma > 0$ such that

$$d_1(\mathcal{W}_n, \mathcal{N}(0, 1)) \leq \gamma \sum_i \sum_{j, \ell \in \mathcal{N}(i)} E[|Y_i Y_j Y_\ell|] + E[|Y_i Y_j|] E[|Y_\ell|]. \quad (2.1)$$

2.2. Convergence to Stationary Ornstein-Uhlenbeck Process

Let $D_{\mathbb{R}}[0, \infty)$ denote the set of maps $x : [0, \infty) \rightarrow \mathbb{R}$ which are right continuous with left limits (RCLL). Let $(D_{\mathbb{R}}[0, \infty), r)$ denote the Skorokhod space with $r$ being the Skorokhod metric. The following result gives sufficient conditions for a sequence of $(D_{\mathbb{R}}[0, \infty), r)$ valued processes to converge in distribution to the stationary Ornstein-Uhlenbeck process. Proof of this result follows from Theorem 7.8, Page 131; Theorem 8.6, Page 137; and Theorem 8.8, Page 139 of [6].

Theorem 2.2. Let $\{X_n(t) : t \geq 0\}$ be the $n$-th element of a sequence of $(D_{\mathbb{R}}[0, \infty), r)$ valued stochastic processes that satisfy the following conditions.

- Convergence of Finite Dimensional Distributions: For any $t_1, \ldots, t_m \geq 0$, as $n \rightarrow \infty$,

$$\left( X_n(t_1), \ldots, X_n(t_m) \right) \Rightarrow \left( U_\lambda(t_1), \ldots, U_\lambda(t_m) \right).$$
• **Tightness:** The sequence \( \{ \{ X_n(t) : t \geq 0 \} : n \geq 1 \} \) is tight or alternatively the following conditions hold true.

\( \mathbf{C}_1 \). There exists \( \Upsilon > 0 \) such that

\[
\lim_{\delta \to 0} \limsup_{n \to \infty} E[|X_n(\delta) - X_n(0)|^\Upsilon] = 0.
\]

\( \mathbf{C}_2 \). For each \( T > 0 \), there exist constants \( \Upsilon_1 > 0, \Upsilon_2 > 1, \) and \( K > 0 \) such that for each \( n \),

\[
E \left[ |X_n(t+h) - X_n(t)|^{\Upsilon_1} |X_n(t) - X_n(t-h)|^{\Upsilon_2} \right] \leq Kh^{\Upsilon_2}
\]

for \( 0 \leq t \leq T + 1 \) and \( 0 \leq h \leq t \).

Then \( \{ X_n(t) : t \geq 0 \} \Rightarrow \{ U_\lambda(t) : t \geq 0 \} \) as \( n \to \infty \).

### 3. Preliminary Results

Here we investigate the limiting behaviour of \( \text{Var}[f_{n,k}(t)] \), \( \text{Var}[\chi_n(t)] \), and \( \text{Var}[\beta_{n,k}(t)] \), where \( f_{n,k}(t), \chi_n(t), \) and \( \beta_{n,k}(t) \) are as in Section [1]. Note that because of stationarity the above three variances are independent of \( t \). We will henceforth use \( [n] := \{ 1, \ldots, n \} \) to denote the vertex set of the dynamic Erdős-Rényi graph.

Let \( \binom{n}{k+1} \) denote the collection of all subsets of \( [n] \) of size \( k+1 \). Of course, \( \binom{n}{k+1} \) is the usual binomial coefficient. For \( A \in \binom{n}{k+1} \), let \( 1_A(t) = 1 \) if \( A \) is a \((k+1)-\)clique in \( G(n,p,t) \) and zero otherwise. Using these notations, we have

\[
f_{n,k}(t) = \sum_{A \in \binom{n}{k+1}} 1_A(t). \tag{3.1}
\]

Hence, it follows that

\[
E[f_{n,k}(t)] = \binom{n}{k+1} p^{(k+1)/2}, \tag{3.2}
\]

and

\[
\text{Var}[f_{n,k}(t)] = \binom{n}{k+1} \sum_{i=0}^{k+1} \binom{k+1}{i} \binom{n-k-1}{k+1-i} \frac{p^{(k+1)/2}}{p^{(i/2)}} - \binom{n}{k+1}^2 p^{(k+1)/2}. \tag{3.3}
\]

The below result gives the limiting behaviour of \( \text{Var}[f_{n,j}(t)] \) for each \( 1 \leq j \leq n - 1 \).
Lemma 3.1. Fix $k \geq 1$, $1 \leq j \leq n - 1$, and $t \geq 0$. Let $p = n^\alpha$, $\alpha \in \left(-\frac{1}{k}, \frac{1}{k+1}\right)$.

(i) If $j = 2k - 1$ with $\alpha \in \left[-\frac{1}{k+0.5}, -\frac{1}{k+1}\right)$ or if $j \leq 2k - 2$, then
\[
\text{Var}[f_{n,j}(t)] = \Theta(n^{2j}p^{2(\frac{j+1}{2})-1}).
\]

(ii) If $j = 2k - 1$ with $\alpha \in \left(-\frac{1}{k}, -\frac{1}{k+0.5}\right]$, or if $2k \leq j \leq n - 1$, then
\[
\text{Var}[f_{n,j}(t)] = \Theta(n^{j+1}p^{\frac{j+1}{2}}).
\]

Proof. Since \(\binom{n}{j+1} = \sum_{i=0}^{j+1} \binom{j+1}{i} \binom{n-j-1}{j+1-i}\), it follows from (3.3) that
\[
\text{Var}[f_{n,j}(t)] = \sum_{i=2}^{j+1} \binom{j+1}{i} \binom{n}{j+1-i} \left[p^{2(\frac{j+1}{2})-\left(\frac{i}{2}\right)} - p^{2(\frac{i+1}{2})}\right].
\]

But \(\binom{n}{j+1-i} = \Theta(n^{2j+2-i})\). Further, $p = n^\alpha$ with $\alpha < 0$, i.e. $p^{2(\frac{j+1}{2})} \leq p^{2(\frac{j+1}{2})-\left(\frac{i}{2}\right)}$. Hence, it suffices to obtain bounds for $\sum_{i=2}^{j+1} \binom{j+1}{i} n^{\phi(i)}$, where $\phi(i) = 2j + 2 - i + \alpha\left[2(\frac{j+1}{2}) - \left(\frac{i}{2}\right)\right]$. Since $\alpha < 0$, note that $\phi$ is a convex function. Hence, one of $\phi(2)$ or $\phi(j+1)$ maximizes $\phi(i)$ for $i \in \{2, \ldots, j+1\}$. When conditions in Part (i) of the lemma hold, it is easy to check that $\phi(2) \geq \phi(j+1)$. Similarly, when (ii) holds, it is easy to check that $\phi(j+1) \geq \phi(2)$. Further, at $\alpha = -\frac{1}{k+0.5}$, $\phi(2) = \phi(j+1)$. The desired result now follows easily.

The following two result are now immediate from Lemma 3.1.

Corollary 3.1. Fix $k \geq 1$. Let $p = n^\alpha$, $\alpha \in \left(-\frac{1}{k}, -\frac{1}{k+1}\right)$. Then, for each $t \geq 0$,
\[
\text{Var}[f_{n,k}(t)] = \Theta(n^{2k}p^{2(k+1)-1}).
\]

Corollary 3.2. Fix $k \geq 1$, $1 \leq j \leq n - 1$, and $t \geq 0$. Let $p = n^\alpha$, $\alpha \in \left(-\frac{1}{k}, -\frac{1}{k+1}\right)$.

1. If $j \neq k$, then
\[
\lim_{n \to \infty} \frac{\text{Var}[f_{n,j}(t)]}{\text{Var}[f_{n,k}(t)]} = 0.
\]
2. If $j \geq 2k + 3$, then

$$\text{Var}[f_{n,j}(t)] = O(\text{Var}[f_{n,2k+2}(t)]).$$

3. \( \lim_{n \to \infty} n^{2} \frac{\text{Var}[f_{n,2k+3}(t)]}{\text{Var}[f_{n,k}(t)]} = 0. \)

The next two results compare the limiting behaviour of \( \text{Var}[f_{n,k}(t)] \) with \( \text{Var}[\chi_n(t)] \).

