Jordan Blocks of $H^2(\mathbb{D}^n)$

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Abstract. We develop a several variables analog of the Jordan blocks of the Hardy space $H^2(\mathbb{D})$. In this consideration, we obtain a complete characterization of the doubly commuting quotient modules of the Hardy module $H^2(\mathbb{D}^n)$. We prove that a quotient module $Q$ of $H^2(\mathbb{D}^n)$ is doubly commuting if and only if

$$Q = Q_{\Theta_1} \otimes \cdots \otimes Q_{\Theta_n},$$

where each $Q_{\Theta_i}$ is either a one variable Jordan block $H^2(\mathbb{D})/\Theta_i H^2(\mathbb{D})$ for some inner function $\Theta_i$ or the Hardy module $H^2(\mathbb{D})$ on the unit disk for all $i = 1, \ldots, n$. We say that a submodule $S$ of $H^2(\mathbb{D}^n)$ is co-doubly commuting if the quotient module $H^2(\mathbb{D}^n)/S$ is doubly commuting. We obtain a Beurling like theorem for the class of co-doubly commuting submodules of $H^2(\mathbb{D}^n)$. We prove that a submodule $S$ of $H^2(\mathbb{D}^n)$ is co-doubly commuting if and only if

$$S = \sum_{i=1}^{m} \Theta_i H^2(\mathbb{D}^n),$$

for some integer $m \leq n$ and one variable inner functions $\{\Theta_i\}_{i=1}^m$.

1. Introduction

Let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ be the $n$-dimensional complex Euclidean space with $n \geq 1$ and $\mathbb{D}^n = \{(z_1, \ldots, z_n) : |z_i| < 1, i = 1, \ldots, n\}$ be the open unit polydisc. We denote the elements of $\mathbb{C}^n$ by $z = (z_1, \ldots, z_n)$ where $z_i \in \mathbb{C}$ for all $i = 1, \ldots, n$. The Hardy space $H^2(\mathbb{D}^n)$ on the polydisc is the Hilbert space of all holomorphic functions $f$ on $\mathbb{D}^n$ such that

$$\|f\|_{H^2(\mathbb{D}^n)} := \left( \sup_{0 \leq r < 1} \int_{\mathbb{T}^n} |f(rz)|^2 d\theta \right)^{\frac{1}{2}} < \infty,$$

where $d\theta$ is the normalized Lebesgue measure on the torus $\mathbb{T}^n$, the distinguished boundary of $\mathbb{D}^n$ and $rz := (rz_1, \ldots, rz_n)$ (cf. [14], [7]).

The multiplication operators by the coordinate functions turns $H^2(\mathbb{D}^n)$ into a Hilbert module over $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$, the ring of polynomials in $n$ variables with complex coefficients in the following sense:

$$\mathbb{C}[z] \times H^2(\mathbb{D}^n) \to H^2(\mathbb{D}^n), \quad (p, f) \mapsto p(M_{z_1}, \ldots, M_{z_n})f,$$

for all $p \in \mathbb{C}[z]$ and $f \in H^2(\mathbb{D}^n)$ (cf. [6]). We also call the Hilbert module $H^2(\mathbb{D}^n)$ over $\mathbb{C}[z]$ as the Hardy module. A closed subspace $S \subseteq H^2(\mathbb{D}^n)$ is said to be a submodule of $H^2(\mathbb{D}^n)$ if
Therefore, that module multiplication operators
paper.

The above fact is one of the motivations to introduce the following notion and the title of the

for all $i = 1, \ldots, n$. A closed subspace $Q \subseteq H^2(\mathbb{D}^n)$ is said to be a quotient module
of $H^2(\mathbb{D}^n)$ if $Q^\perp (\cong H^2(\mathbb{D}^n)/Q)$ is a submodule of $H^2(\mathbb{D}^n)$.

Let $S$ be a submodule and $Q$ be a quotient module of $H^2(\mathbb{D}^n)$. Then the module multiplication
operators on $S$ and $Q$ are given by the restrictions ($R_{z_1}, \ldots, R_{z_n}$) and the compressions
($C_{z_1}, \ldots, C_{z_n}$) of the module multiplications of $H^2(\mathbb{D}^n)$, respectively. That is,

$$R_{z_i} = M_{z_i}|_S \quad \text{and} \quad C_{z_i} = P_Q M_{z_i}|_Q,$$

for all $i = 1, \ldots, n$. Here, for a given closed subspace $M$ of a Hilbert space $H$, we denote the
orthogonal projection of $H$ onto $M$ by $P_M$. Note that

$$R_{z_i}^* = P_S M_{z_i}^*|_S \quad \text{and} \quad C_{z_i}^* = M_{z_i}^*|_Q,$$

for all $i = 1, \ldots, n$.