**Lemma 3.2.** Fix \( k \geq 1 \). Let \( p = n^{\alpha}, \alpha \in \left( -\frac{1}{k}, -\frac{1}{k+1} \right) \). Then, for each \( t \geq 0 \),

$$\lim_{n \to \infty} \frac{\text{Var}[(−1)^k \chi_n(t) - f_{n,k}(t)]}{\text{Var}[f_{n,k}(t)]} = 0.$$

**Proof.** From (1.5), we have

\[
\text{Var}[(−1)^k \chi_n(t) - f_{n,k}(t)] \leq \sum_{i \in [n]\backslash k} \text{Var}[f_{n,i}(t)] + 2 \sum_{i<j \in [n]\backslash k} |\text{Cov}[f_{n,i}(t), f_{n,j}(t)]| \leq \sum_{i \in [n]\backslash k} \text{Var}[f_{n,i}(t)] + 2 \sum_{i<j \in [n]\backslash k} \sqrt{\text{Var}[f_{n,i}(t)]\text{Var}[f_{n,j}(t)]}. \]

The desired result now follows from Corollary 3.2 and the constraints on \( \alpha \). \( \square \)

**Lemma 3.3.** Fix \( k \geq 1 \). Let \( p = n^{\alpha}, \alpha \in \left( -\frac{1}{k}, -\frac{1}{k+1} \right) \). Then, for each \( t \geq 0 \),

$$\lim_{n \to \infty} \frac{\text{Var}[\chi_n(t)]}{\text{Var}[f_{n,k}(t)]} = 1.$$

**Proof.** By adding and subtracting \( f_{n,k}(t) \), we get

$$\text{Var}[\chi_n(t)] = \text{Var}[f_{n,k}(t)] + \text{Var}[(−1)^k \chi_n(t) - f_{n,k}(t)] + 2\text{Cov}[(−1)^k \chi_n(t) - f_{n,k}(t), f_{n,k}(t)].$$

Hence it follows that

$$\left| \frac{\text{Var}[\chi_n(t)]}{\text{Var}[f_{n,k}(t)]} - 1 \right| \leq \frac{\text{Var}[(−1)^k \chi_n(t) - f_{n,k}(t)]}{\text{Var}[f_{n,k}(t)]} + \sqrt{\frac{\text{Var}[(−1)^k \chi_n(t) - f_{n,k}(t)]}{\text{Var}[f_{n,k}(t)]}}.$$

The desired result now follows from Lemma 3.2. \( \square \)
In a similar spirit to the above two results, Lemmas 3.4 and 3.5 compare the limiting behaviour $\text{Var}[\chi_n(t)]$ with $\text{Var}[\beta_{n,k}(t)]$.

**Lemma 3.4** ([14], Page 2). Fix $k \geq 1$. Let $p = n^\alpha$, $\alpha \in \left(-\frac{1}{k}, -\frac{1}{k+1}\right)$. Then, for each $t \geq 0$,

$$\lim_{n \to \infty} \frac{\text{Var}[\beta_{n,k}(t) - (-1)^k \chi_n(t)]}{\text{Var}[\chi_n(t)]} = 0.$$  

**Lemma 3.5** ([14], Page 2). Fix $k \geq 1$. Let $p = n^\alpha$, $\alpha \in \left(-\frac{1}{k}, -\frac{1}{k+1}\right)$. Then, for each $t \geq 0$,

$$\lim_{n \to \infty} \frac{\text{Var}[\beta_{n,k}(t)]}{\text{Var}[\chi_n(t)]} = 1.$$  

### 4. Covariance

We now investigate the asymptotic covariance of the processes $\{\bar{f}_{n,k}(t) : t \geq 0\}$, $\{\chi_n(t) : t \geq 0\}$, and $\{\bar{\beta}_{n,k}(t) : t \geq 0\}$. This will be needed in Section 5 to show that finite dimensional distributions of $\{\bar{\beta}_{n,k}(t) : t \geq 0\}$ converge to those of the Ornstein-Uhlenbeck process.

**Lemma 4.1**. Fix $k \geq 1$. Let $p = n^\alpha$, $\alpha \in \left(-\frac{1}{k}, -\frac{1}{k+1}\right)$. Then, for any $t_1, t_2 \geq 0$,

$$\lim_{n \to \infty} \text{Cov}[\bar{f}_{n,k}(t_1), \bar{f}_{n,k}(t_2)] = e^{-\lambda|t_1-t_2|}.$$  

**Proof.** Fix arbitrary $t_1, t_2 \geq 0$, and let $K = e^{-\lambda|t_1-t_2|}$. We need to show that

$$\lim_{n \to \infty} \text{Cov}[\bar{f}_{n,k}(t_1), \bar{f}_{n,k}(t_2)] = K.$$  

Using (1.1) and (3.3), it follows that

$$\text{E}[f_{n,k}(t_1)f_{n,k}(t_2)] = \binom{n}{k+1} \sum_{i=0}^{k+1} \binom{k+1}{i} \binom{n-k-1}{k+1-i} p^{\frac{k+1}{2}} \left[p + (1-p)K\right]^i.$$  

Combing this with (3.2) and (3.3), it is easy to see that

$$\text{Cov}[\bar{f}_{n,k}(t_1), \bar{f}_{n,k}(t_2)] = \frac{\sum_{i=0}^{k+1} \binom{k+1}{i} \binom{n-k-1}{k+1-i} p^{\frac{k+1}{2}} \left[N + \frac{(1-p)K}{p}\right] - \binom{n}{k+1}}{\sum_{i=0}^{k+1} \binom{k+1}{i} \binom{n-k-1}{k+1-i} p^{\frac{k+1}{2}} \left[p + (1-p)K\right] - \binom{n}{k+1}}.$$
Now using the fact that \( \binom{n}{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} \binom{n-k-1}{k+1-i} \), we have
\[
Cov[\bar{f}_{n,k}(t_1), \bar{f}_{n,k}(t_2)] = \frac{\sum_{i=2}^{k+1} \binom{k+1}{i} \binom{n-k-1}{k+1-i} [(1 + \frac{1-p}{p} K)^{\frac{i}{2}} - 1]}{\sum_{i=2}^{k+1} \binom{k+1}{i} \binom{n-k-1}{k+1-i} [\frac{1-p}{p} ^{\frac{i}{2}} - 1]}. 
\]

By expanding terms inside the square brackets and cancelling out \( \frac{1-p}{p} \), we get
\[
Cov[\bar{f}_{n,k}(t_1), \bar{f}_{n,k}(t_2)] = K \frac{\sum_{i=2}^{k+1} \binom{k+1}{i} \binom{n-k-1}{k+1-i} \left[ \sum_{j=1}^{(\frac{i}{2})} c_{ij} \left( \frac{1-p}{p} K \right)^{j-1} \right]}{\sum_{i=2}^{k+1} \binom{k+1}{i} \binom{n-k-1}{k+1-i} \left[ \sum_{j=1}^{(\frac{i}{2})} \left( \frac{1}{p} \right)^{j-1} \right]},
\]
where \( c_{ij} = \binom{\frac{i}{2}}{j} \). Now observe that the term corresponding to \( i = 2 \) inside the summation in both the numerator as well as denominator is the same. Hence,
\[
Cov[\bar{f}_{n,k}(t_1), \bar{f}_{n,k}(t_2)] = K + K \frac{\sum_{i=3}^{k+1} \binom{k+1}{i} \binom{n-k-1}{k+1-i} \left[ \sum_{j=1}^{(\frac{i}{2})} c_{ij} \left( (1-p) K \right)^{j-1} - 1 \right] \left[ \sum_{j=1}^{(\frac{i}{2})} \left( \frac{1}{p} \right)^{j-1} \right]}{\sum_{i=2}^{k+1} \binom{k+1}{i} \binom{n-k-1}{k+1-i} \left[ \sum_{j=1}^{(\frac{i}{2})} \left( \frac{1}{p} \right)^{j-1} \right]},
\]
\[
:= K(1 + \mathcal{Z}_{n,k}). \tag{4.1}
\]

To prove the desired result, it suffices to show that \( \mathcal{Z}_{n,k} \to 0 \) as \( n \to \infty \). If \( k = 1 \), then \( \mathcal{Z}_{n,k} = 0 \) for each \( n \) and hence \( \lim_{n \to \infty} \mathcal{Z}_{n,k} = 0 \) trivially. Suppose that \( k \geq 2 \). By multiplying the numerator and denominator of \( \mathcal{Z}_{n,k} \) by \( p^{(\frac{k+1}{2})-1} \), observe that one can rewrite \( \mathcal{Z}_{n,k} \) as
\[
\mathcal{Z}_{n,k} = \frac{\sum_{i=3}^{k+1} \sum_{j=1}^{(\frac{i}{2})} \omega_{ij} \binom{n-k-1}{k+1-i} p^{(\frac{k+1}{2})-j}}{\sum_{i=2}^{k+1} \sum_{j=1}^{(\frac{i}{2})} \xi_{ij} \binom{n-k-1}{k+1-i} p^{(\frac{k+1}{2})-j}}
\]
for some real constants \( \{\omega_{ij}\} \) and \( \{\xi_{ij}\} \). Since \( \binom{n-k-1}{k+1-i} = \Theta(n^{k+1-i}) \), it follows that to show \( \lim_{n \to \infty} \mathcal{Z}_{n,k} = 0 \) one only needs to show that \( \lim_{n \to \infty} \mathcal{Z}'_{n,k} = 0 \),
where

\[ Z'_{n,k} := \frac{k+1}{\sum_{i=3}^{k+1} \sum_{j=1}^{(i)} \omega_{ij} n^{k+1-i} p^{(k+1)-j}}. \]

Since \( p = n^\alpha \), the power of \( n \) in the summand of numerator as well as denominator of \( Z'_{n,k} \) is of the form

\[ k + 1 - i + \alpha \left( \binom{k+1}{2} - j \right). \]

Because \( \alpha < 0 \), we have

\[ \arg \max_{1 \leq j \leq \binom{i}{2}} \left( k + 1 - i + \alpha \left( \binom{k+1}{2} - j \right) \right) = \binom{i}{2}. \] (4.2)

Further, the restriction that \( \alpha > -1/k \) shows that, for each \( i < k \),

\[ k + 1 - i + \alpha \left( \binom{k+1}{2} - \binom{i}{2} \right) \geq k + 1 - (i + 1) + \alpha \left( \binom{k+1}{2} - \binom{i+1}{2} \right). \] (4.3)

From (4.2) and (4.3), it follows that the largest power of \( n \) in the numerator of \( Z'_{n,k} \) is

\[ k + 1 - 3 + \alpha \left( \binom{k+1}{2} - \binom{3}{2} \right), \] (4.4)

while, in the denominator, it is

\[ k + 1 - 2 + \alpha \left( \binom{k+1}{2} - \binom{2}{2} \right). \] (4.5)

Because \( k \geq 2 \) and consequently \( \alpha \geq -1/2 \), it follows that the term in (4.5) is larger than the term in (4.4). This shows that \( \lim_{n \to \infty} Z'_{n,k} = 0 \) as desired.

This completes the proof. \( \square \)

**Lemma 4.2.** Fix \( k \geq 1 \). Let \( p = n^\alpha, \alpha \in \left( -\frac{1}{k}, -\frac{1}{k+1} \right) \). Then, for any \( t_1, t_2 \geq 0 \),

\[ \lim_{n \to \infty} \Cov[\tilde{\chi}_n(t_1), \tilde{\chi}_n(t_2)] = e^{-\lambda|t_1-t_2|}. \]

**Proof.** We need to show that

\[ \lim_{n \to \infty} \frac{\Cov[\chi_n(t_1), \chi_n(t_2)]}{\sqrt{\Var[\chi_n(t_1)] \Var[\chi_n(t_2)]}} = e^{-\lambda|t_1-t_2|}. \]
However, since Lemma 3.3 holds, it suffices to show that
\[
\lim_{n \to \infty} \frac{\text{Cov}[\chi_n(t_1), \chi_n(t_2)]}{\sqrt{\text{Var}[f_{n,k}(t_1)] \text{Var}[f_{n,k}(t_2)]}} = e^{-\lambda|t_1 - t_2|}.
\]
But the term inside limit on the left hand side equals
\[
\sqrt{\text{Var}[f_{n,k}(t_1)] \text{Var}[f_{n,k}(t_2)]}.
\]
Lemma 4.1 shows that the first term converges to \(e^{-\lambda|t_1 - t_2|}\) as \(n \to \infty\). The remaining three terms converge to zero because of Lemma 3.2. The desired result thus follows. \(\square\)

**Theorem 4.1.** Fix \(k \geq 1\). Let \(p = n^\alpha, \alpha \in (-\frac{1}{k}, -\frac{1}{k+1})\). Then, for any \(t_1, t_2 \geq 0\),
\[
\lim_{n \to \infty} \text{Cov}[\bar{\beta}_{n,k}(t_1), \bar{\beta}_{n,k}(t_2)] = e^{-\lambda|t_1 - t_2|}.
\]

**Proof.** Because of Lemma 3.5 it suffices to show that
\[
\lim_{n \to \infty} \frac{\text{Cov}[\beta_{n,k}(t_1), \beta_{n,k}(t_2)]}{\sqrt{\text{Var}[\chi_n(t_1)] \text{Var}[\chi_n(t_1)]}} = e^{-\lambda|t_1 - t_2|}.
\]
But the term inside limit on the left hand side equals
\[
\sqrt{\text{Var}[\chi_n(t_1)] \text{Var}[\chi_n(t_1)]}.
\]
Lemma 4.2 shows that the first term converges to \(e^{-\lambda|t_1 - t_2|}\) as \(n \to \infty\). The remaining three terms converge to zero because of Lemma 3.4. The desired result thus follows. \(\square\)
5. Convergence of Finite Dimensional Distributions

In this section we show that the finite dimensional distributions of \( \tilde{\beta}_{n,k}(t) : t \geq 0 \) converge to those of the stationary Ornstein-Uhlenbeck process. We establish this result by first showing that the finite dimensional distributions of \( \tilde{f}_{n,k}(t) : t \geq 0 \), and consequently \( \tilde{\chi}_n(t) : t \geq 0 \), converge to that of the stationary Ornstein-Uhlenbeck process. Note that for two real valued random variables \( X \) and \( Y \), we will write \( X \overset{d}= Y \) to indicate that \( X \) and \( Y \) have the same distribution. Further, we will use \( a_{12} \) to denote the number of vertices common to sets \( A_1 \) and \( A_2 \), \( a_{123} \) to denote the number of vertices common to \( A_1, A_2, \) and \( A_3 \), and so on.

Lemma 5.1. Fix \( k \geq 1 \). Let \( p = \alpha \in \left( -\frac{1}{k}, -\frac{1}{k+1} \right) \). Then, for any \( m \in \mathbb{N} \) and any \( t_1, \ldots, t_m \), as \( n \to \infty \),

\[
(\tilde{f}_{n,k}(t_1), \ldots, \tilde{f}_{n,k}(t_m)) \overset{d}{\Rightarrow} (U_{\lambda}(t_1), \ldots, U_{\lambda}(t_m)).
\]

Proof. Fix \( m \in \mathbb{N} \), arbitrary \( t_1, \ldots, t_m \geq 0 \), and arbitrary \( \omega_1, \ldots, \omega_m \in \mathbb{R} \). As a consequence of the Cramér-Wold theorem, to prove the desired result it suffices to show that, as \( n \to \infty \),

\[
\omega_1 \tilde{f}_{n,k}(t_1) + \cdots + \omega_m \tilde{f}_{n,k}(t_m) \overset{d}{\Rightarrow} \omega_1 U_{\lambda}(t_1) + \cdots + \omega_m U_{\lambda}(t_m).
\]  
(5.1)

But, by definition, \( \{U_{\lambda}(t) : t \geq 0\} \) is a zero mean Gaussian process with \( \text{Cov}[U_{\lambda}(t_i), U_{\lambda}(t_j)] = e^{-|t_i - t_j|} \). Hence,

\[
\frac{\omega_1 U_{\lambda}(t_1) + \cdots + \omega_m U_{\lambda}(t_m)}{\sqrt{\omega_1^2 + \cdots + \omega_m^2 + 2 \sum_{i<j} \omega_i \omega_j e^{-|t_i - t_j|}}} \overset{d}{\Rightarrow} \mathcal{N}(0, 1).
\]

Further, Lemma 4.1 shows that

\[
\lim_{n \to \infty} \frac{\sqrt{\text{Var}\left[\sum_{i=1}^{m} \omega_i \tilde{f}_{n,k}(t_i)\right]}}{\sqrt{\omega_1^2 + \cdots + \omega_m^2 + 2 \sum_{i<j} \omega_i \omega_j e^{-|t_i - t_j|}}} = 1.
\]  
(5.2)

Hence, it follows that to prove (5.1) one only needs to show that, as \( n \to \infty \),

\[
\mathcal{W}_{n,k} := \frac{\omega_1 \tilde{f}_{n,k}(t_1) + \cdots + \omega_m \tilde{f}_{n,k}(t_m)}{\sqrt{\text{Var}\left[\sum_{i=1}^{m} \omega_i \tilde{f}_{n,k}(t_i)\right]}} \overset{d}{\Rightarrow} \mathcal{N}(0, 1).
\]  
(5.3)
Let $S_{n,k,m} = \binom{m}{k+1} \times [m]$, where $[m] := \{1, \ldots, m\}$. For $(A_1, i) \in S_{n,k,m}$, let

$$
\aleph(A_1, i) \equiv \aleph_{n,k,m}(A_1, i) := \{(A_2, j) \in S_{n,k,m} : a_{12} \geq 2\}.
$$

(5.4)

For each $n$, it follows from (1.7) and (3.1) that

$$
\mathcal{W}_{n,k} = \frac{\sum_{(A,i)\in S_{n,k,m}} \omega_i \bar{I}_A(t_i)}{\sqrt{\text{Var}[\sum_{i=1}^{m} \omega_i \bar{f}_{n,k}(t_i)]}},
$$

where $\bar{I}_A(t_i) = \left( \frac{I_A(t_i) - \text{Var}[I_A(t_i)]}{\sqrt{\text{Var}[I_A(t_i)]}} \right)$. Clearly, $\text{Var}[\bar{I}_A(t_i)] = 0$ and $\mathbb{E}[\mathcal{W}_{n,k}^2] = 1$.