**Jordan blocks of $H^2(\mathbb{D})$:** A closed subspace $Q \subseteq H^2(\mathbb{D})$ is said to be a Jordan block of $H^2(\mathbb{D})$
if $Q$ is a quotient module and $Q \neq H^2(\mathbb{D})$ (see [13], [12]). By Beurling’s theorem [3], a closed
subspace $Q(\neq H^2(\mathbb{D}))$ is a quotient module of $H^2(\mathbb{D})$ if and only if the submodule $Q^\perp$ is given
by $Q^\perp = \Theta H^2(\mathbb{D})$ for some inner function $\Theta \in H^\infty(\mathbb{D})$. In other words, the quotient modules
and hence the Jordan blocks of $H^2(\mathbb{D})$ are precisely given by

$$Q_\Theta := H^2(\mathbb{D})/\Theta H^2(\mathbb{D}),$$

for inner functions $\Theta \in H^\infty(\mathbb{D})$.

**Jordan blocks of $H^2(\mathbb{D}^n)$ ($n > 1$):** First we note that the Hardy module $H^2(\mathbb{D}^n)$ (with $n > 1$)
can be identified with the $n$-fold Hilbert space tensor product of the Hardy space $H^2(\mathbb{D})$ on the
disc

$$H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D}),$$

via the unitary map $U : H^2(\mathbb{D}^n) \to H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$, where

$$U(z_1^{l_1} \cdots z_n^{l_n}) := z_1^{l_1} \otimes \cdots \otimes z_n^{l_n}$$

for all $l_1, \ldots, l_n \in \mathbb{N}$. Moreover, $H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$ is a Hilbert module over $\mathbb{C}[z]$ with the
module multiplication operators

$$\{I_{H^2(\mathbb{D})} \otimes \cdots \otimes M_z \otimes \cdots \otimes I_{H^2(\mathbb{D})}\}_{i=1}^n.$$

Therefore, that $U$ is a module map

$$UM_{z_i} = (I_{H^2(\mathbb{D})} \otimes \cdots \otimes M_z \otimes \cdots \otimes I_{H^2(\mathbb{D})})U,$$

for all $1 \leq i \leq n$. It is easy to see that

$$M_z M_{z_j}^* = M_{z_j}^* M_z,$$

for all $i \neq j$.

The above fact is one of the motivations to introduce the following notion and the title of the paper.
Definition 1.1. Let $Q$ be a quotient module of $H^2(\mathbb{D}^n)$ and $n > 1$. Then $Q$ is said to be a Jordan block of $H^2(\mathbb{D}^n)$ if $Q$ is doubly commuting, that is, $C_{zi} C_{zj} = C_{zj} C_{zi}$, for all $1 \leq i < j \leq n$ and $Q \neq H^2(\mathbb{D}^n)$. Also a closed subspace $S$ of $H^2(\mathbb{D}^n)$ is said to be co-doubly commuting submodule of $H^2(\mathbb{D}^n)$ if $H^2(\mathbb{D}^n)/S$ is a doubly commuting quotient module.

However, in most of the following we will simply regard a Jordan block of $H^2(\mathbb{D}^n)$ as a doubly commuting quotient module of $H^2(\mathbb{D}^n)$.

The study of the doubly commuting quotient modules of the Hardy module $H^2(\mathbb{D}^2)$ was initiated by Douglas and Yang in [4] and [5] (also see [2]). Later in [11] Izuchi, Nakazi and Seto obtained a classification of the doubly commuting quotient modules of the Hardy module $H^2(\mathbb{D}^2)$ (see Theorems 2.1 and 3.1 in [11]).

In this paper we completely classify the doubly commuting quotient modules of $H^2(\mathbb{D}^n)$ for any $n \geq 2$. In this consideration, we provide a more refined analysis compared to [11]. More specifically, our method is based on the Hilbert tensor product structure of the Hardy module $H^2(\mathbb{D}^n)$ which also yield new proofs of earlier results by Izuchi, Nakazi and Seto [11] concerning the base case $n = 2$.