Also, $\bar{I}_A(t_j)$ is independent of $\bar{I}_A(t_i)$ if and only if $(A_2, j) \notin \aleph(A_1, i)$. Hence, from Theorem 2.1 we have

$$
d_1(\mathcal{W}_{n,k}, \mathcal{N}(0, 1)) \leq \frac{\gamma \omega^3}{(\text{Var}[\sum_{i=1}^{m} \omega_i \bar{f}_{n,k}(t_i)])^{3/2}} \times \sum_{(A_1, i)\in S_{n,k,m}} \sum_{(A_2, j), (A_3, t) \notin \aleph(A_1, i)} \mathbb{E}[\bar{I}_A(t_i) \bar{I}_A(t_j)]
$$

$$
\times \mathbb{E}[\bar{I}_A(t_i) \bar{I}_A(t_j) \bar{I}_A(t_t)] + \mathbb{E}[\bar{I}_A(t_i) \bar{I}_A(t_j)] \mathbb{E}[\bar{I}_A(t_t)] \leq \frac{16 \mathbb{E}[1_{A_1(t_i)} 1_{A_2(t_j)} 1_{A_3(t_t)}]}{\sqrt{\text{Var}[f_{n,k}(t_i)] \text{Var}[f_{n,k}(t_j)] \text{Var}[f_{n,k}(t_t)]}}.
$$

we have

$$
d_1(\mathcal{W}_{n,k}, \mathcal{N}(0, 1)) \leq \frac{16 \gamma^3}{(\text{Var}[\sum_{i=1}^{m} \omega_i \bar{f}_{n,k}(t_i)])^{3/2}} \frac{\sum_{(A_1, i)\in S_{n,k,m}} \sum_{(A_2, j), (A_3, t) \notin \aleph(A_1, i)} \mathbb{E}[1_{A_1(t_i)} 1_{A_2(t_j)} 1_{A_3(t_t)}]}{\sqrt{\text{Var}[f_{n,k}(t_i)] \text{Var}[f_{n,k}(t_j)] \text{Var}[f_{n,k}(t_t)]}}.
$$

Combining this with (5.2) and defining

$$
\mathcal{R}_{n,k} := \frac{\sum_{(A_1, i)\in S_{n,k,m}} \sum_{(A_2, j), (A_3, t) \notin \aleph(A_1, i)} \mathbb{E}[1_{A_1(t_i)} 1_{A_2(t_j)} 1_{A_3(t_t)}]}{\sqrt{\text{Var}[f_{n,k}(t_i)] \text{Var}[f_{n,k}(t_j)] \text{Var}[f_{n,k}(t_t)]}},
$$

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it follows that to establish (5.3) one only needs to show that
\[ \lim_{n \to \infty} R_{n,k} = 0. \] (5.5)

Fix arbitrary \( t \geq 0 \) and let
\[
R'_{n,k} := \sum_{A_1 \in \left( [n] \right)_{k+1}} \sum_{A_2, A_3 \in \mathcal{A}(A_1)} \mathbb{E} \left[ 1_{A_1}(t)1_{A_2}(t)1_{A_3}(t) \right] \times \frac{\left( \text{Var}[f_{n,k}(t)] \right)^{3/2}}{\left( \text{Var}[f_{n,k}(t)] \right)^{3/2}},
\]
where \( \mathcal{A}(A_1) \equiv \mathcal{A}_{n,k}(A_1) := \{ A_2 \in \left( [n] \right)_{k+1} : a_{12} \geq 2 \} \). In [13], as part of proof of Claim 2.5 (ii), it was shown that \( \lim_{n \to \infty} R'_{n,k} = 0 \). In the remaining part of this proof, we will show that
\[ R_{n,k} \leq m^3 R'_{n,k}. \] (5.6)

This is clearly sufficient to establish (5.5).

Recall from (3.3) that \( \text{Var}[f_{n,k}(t)] \) is independent of \( t \). Hence, it follows that the denominators in \( R_{n,k} \) and \( R'_{n,k} \) are identical. Now using (1.1) and (1.3) and the fact that \( p + (1 - p)e^{-\tau} \leq 1 \) for any \( \tau \geq 0 \), observe that
\[
\mathbb{E} \left[ 1_{A_1}(t_i)1_{A_2}(t_j)1_{A_3}(t_k) \right] \leq p^{a_{12} - a_{13} - a_{23} + a_{123}} = \mathbb{E} \left[ 1_{A_1}(t)1_{A_2}(t)1_{A_3}(t) \right].
\]

From this and the definition of \( R_{n,k} \), (5.6) follows easily. The desired result thus follows.

**Lemma 5.2.** Fix \( k \geq 1 \). Let \( p = n^\alpha, \alpha \in \left( -\frac{1}{k}, -\frac{1}{k+1} \right) \). Then, for any \( m \in \mathbb{N} \) and any \( t_1, \ldots, t_m \), as \( n \to \infty \),
\[
\left( \tilde{x}_n(t_1), \ldots, \tilde{x}_n(t_m) \right) \Rightarrow (\mathcal{U}_\lambda(t_1), \ldots, \mathcal{U}_\lambda(t_m)).
\]

**Proof.** As in the proof of Lemma 5.1, it suffices to show that for arbitrary \( \omega_1, \ldots, \omega_m \in \mathbb{R} \), as \( n \to \infty \),
\[
\frac{\omega_1 \tilde{x}_n(t_1) + \cdots + \omega_m \tilde{x}_n(t_m)}{\sqrt{\text{Var}[\sum_{i=1}^m \omega_i \tilde{x}_n(t_i)]}} \Rightarrow \mathcal{N}(0, 1).
\]
Since Lemmas 4.1 and 4.2 hold, it in fact suffices to show that
\[
\frac{\omega_1 \bar{x}_n(t_1) + \cdots + \omega_m \bar{x}_n(t_m)}{\sqrt{\text{Var}[\sum_{i=1}^{m} \omega_i \bar{f}_{n,k}(t_i)]}} \Rightarrow N(0,1) .
\] (5.7)

Note that from Lemma 3.2 and using Cauchy-Schwarz inequality we have that
for all \(i\),
\[
\text{Var}[-1 \bar{\chi}_n(t_i) - \bar{f}_{n,k}(t_i)] = \frac{1}{\sqrt{\text{Var}[\chi_n(t_i)]}} - \frac{1}{\sqrt{\text{Var}[f_{n,k}(t_i)]}} + \frac{\text{Var}[-1 \bar{\chi}_n(t_i) - \bar{f}_{n,k}(t_i)]}{\text{Var}[f_{n,k}(t_i)]} \to 0,
\]
as \(n \to \infty\). Using the above estimate and again Cauchy-Schwarz inequality, we have that
we have that
\[
\text{Var} \left[ \sum_{i=1}^{m} \omega_i \left( [-1 \bar{\chi}_n(t_i) - \bar{f}_{n,k}(t_i)] \right) \right] \to 0.
\]
Combining the above result with Slutsky’s theorem and Lemma 5.1 follows and so does the proof. □

**Theorem 5.1.** Fix \(k \geq 1\). Let \(p = n^\alpha, \alpha \in \left( -\frac{1}{k}, -\frac{1}{k+1} \right) \). Then, for any \(m \in \mathbb{N}\) and any \(t_1, \ldots, t_m\), as \(n \to \infty\),
\[
(\hat{\beta}_{n,k}(t_1), \ldots, \hat{\beta}_{n,k}(t_m)) \Rightarrow (U_{\lambda}(t_1), \ldots, U_{\lambda}(t_m)).
\]

**Proof.** Using Lemma 3.4 and bounds as above in Lemma 5.2, we obtain that
for any \(\omega_1, \ldots, \omega_m \in \mathbb{R}\),
\[
\lim_{n \to \infty} \text{Var} \left[ \sum_{i=1}^{m} \omega_i \left( [-1 \bar{\chi}_n(t_i) - \bar{f}_{n,k}(t_i)] \right) \right] = 0.
\]
Arguing now as in the proof of Lemma 5.2 and making use of Lemma 4.2 and Theorem 4.1, the desired result is easy to see. □
6. Tightness

In this section, we show that the sequence of processes \( \{\tilde{\beta}_{n,k}(t) : t \geq 0\} : n \geq 1 \) is tight. By Theorem 2.2, it suffices to establish the two conditions \( \mathbf{C}_1 \) and \( \mathbf{C}_2 \).

**Lemma 6.1.** Fix \( k \geq 1 \). Let \( p = n^\alpha, \alpha \in \left( -\frac{1}{k}, -\frac{1}{k+1} \right) \). Then, for the process \( \{\tilde{\beta}_{n,k}(t) : t \geq 0\} \), condition \( \mathbf{C}_1 \) holds true with \( \Upsilon = 2 \), i.e.

\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{E}[\tilde{\beta}_{n,k}(\delta) - \tilde{\beta}_{n,k}(0)]^2 = 0.
\]

**Proof.** Clearly,

\[
\mathbb{E}[\tilde{\beta}_{n,k}(\delta) - \tilde{\beta}_{n,k}(0)]^2 = 2 - 2\text{Cov}[\tilde{\beta}_{n,k}(\delta), \tilde{\beta}_{n,k}(0)].
\]

Now using 4.1 we have that

\[
\lim_{n \to \infty} \mathbb{E}[\tilde{\beta}_{n,k}(\delta) - \tilde{\beta}_{n,k}(0)]^2 = 2 - 2e^{-\delta}.
\]

The desired result is immediate. \( \square \)

Note that using the same argument as above, it also follows that \( \{\tilde{f}_{n,k}(t) : t \geq 0\} \) and \( \{\tilde{\chi}_{n}(t) : t \geq 0\} \) satisfy condition \( \mathbf{C}_1 \). We now work towards showing that condition \( \mathbf{C}_2 \) holds true for the process \( \{\tilde{\beta}_{n,k}(t) : t \geq 0\} \). Our approach will be to first establish this result for \( \{\tilde{f}_{n,k}(t) : t \geq 0\} \) and \( \{\tilde{\chi}_{n}(t) : t \geq 0\} \) and from this conclude the same for \( \{\tilde{\beta}_{n,k}(t) : t \geq 0\} \).

**Lemma 6.2.** Fix \( k \geq 1 \). Let \( p = n^\alpha, \alpha \in \left( -\frac{1}{k}, -\frac{1}{k+1} \right) \). Then, for the process \( \{\tilde{f}_{n,k}(t) : t \geq 0\} \), condition \( \mathbf{C}_2 \) holds with \( \Upsilon_1 = \Upsilon_2 = 2 \), i.e., for any \( T > 0 \), there exists \( K_f > 0 \) such that for all \( n \geq 1 \),

\[
\mathbb{E} \left[ \tilde{f}_{n,k}(t+h) - \tilde{f}_{n,k}(t) \right]^2 \left[ \tilde{f}_{n,k}(t) - \tilde{f}_{n,k}(t-h) \right] \leq K_f h^2
\]

for all \( 0 \leq t \leq T + 1 \) and \( 0 \leq h \leq t \).

This follows from the next result and hence we prove only that.
Lemma 6.3. Fix $k \geq 1$. Let $p = n^\alpha$, $\alpha \in \left(-\frac{1}{k}, -\frac{1}{k+1}\right)$. Then, for the process \{\bar{\chi}_n(t) : t \geq 0\}, condition $\square_2$ holds with $\Upsilon_1 = \Upsilon_2 = 2$, i.e., for any $T > 0$, there exists $K_\chi > 0$ such that for all $n \geq 1$,

$$E[\bar{\chi}_{n,k}(t + h) - \bar{\chi}_{n,k}(t)]^2 [\bar{\chi}_{n,k}(t) - \bar{\chi}_{n,k}(t - h)]^2 \leq K_\chi h^2$$

for all $0 \leq t \leq T + 1$ and $0 \leq h \leq t$.

We first establish some notations and lemmas that will be useful in proving the above main result. Fix arbitrary $n, k \geq 1$ and let $p$ be as in Lemma 6.3. Also fix $i$ and $j$ such that $0 \leq i, j \leq n - 1$ and let

$$\xi_{ij}(h) := E[f_{n,i}(2h) - f_{n,i}(h)]^2 [f_{n,j}(h) - f_{n,j}(0)]^2. \quad (6.1)$$

For $\bar{\Lambda} = (A_1, A_2, A_3, A_4) \in \left(\binom{n}{i+1}\times\binom{n}{j+1}\right)^2$, let $a_q$ denote the number of vertices in $A_q$, $a_{qr}$ the number of vertices common to both $A_q$ and $A_r$, and so on. Note that inequalities such as $a_{1234} \leq a_{qrs} \leq a_q$ for any $q, r, s \in \{1, \ldots, 4\}$ hold trivially. Let

$$\tau(\bar{\Lambda}) = (a_1, \ldots, a_4, a_{12}, \ldots, a_{34}, a_{13}, \ldots, a_{234}, a_{1234}),$$

$$\text{ver}(\bar{\Lambda}) = \sum_{q=1}^4 a_q - \sum_{1 \leq q < r \leq 4} a_{qr} + \sum_{1 \leq q < r < s \leq 4} a_{qrs} - a_{1234}, \quad (6.2)$$

$$\text{pair}(\bar{\Lambda}) = \sum_{q=1}^4 \left(\begin{array}{c} a_q \\ 2 \end{array}\right) - \sum_{1 \leq q < r \leq 4} \left(\begin{array}{c} a_{qr} \\ 2 \end{array}\right) + \sum_{1 \leq q < r < s \leq 4} \left(\begin{array}{c} a_{qrs} \\ 2 \end{array}\right) - \left(\begin{array}{c} a_{1234} \\ 2 \end{array}\right), \quad (6.3)$$

and

$$g(h; \bar{\Lambda}) := [1_{A_2}(2h) - 1_{A_1}(h)][1_{A_2}(2h) - 1_{A_2}(h)]$$

$$\times [1_{A_3}(h) - 1_{A_3}(0)][1_{A_4}(h) - 1_{A_4}(0)]. \quad (6.4)$$

Here $\tau(\bar{\Lambda})$ denotes the intersection type of $\bar{\Lambda}$, while $\text{ver}(\bar{\Lambda})$ and $\text{pair}(\bar{\Lambda})$ denote respectively the sum of vertices and maximum possible edges in $A_1, \ldots, A_4$ with common vertices and edges counted only once. Terms of the form $g(h; \bar{\Lambda})$ appear in the expansion of $\xi_{ij}(h)$ and hence will be useful later.
For $\vec{A} \in \binom{[n]}{i+1}^2 \times \binom{[n]}{j+1}^2$, we will say $\vec{A} \sim \vec{B}$ if there exists a permutation $\pi$ of the sets in $\vec{B}$ such that $\tau(\vec{A}) = \tau(\pi(\vec{B}))$. Clearly, $\sim$ is an equivalence relation. Let $\Gamma_{ij} := \{[\vec{A}]\}$ denote the quotient of $\binom{[n]}{i+1}^2 \times \binom{[n]}{j+1}^2$ under $\sim$, where $[\vec{A}]$ denotes the equivalence class of $\vec{A}$. Since each $a_{qr}, a_{qrs}$, and $a_{1234}$ (11 variables in total) is a number between 0 and $\max \{i + 1, j + 1\} \leq (i + j + 1)$, it is easy to see that the number of equivalence classes in $\Gamma_{ij}$ satisfies

$$|\Gamma_{ij}| \leq (i + j + 1)^{11}.$$  

(6.5)

We will say $\vec{A} \in \binom{[n]}{i+1}^2 \times \binom{[n]}{j+1}^2$ has an independent set if there exists $q \in \{1, 2, 3, 4\}$ such that $a_{qr} \leq 1$ for all $r \neq q$. That is, there exists a special set among $A_1, \ldots, A_4$ which shares at most one vertex with the remaining three sets. Clearly, the indicator associated with this special set is independent of the indicator associated with the other three sets. Based on this description, let

$$\mathcal{S}_{ij} := \{[\vec{A}] \in \Gamma_{ij} : \exists q \in \{1, 2, 3, 4\} \text{ such that } \forall r \neq q, a_{qr} \leq 1\}.$$  

(6.6)

**Lemma 6.4.** Fix arbitrary $n, k \geq 1$, and let $p$ be as in Lemma 6.3. Also fix $i$ and $j$ such that $0 \leq i, j \leq n - 1$. Fix $\vec{A} \in \binom{[n]}{i+1}^2 \times \binom{[n]}{j+1}^2$.

1. If $[\vec{A}] \in \mathcal{S}_{ij}$, then $E[g(h; \vec{A})] \equiv 0$.

2. If $[\vec{A}] \in \Gamma_{ij} \setminus \mathcal{S}_{ij}$, then there exists some universal constant $\gamma \geq 0$ (independent of $\vec{A}$, $i$, $j$, $k$, and $n$) such that

$$|E[g(h; \vec{A})]| \leq \gamma(i + j + 1)^4 p^{pair(\vec{A})} h^2$$

for all $0 \leq h \leq 1$.

**Proof.** The first statement is trivial. So we discuss only the second one.

Consider $\vec{A} \in \binom{[n]}{i+1}^2 \times \binom{[n]}{j+1}^2$ with $[\vec{A}] \in \Gamma_{ij} \setminus \mathcal{S}_{ij}$. For such a $\vec{A}$, $g(h; \vec{A})$ clearly satisfies the relation given in (A.1). It is not difficult to see using (1.1) and (1.3) that

$$E[g(h; \vec{A})] = p^{pair(\vec{A})} \Phi(h; \vec{A}),$$

(6.7)

where $\Phi(h; \vec{A})$ is as in (A.2). Observe that

$$\Phi(h; \vec{A}) = \sum_{\ell=1}^{16} \phi_\ell(h; \vec{A}),$$

(6.8)
where each $\phi_\ell(h; \bar{A})$ is of the form

$$
\phi_\ell(h; \bar{A}) = \pm ((1 - p)e^{-h} + p)c_1(\ell)((1 - p)e^{-2h} + p)c_2(\ell)
$$

with $c_1(\ell)$ and $c_2(\ell)$ being non-negative real numbers bounded above by

$$
\sum_{1 \leq q < r \leq 4} \left( \frac{a_{qr}}{2} \right) + \left( \frac{a_{1234}}{2} \right) \leq 7(i + j + 1)^2. \tag{6.9}
$$

Now it is not difficult to see that

$$
\Phi(h; \bar{A}) \bigg|_{h=0} = 0
$$

and

$$
\frac{\partial \Phi(h; \bar{A})}{\partial h} \bigg|_{h=0} = 0.
$$

Because of the above two facts, expanding $\Phi(h; \bar{A})$ using the Lagrangian form of Taylor series shows that for each $0 \leq h \leq 1$ there exists $c \in [0, h]$ such that

$$
\Phi(h; \bar{A}) = \frac{1}{2} h^2 \frac{\partial^2 \Phi(h; \bar{A})}{\partial h^2} \bigg|_{h=c}. \tag{6.10}
$$

Now using (6.9) and the fact that both $((1 - p)e^{-h} + p)$ and $((1 - p)e^{-2h} + p)$ are bounded from above by 1 for $h \geq 0$, it is not difficult to see that there exists some universal constant $\gamma_1 \geq 0$ (independent of $\bar{A}, i, j, k$, and $n$) such that

$$
\max_{1 \leq \ell \leq 16} \sup_{h \geq 0} \left| \frac{\partial^2 \phi_\ell(h; \bar{A})}{\partial h^2} \right| \leq \gamma_1(i + j + 1)^4.
$$

Combining this with (6.8) and (6.10), it follows that

$$
|\Phi(h; \bar{A})| \leq 8\gamma_1(i + j + 1)^4h^2.
$$

The desired result follows.