A key example of doubly commuting quotient modules over $\mathbb{C}[z]$ is the Hilbert tensor product of quotient modules of the Hardy module $H^2(\mathbb{D})$. That is, if we consider $n$ quotient modules $\{Q_i\}_{i=1}^n$ of the Hardy module $H^2(\mathbb{D})$ then
\[ Q = Q_1 \otimes \cdots \otimes Q_n, \]
is a doubly commuting quotient module of $H^2(\mathbb{D}^n)$ with the module multiplication operators as
\[ \{I_{Q_1} \otimes \cdots \otimes P_{Q_i} M_z | Q_1 \otimes \cdots \otimes I_{Q_n}\}_{i=1}^n. \]

We prove that a doubly commuting quotient module of $H^2(\mathbb{D}^n)$ can be also represented by the Hilbert space tensor product of quotient modules of $H^2(\mathbb{D})$ in the above form. This result is then used to prove a Beurling type theorem for the co-doubly commuting submodules.

We now summarize the contents of this paper. In Section 2, we give relevant background for the main results of this paper. In Section 3, we prove that a quotient module $Q$ of $H^2(\mathbb{D}^n)$ is doubly commuting if and only if $Q$ is the $n$ times Hilbert tensor product of quotient modules of the Hardy module $H^2(\mathbb{D})$. In Section 4, we characterize the class of co-doubly commuting submodules of $H^2(\mathbb{D}^n)$.

2. Preparatory results

In this section, we gather together some concepts and results concerning various aspects of the Hardy modules that are used frequently in the rest of this paper. Some of the results of the present section are of independent interest.

We first recall that the module multiplication of the Hardy module $H^2(\mathbb{D}^n)$ satisfies the following relation
\[ M_z M_z^* = I_{H^2(\mathbb{D})} - P_{C}. \]
where $P_C$ denotes the orthogonal projection of $H^2(\mathbb{D})$ onto the space of constant functions. Moreover, if $Q_\Theta = H^2(\mathbb{D})/\Theta H^2(\mathbb{D})$ is a Jordan block for some inner function $\Theta \in H^\infty(\mathbb{D})$, then we have

$$P_{Q_\Theta} = I_{H^2(\mathbb{D})} - M_\Theta M_\Theta^*$$ and $$P_{\Theta H^2(\mathbb{D})} = M_\Theta M_\Theta^*.$$ We also have

$$I_Q - C_z C_z^* = P_Q (I_{H^2(\mathbb{D})} - M_z M_z^*)|_Q = P_Q P_C|_Q,$$ where $Q$ is a quotient module of $H^2(\mathbb{D})$.

The following lemma is well known.

**Lemma 2.1.** Let $Q_\Theta$ be a Jordan block of $H^2(\mathbb{D})$ for some inner function $\Theta \in H^\infty(\mathbb{D})$. Then

$$P_{Q_\Theta} 1 = 1 - \Theta(0)\Theta,$$

and

$$(P_{Q_\Theta} P_C P_{Q_\Theta}) 1 = (1 - |\Theta(0)|^2) (1 - \overline{\Theta(0)}\Theta).$$

**Proof.** By virtue of $M_\Theta^* 1 = \Theta(0)$ we have

$$P_{Q_\Theta} 1 = (I_{H^2(\mathbb{D})} - M_\Theta M_\Theta^*) 1 = 1 - M_\Theta (M_\Theta^* 1) = 1 - \Theta(0)\Theta.$$ For the second equality, we compute

$$(P_{Q_\Theta} P_C P_{Q_\Theta}) 1 = (P_{Q_\Theta} P_C) (1 - \Theta(0)\Theta) = P_{Q_\Theta} (1 - |\Theta(0)|^2) = (1 - |\Theta(0)|^2) (1 - \Theta(0)\Theta).$$

This completes the proof.

**Corollary 2.2.** Let $Q$ be a quotient module of $H^2(\mathbb{D})$. Then

$$P_Q 1 \in \text{ran}(P_Q P_C P_Q).$$

**Proof.** If $Q = H^2(\mathbb{D})$ then the result follows trivially. If $Q \neq H^2(\mathbb{D})$ then $Q$ is a Jordan block and hence the conclusion follows from Lemma 2.1.

The following lemma is a variation on the theme of the isometric dilation theory of contractions.