**Lemma 6.5.** Fix arbitrary $n, k \geq 1$, and let $p$ be as in Lemma 6.3. Also fix $i$ and $j$ such that $0 \leq i, j \leq n - 1$. Fix $\bar{A} \in \binom{[n]}{i+1} \times \binom{[n]}{j+1}$.

(i) If $[\bar{A}] \in \Gamma_{ij} \setminus \mathcal{J}_{ij}$, then

$$
\frac{n^{\text{vert}(\bar{A})p\text{pair}(\bar{A})}}{n^{4k}p^k(\frac{k+1}{2})^{-2}} \leq 1.
$$
(ii) If $\bar{A} \in \Gamma_{ij} \setminus \mathcal{I}_{ij}$ and $(i + j) \geq 16k + 15$, then

$$\frac{n_{\text{ver}(\bar{A})}p_{\text{pair}(\bar{A})}}{n^{4k}p^{\binom{k+1}{2} - 2}} \leq \frac{1}{n^{2k+2(i+j-16k-15)}}.$$ 

Proof. Consider Statement (i). Note from (6.6) that, since $\bar{A} \in \Gamma_{ij} \setminus \mathcal{I}_{ij}$, one of the following cases must necessarily be true.

**Case A:** Either $a_{12}, a_{34} \geq 2$, or $a_{13}, a_{24} \geq 2$, or $a_{14}, a_{23} \geq 2$.

**Case B:** There exists $q \in \{1, \ldots, 4\}$ such that $a_{qr} \geq 2$ for all $r \neq q$.

In both cases, using the same arguments as those used to arrive at (8) in [14], it is easy to see that

$$n_{\text{ver}(\bar{A})}p_{\text{pair}(\bar{A})} \leq n^{4k}p^{\binom{k+1}{2} - 2}$$

which proves the desired result. The only differences in the two arguments being the following.

- There (i.e. in [14]) one dealt with intersection of three sets while here we need to deal with intersection of four sets. In both cases, however, note that independent sets are absent, i.e., each set has at least two vertices in common with one of the remaining sets.

- There an upper bound for $n_{\text{ver}(\bar{A})}p_{\text{pair}(\bar{A})}$, with $\text{ver}(\bar{A})$ and $\text{pair}(\bar{A})$ appropriately defined, was obtained by sequentially dealing with the number of vertices in the third set, then the second set, and so on. Here we have to repeat the same idea by first dealing with the number of vertices in the fourth set, then the third set, and so on.

Now consider Statement (ii). From (6.2), (6.3), and (6.6), we have

$$\text{ver}(\bar{A}) \leq 2i + 2j$$

and

$$\text{pair}(\bar{A}) \geq \max \left\{ \binom{i+1}{2}, \binom{j+1}{2} \right\}.$$ 

Using these two relations and fact that $\alpha \in \left(-\frac{1}{k}, -\frac{1}{k+1}\right)$, we have

$$\text{ver}(\bar{A}) + \alpha \text{pair}(\bar{A}) \leq 2(i + j) + \alpha \max \left\{ \binom{i+1}{2}, \binom{j+1}{2} \right\}.$$
Since \( \max \left\{ \binom{i+1}{2}, \binom{j+1}{2} \right\} \geq \binom{i+j+1}{2}/4 \), it follows that

\[
\text{ver}(\bar{A}) + \alpha\text{pair}(\bar{A}) \leq 2(i + j) + \alpha\binom{i+j+1}{2}/4.
\]

Consequently, to prove the desired result, it suffices to show that for \( i + j \geq 16k + 15 \),

\[
\frac{n^{2(i+j)+\alpha\binom{i+j+1}{2}/4}}{n^{4k}p^4\binom{k+1}{2}^2 - 2} \leq \frac{1}{n^{2k+2(i+j-16k+15)}}. \tag{6.11}
\]

Now observe that if \( i + j = 16k + 15 \), then

\[
\frac{n^{2(i+j)+\alpha\binom{i+j+1}{2}/4}}{n^{4k}p^4\binom{k+1}{2}^2} \leq \frac{1}{n^{2k}}.
\]

Suppose that for \( i' \) and \( j' \) with \( (i' + j') \geq 16k + 15 \), the desired result holds. Now consider \( i \) and \( j \) satisfying \( (i + j) = (i' + j') + 1 \). Clearly,

\[
2(i + j) - 2(i' + j') + \alpha \left( \binom{i+j+1}{2}/4 - \binom{i'+j'+1}{2}/4 \right) = 2 + \alpha (i' + j' + 1)/4
\]

\[
\leq -2,
\]

where inequality follows because \( (i' + j') \geq 16k + 15 \). By induction, (6.11) follows and consequently the proof is complete. \( \square \)

**Lemma 6.6.** Fix arbitrary \( n, k \geq 1 \), and let \( p \) be as in Lemma 6.3. Also fix \( i \) and \( j \) such that \( 0 \leq i, j \leq n-1 \). Let \( \xi_{ij}(h) \) be as in (6.1) and \( \gamma \) as in Claim 6.4.

(i) If \( (i + j) < 16k + 15 \), then

\[
\frac{\xi_{ij}(h)}{n^{4k}p^4\binom{k+1}{2}^2 - 2} \leq \gamma (i + j + 1)^{15}h^2.
\]

(ii) If \( (i + j) \geq 16k + 15 \), then

\[
\frac{\xi_{ij}(h)}{n^{4k}p^4\binom{k+1}{2}^2 - 2} \leq \gamma \frac{(i + j + 1)^{15}}{n^{2k+2(i+j-16k+15)}}h^2.
\]

**Proof.** From (6.1) and (6.4), it is easy to see that

\[
\xi_{ij}(h) = \sum_{\bar{A} \in \binom{i+1}{2} \times \binom{j+1}{2}} \mathbb{E}[g(h; \bar{A})].
\]

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Segregating terms based on their equivalence classes under ∼, it follows that

\[
\xi_{ij}(h) = \sum_{[\bar{B}] \in \Gamma_{ij}} \sum_{\bar{A} \sim \bar{B}} E[g(h; \bar{A})].
\]

Applying Claim 6.4 gives

\[
\xi_{ij}(h) \leq \gamma (i + j + 1)^4 h^2 \sum_{[\bar{B}] \in \Gamma_{ij}\setminus \mathcal{Z}_{ij}} \sum_{\bar{A} \sim \bar{B}} \sum_{\bar{A} \sim \bar{B}} p^\text{pair}(\bar{A}).
\]

Now note from (6.2) and (6.3) that, if \(\bar{A} \sim \bar{B}\), then \(\text{ver}(\bar{A}) = \text{ver}(\bar{B})\) and \(\text{pair}(\bar{A}) = \text{pair}(\bar{B})\). Further, the cardinality of the set

\[
\{\bar{A} \in (i+1)^2 \times (j+1)^2 : \bar{A} \sim \bar{B}\}
\]

is bounded above by \(n^{\text{ver}(\bar{B})}\). From these observations, it follows that

\[
\xi_{ij}(h) \leq \gamma (i + j + 1)^4 h^2 \sum_{[\bar{B}] \in \Gamma_{ij}\setminus \mathcal{Z}_{ij}} n^{\text{ver}(\bar{B})} p^\text{pair}(\bar{B}).
\]

Using (6.5) and claim 6.5 both the desired statements are now easy to see. □

Proof of Lemma 6.3. Since \(\{G(n, p, t) : t \geq 0\}\) and hence \(\{\tilde{\chi}_n(t) : t \geq 0\}\) is stationary, to prove the desired result, it suffices to show that there exists \(K_\chi > 0\) such that

\[
\mathbb{E}[\tilde{\chi}_n(2h) - \tilde{\chi}_n(h)]^2 [\tilde{\chi}_n(h) - \tilde{\chi}_n(0)]^2 \leq K_\chi h^2 \tag{6.12}
\]

for \(0 \leq h \leq 1\) and \(n \geq 1\). From Corollary 3.1 and Lemma 3.3 we have \(\text{Var}[\chi_n(t)] = \Theta(n^{2k} p^2 \binom{k+1}{2}^{-1})\). Consequently, to prove the desired result, it suffices to show that there exists \(K_\chi > 0\) such that

\[
\Omega_{n,k}(h) := \frac{\mathbb{E}[\chi_n(2h) - \chi_n(h)]^2 [\chi_n(h) - \chi_n(0)]^2}{n^{4k} p^4 \binom{k+1}{2}^{-2}} \leq K_\chi h^2.
\]

Using (1.5) and the triangle inequality, we get

\[
\sqrt{\Omega_{n,k}(h)} \leq \sum_{0 \leq i,j \leq n-1} \sqrt{\frac{\xi_{ij}(h)}{n^{4k} p^4 \binom{k+1}{2}^{-2}}},
\]

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where \( \xi_{ij}(h) \) is as in (6.1). Segregating based on the sum \((i+j)\), one gets

\[
\sqrt{\Omega_{n,k}(h)} \leq \sum_{0 \leq \ell \leq 2(n-1)} \sum_{(i+j) = \ell} \frac{\xi_{ij}(h)}{n^{4k}p^{(k+1)/2} - 2}.
\]

This implies that

\[
\sqrt{\Omega_{n,k}(h)} \leq \sum_{0 \leq \ell < \infty} \sum_{(i+j) = \ell} \frac{\xi_{ij}(h)}{n^{4k}p^{(k+1)/2} - 2}.
\]

From this it follows that

\[
\sqrt{\Omega_{n,k}(h)} \leq \text{Term}_1 + \text{Term}_2,
\]

where

\[
\text{Term}_1 := \sum_{0 \leq \ell < 16k+15} \sum_{(i+j) = \ell} \frac{\xi_{ij}(h)}{n^{4k}p^{(k+1)/2} - 2}
\]

and

\[
\text{Term}_2 := \sum_{16k+15 \leq \ell < \infty} \sum_{(i+j) = \ell} \frac{\xi_{ij}(h)}{n^{4k}p^{(k+1)/2} - 2}.
\]

Using part (i) of Claim 6.6 and the fact that the cardinality of the set \(\{(i,j) : i, j \geq 0, i+j = \ell\}\) is \(\ell + 1\), it is easy to see that

\[
\text{Term}_1 \leq \sqrt{\gamma h^2 K_1},
\]

where

\[
K_1 := \sum_{0 \leq \ell < 16k+14} (\ell + 1)^9.
\]

Note that \(K_1\) is a constant independent of \(n\) and \(0 \leq h \leq 1\). Similarly, using Statement 2. of Claim 6.6 we obtain

\[
\text{Term}_2 \leq \sqrt{\gamma h^2 K_2(n)},
\]

where

\[
K_2(n) := \sum_{16k+14 \leq \ell < \infty} \frac{(\ell + 1)^9}{n^{k+(\ell-16k-15)}}.
\]

Clearly \(K_2(n)\) is finite for each \(n > 1\) and is monotonically decreasing. Consequently, if we let \(K := \gamma(K_1 + K_2(2))^2\), then the desired result follows. \(\Box\)
Theorem 6.1. Fix $k \geq 1$. Let $p = n^\alpha$, $\alpha \in \left(-\frac{1}{2}, -\frac{1}{2k+1}\right)$. Then, for the process \{\(\bar{\beta}_{n,k}(t) : t \geq 0\)\}, condition $\mathcal{C}_2$ holds with $\Upsilon_1 = \Upsilon_2 = 2$, viz. for any $T > 0$, there exists $K_\beta > 0$ such that, for all $n \geq 1$,

\[ E \left[ \bar{\beta}_{n,k}(t + h) - \bar{\beta}_{n,k}(t) \right]^2 \left[ \bar{\beta}_{n,k}(t) - \bar{\beta}_{n,k}(t - h) \right]^2 \leq K_\beta h^2, \]

for all $0 \leq t \leq T + 1$ and $0 \leq h \leq t$.

Proof. From Lemmas 3.3 and 3.5 and Corollary 3.1, we have
\[
\text{Var}[\beta_{n,k}(t)] = \Theta(n^{2k}p^{2(k+1)/2} - 1).
\]
Consequently, as discussed in Lemma 6.3 to prove the desired result it suffices to show that there exists $K_\beta > 0$ such that
\[
\Omega_{n,k}(h) := \frac{E[\beta_{n,k}(2h) - \beta_{n,k}(h)]^2[\beta_{n,k}(h) - \beta_{n,k}(0)]^2}{n^{4k}p^{4(k+1)/2} - 2} \leq K_\beta h^2
\]
for all $n \geq 1$ and $0 \leq h \leq 1$.

Now fix an arbitrary $h \in [0, 1]$ and consider the event
\[
E = \{(\frac{1}{n})^{\chi_n(0)} = \beta_{n,k}(0)\} \cap \{(\frac{1}{n})^{\chi_n}(h) = \beta_{n,k}(h)\} \cap \{(\frac{1}{n})^{\chi_n}(2h) = \beta_{n,k}(2h)\}. \tag{6.13}
\]
Then, observe that
\[
E[\beta_{n,k}(2h) - \beta_{n,k}(h)]^2[\beta_{n,k}(h) - \beta_{n,k}(0)]^2 = \text{Term}_1 + \text{Term}_2, \tag{6.14}
\]
where
\[
\text{Term}_1 = E[\beta_{n,k}(2h) - \beta_{n,k}(h)]^2[\beta_{n,k}(h) - \beta_{n,k}(0)]^21_E \tag{6.15}
\]
and
\[
\text{Term}_2 = E[\beta_{n,k}(2h) - \beta_{n,k}(h)]^2[\beta_{n,k}(h) - \beta_{n,k}(0)]^21_{E^c}. \tag{6.16}
\]
Clearly,
\[
\text{Term}_1 = E[\chi_n(2h) - \chi_n(h)]^2[\chi_n(h) - \chi_n(0)]^21_E \tag{6.17}
\]
and hence
\[
\text{Term}_1 \leq E[\chi_n(2h) - \chi_n(h)]^2[\chi_n(h) - \chi_n(0)]^2. \tag{6.18}
\]
Using Lemma 6.3 it follows that

\[ \frac{\text{Term}_1}{n^{4k}p^4\left(\frac{k+1}{2}\right)^{-2}} \leq K^{\prime}h^2. \]  

(6.19)

To obtain a bound on \( \text{Term}_2 \), we consider an alternate but equivalent description of the dynamic Erdős-Rényi graph. Specifically, to each edge \( e \), independently associate two independent sequences \( T^e := \{T^e_i \}_{i \geq 1} \) and \( I^e := \{I^e_i \}_{i \geq 0} \), where \( T^e \) are arrival times of a Poisson process with parameter \( \lambda \) and \( I^e \) are i.i.d. Bernoulli random variables which take the ‘on’ state with probability \( p \) and ‘off’ state with probability \( 1 - p \). Let \( T^e_0 = 0 \). If we define

\[ 1_e(t) := \sum_{i \geq 0} 1\{T^e_i \leq t < T^e_{i+1}\} I^e_i, \]

then clearly the behaviour of edge \( e \) is equivalent to that given in Definition 1.1.

Let \( S_{0,h} := \sum_{e} \sum_{i \geq 1} 1\{T^e_i \leq h\} \) denote the sum of arrivals that happened across each edge in time \((0, h]\). Let \( \tau_1, \tau_2, \ldots \), with \( \tau_i \leq \tau_{i+1} \), denote the sequence of arrival times in \((0, h]\) at which these \( S_{0,h} \) arrivals occurred. Note that \( \tau_i \) and \( \tau_{i+1} \) could correspond to arrivals along different edges. Separately, let \( \tau_0 = 0 \).

Let \( \mathcal{P}_0 \) denote the event that no arrival occurs at time 0, i.e. for all \( i \geq 1, \tau_i > 0 \). Then,

\[ |\beta_{n,k}(h) - \beta_{n,k}(0)|_{\mathcal{P}_0} \leq \sum_{i=1}^{S_{0,h}} |\beta_{n,k}(\tau_i) - \beta_{n,k}(\tau_{i-1})|_{\mathcal{P}_0}. \]

Using Lemma 2.2 from [17], it then follows that

\[ |\beta_{n,k}(h) - \beta_{n,k}(0)|_{\mathcal{P}_0} \leq \sum_{i=1}^{S_{0,h}} |f_{n,k}(\tau_i) - f_{n,k}(\tau_{i-1})|_{\mathcal{P}_0} + \sum_{i=1}^{S_{0,h}} |f_{n,k+1}(\tau_i) - f_{n,k+1}(\tau_{i-1})|_{\mathcal{P}_0}. \]

But \( |f_{n,k}(\tau_i) - f_{n,k}(\tau_{i-1})| \leq \binom{n}{k+1} \) and \( |f_{n,k+1}(\tau_i) - f_{n,k}(\tau_{i-1})| \leq \binom{n}{k+2} \). Hence,

\[ |\beta_{n,k}(h) - \beta_{n,k}(0)|_{\mathcal{P}_0} \leq \left( \binom{n}{k+1} + \binom{n}{k+2} \right) S_{0,h} 1_{\mathcal{P}_0} \leq 2n^{k+2} S_{0,h}. \]

Similarly, if we let \( S_{h,2h} \) denote the total arrivals across edges in \((h, 2h]\), then

\[ |\beta_{n,k}(2h) - \beta_{n,k}(h)|_{\mathcal{P}_0} \leq 2n^{k+2} S_{h,2h}. \]
where $\mathcal{P}_h$ denotes the event that no arrivals happened at time $h$. Since $1_{\mathcal{P}_h}$ and $1_{\mathcal{P}_h}$ are almost sure events, the above inequalities combined with \[6.16\] show that

$$
\text{Term}_2 \leq 16n^{4k+8}\mathbb{E}[S_{0,h}^2S_{h,2h}^2 1_{E^c}].
$$

Now using \[6.13\], note that

$$
1_{E^c} \leq 1_{\{(−1)^k\chi_n(0)\neq \beta_{n,k}(0)\}} + 1_{\{(−1)^k\chi_n(h)\neq \beta_{n,k}(h)\}} + 1_{\{(−1)^k\chi_n(2h)\neq \beta_{n,k}(2h)\}}.
$$

Consequently, we have

$$
\text{Term}_2 \leq 16n^{4k+8}\left\{ \mathbb{E}\left[ S_{0,h}^2S_{h,2h}^2 1_{\{(−1)^k\chi_n(0)\neq \beta_{n,k}(0)\}} \right] \\
+ \mathbb{E}\left[ S_{0,h}^2S_{h,2h}^2 1_{\{(−1)^k\chi_n(h)\neq \beta_{n,k}(h)\}} \right] \\
+ \mathbb{E}\left[ S_{0,h}^2S_{h,2h}^2 1_{\{(−1)^k\chi_n(2h)\neq \beta_{n,k}(2h)\}} \right] \right\}.
$$

(6.20)

However, for any $t \geq 0$, note that $1_{\{(−1)^k\chi_n(t)\neq \beta_{n,k}(t)\}}$ is a function of only $G(n, p, t)$ which in turn is a function of only $\{I_{e_i(t)}\}$, where

$$
i_e(t) := \min\{i : T_i^e \leq t < T_{i+1}^e\}.
$$

Since for each $e$, the i.i.d. sequence $\{I_i^e\}$ and the sequence $\{T_i^e\}$ are independent, it is not difficult to see that $\cup_e I_{e_i(t)}$ is independent of $\cup_e T_i^e$. So, $S_{0,h}, S_{h,2h}, (\text{both of which depend only upon } \cup_e I_{e_i(t)} \text{ and } 1_{\{(−1)^k\chi_n(t)\neq \beta_{n,k}(t)\}}$ (which depends only upon $\cup_e I_{e_i(t)}$) are mutually independent for any $t \geq 0.

Since $S_{0,h}^2$ and $S_{h,2h}^2$ are Poisson with parameter $(\frac{n}{2})\lambda h$,

$$
\mathbb{E}[S_{0,h}^2] = \mathbb{E}[S_{h,2h}^2] = (\frac{n}{2})\lambda h + (\frac{n}{2})^2 \lambda^2 h^2 \leq 2n^2\lambda^2 h,
$$

where the last inequality follows since $0 \leq h \leq 1$. Consequently, we have

$$
\text{Term}_2 \leq 64n^{4k+16}\lambda^2 h^2\left\{ \mathbb{P}\{(−1)^k\chi_n(0) \neq \beta_{n,k}(0)\} \\
+ \mathbb{P}\{(−1)^k\chi_n(h) \neq \beta_{n,k}(h)\} + \mathbb{P}\{(−1)^k\chi_n(2h) \neq \beta_{n,k}(2h)\} \right\}.
$$

However, from Theorem 1.2

$$
\mathbb{P}\{(−1)^k\chi_n(t) \neq \beta_n(t)\} \leq n^{-M},
$$

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for any constant \( M > 0 \). Using this, it is not difficult to see that there exists \( K'_\beta > 0 \) such that
\[
\frac{\text{Term}_2}{n^{4k}p^{4(k+1)/2} - 1} \leq K'_\beta h^2.
\]
(6.21)
Combining (6.14), (6.19), and (6.21), the desired result follows.

Putting Lemma 6.1 and Theorem 6.1 together shows that the sequence of processes \( \{\tilde{\beta}_{n,k}(t) : t \geq 0 \} : n \geq 1 \} \) is tight. Combining this with Theorem 5.1 completes the proof for Theorem 1.4 as desired. Note that along the way we have also proved that if \( p = n^\alpha \) with \( \alpha \in (-1/k, -1/(k+1)) \), then the sequences of processes \( \{\tilde{f}_{n,k}(t) : t \geq 0 \} : n \geq 1 \} \) and \( \{\tilde{\chi}_n(t) : t \geq 0 \} : n \geq 1 \} \) converge in distribution to the stationary Ornstein-Uhlenbeck process.

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**Appendix A.**

Consider the notations defined below Lemma 6.3. The aim here is to obtain an explicit expression for \( \mathbb{E}[g(h; \tilde{\Lambda})] \), where \( g(h; \tilde{\Lambda}) \) is as defined in (6.4). Clearly,
\[ g(h; \bar{A}) = 1 \]
\[ + 1_A_1(2h)1_A_2(2h)1_A_3(h)1_A_4(h) + 1_A_1(h)1_A_2(h)1_A_3(h)1_A_4(0) 
+ 1_A_1(2h)1_A_2(h)1_A_3(0)1_A_4(0) + 1_A_1(h)1_A_2(2h)1_A_3(h)1_A_4(h) 
+ 1_A_1(2h)1_A_2(h)1_A_3(0)1_A_4(h) + 1_A_1(h)1_A_2(h)1_A_3(0)1_A_4(h) 
- 1_A_1(2h)1_A_2(2h)1_A_3(h)1_A_4(0) - 1_A_1(2h)1_A_2(2h)1_A_3(0)1_A_4(h) 
- 1_A_1(h)1_A_2(h)1_A_3(h)1_A_4(h) - 1_A_1(h)1_A_2(h)1_A_3(h)1_A_4(0) 
- 1_A_1(2h)1_A_2(h)1_A_3(h)1_A_4(h) - 1_A_1(h)1_A_2(2h)1_A_3(h)1_A_4(h) 
- 1_A_1(2h)1_A_2(h)1_A_3(h)1_A_4(0) - 1_A_1(2h)1_A_2(2h)1_A_3(0)1_A_4(0). \]

(A.1)

Observe that

\[ E[1_A_1(2h)1_A_2(2h)1_A_3(h)1_A_4(h)] = p_{\text{pair}(\bar{A})} \]
\[ \times \left[ (1 - p)e^{-h} + p \right]^{(a_{12}^3) + (a_{23}^3) + (a_{34}^3) - (a_{123}) - (a_{134}) - (a_{234}) + (a_{1234})}, \]

while \[ E[-1_A_1(2h)1_A_2(2h)1_A_3(h)1_A_4(0)] \] equals

\[ - p_{\text{pair}(\bar{A})} \left[ (1 - p)e^{-h} + p \right]^{(a_{12}^3) + (a_{23}^3) + (a_{34}^3) - (a_{123})} \]
\[ \times \left[ (1 - p)e^{-2h} + p \right]^{(a_{12}^4) + (a_{23}^4) - (a_{1234}) - (a_{134}^4) - (a_{234}^4) + (a_{1234}^4)}. \]
Extending these observations, it follows that $E[g(h; \bar{A})] = p^{\text{pair}(\bar{A})} \Phi(h; \bar{A})$, where

$$\Phi(h; \bar{A}) = \left[ \delta(h) \right]^{(12)} + \left[ \delta(h) \right]^{(14)} + \left[ \delta(h) \right]^{(23)} + \left[ \delta(h) \right]^{(24)} - \left[ \delta(h) \right]^{(1324)} - \left[ \delta(h) \right]^{(1234)} + \left[ \delta(h) \right]^{(1243)} + \left[ \delta(h) \right]^{(1423)} + \left[ \delta(h) \right]^{(234)} + \left[ \delta(h) \right]^{(243)} + \left[ \delta(h) \right]^{(234)} + \left[ \delta(h) \right]^{(243)} + \left[ \delta(h) \right]^{(243)} + \left[ \delta(h) \right]^{(243)}$$

$$+ \left[ \delta(2h) \right]^{(1234)} + \left[ \delta(2h) \right]^{(1234)} - \left[ \delta(2h) \right]^{(1324)} - \left[ \delta(2h) \right]^{(1234)}$$

$$+ \left[ \delta(2h) \right]^{(1234)} + \left[ \delta(2h) \right]^{(1234)} - \left[ \delta(2h) \right]^{(1324)} - \left[ \delta(2h) \right]^{(1234)}$$

$$+ \left[ \delta(2h) \right]^{(1234)} + \left[ \delta(2h) \right]^{(1234)} - \left[ \delta(2h) \right]^{(1324)} - \left[ \delta(2h) \right]^{(1234)}$$

$$- \left[ \delta(2h) \right]^{(1234)} + \left[ \delta(2h) \right]^{(1234)} - \left[ \delta(2h) \right]^{(1324)} - \left[ \delta(2h) \right]^{(1234)}$$

$$- \left[ \delta(2h) \right]^{(1234)} + \left[ \delta(2h) \right]^{(1234)} - \left[ \delta(2h) \right]^{(1324)} - \left[ \delta(2h) \right]^{(1234)}$$

$$+ \left[ \delta(h) \right]^{(1234)} + \left[ \delta(h) \right]^{(1234)} - \left[ \delta(h) \right]^{(1324)} - \left[ \delta(h) \right]^{(1234)}$$

$$\text{where} \quad \Phi(h; \bar{A}) = p^{\text{pair}(\bar{A})} \Phi(h; \bar{A})$$

References