**Lemma 2.3.** Let $Q$ be a quotient module of $H^2(\mathbb{D})$ and $L = \text{ran}(I_Q - C_z C_z^*) = \text{ran}(P_Q P_C P_Q)$. Then

$$Q = \bigoplus_{l=0}^\infty P_Q M_z^l L.$$**Proof.** The result is trivial if $Q = \{0\}$. Let $Q \neq \{0\}$. Notice that

$$\bigoplus_{l=0}^\infty P_Q M_z^l L \subseteq Q.$$ Let now $f \in Q$ be such that $f \perp \bigoplus_{l=0}^\infty P_Q M_z^l L$. Then for all $l \geq 0$ we have that $f \perp P_Q M_z^l P_C Q$, or equivalently, $P_C M_z^l f \in Q^\perp$. Since $Q^\perp$ is a proper submodule of $H^2(\mathbb{D})$, it follows that

$$P_C M_z^l f = 0,$$ for all $l \geq 0$. Consequently,

$$f = 0.$$
This concludes the proof.

In the following, we employ the standard multi-index notation that \( \mathbb{N}^n = \{(k_1, \ldots, k_n) : k_i \in \mathbb{N}, i = 1, \ldots, n \} \) and for any \( k = (k_1, \ldots, k_n) \in \mathbb{N}^n \) we denote \( z^k = z_1^{k_1} \cdots z_n^{k_n} \) and \( M^k = M^k_{z_1} \cdots M^k_{z_n} \).

We now present a characterization of \( M_{z_i} \)-reducing subspace of \( H^2(\mathbb{D}^n) \). However the technique used here seems to be well known in the study of the reducing subspaces.

**Proposition 2.4.** Let \( n > 1 \) and \( S \) be a closed subspace of \( H^2(\mathbb{D}^n) \). Then \( S \) is a \( (M_{z_2}, \ldots, M_{z_n}) \)-reducing subspace of \( H^2(\mathbb{D}^n) \) if and only if

\[
S = S_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D}) \quad \text{for some closed subspace } S_1 \text{ of } H^2(\mathbb{D}).
\]

**Proof.** Let \( S \) be a \( (M_{z_2}, \ldots, M_{z_n}) \)-reducing closed subspace of \( H^2(\mathbb{D}^n) \), that is, for all \( 2 \leq i \leq n \) we have

\[
M_{z_i} P_S = P_S M_{z_i}.
\]

Following Agler’s hereditary functional calculus (cf. [1])

\[
\left( \prod_{i=2}^n (1 - z_i \bar{w}_i) \right) (M_z, M_z) = \sum_{0 \leq i_1 < \ldots < i_l \leq n} (-1)^l (z_{i_1} \cdots z_{i_l} \bar{w}_{i_1} \cdots \bar{w}_{i_l}) (M_z, M_z) = \sum_{0 \leq i_1 < \ldots < i_l \leq n, i_1, i_2 \neq 1} (-1)^l M_{z_{i_1}} \cdots M_{z_{i_l}} M^*_{z_{i_1}} \cdots M^*_{z_{i_l}} = P_{H^2(\mathbb{D})} \otimes P_C \otimes \cdots \otimes P_C.
\]

Consequently,

\[
(P_{H^2(\mathbb{D})} \otimes P_C \otimes \cdots \otimes P_C) P_S = P_S (P_{H^2(\mathbb{D})} \otimes P_C \otimes \cdots \otimes P_C),
\]

which yields that \( P_S (P_{H^2(\mathbb{D})} \otimes P_C \otimes \cdots \otimes P_C) \) is an orthogonal projection and

\[
P_S (P_{H^2(\mathbb{D})} \otimes P_C \otimes \cdots \otimes P_C) = (P_{H^2(\mathbb{D})} \otimes P_C \otimes \cdots \otimes P_C) P_S = P_{\mathcal{S}_1},
\]

where \( \mathcal{S}_1 := (H^2(\mathbb{D}) \otimes C \otimes \cdots \otimes C) \cap S \). Let

\[
\mathcal{S}_1 = S_1 \otimes C \otimes \cdots \otimes C,
\]

for some closed subspace \( S_1 \) of \( H^2(\mathbb{D}) \). We claim that

\[
S = \text{span} \{ M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} \mathcal{S}_1 : l_2, \ldots, l_n \in \mathbb{N} \} = S_1 \otimes H^2(\mathbb{D}^{n-1}).
\]

Now for any

\[
f = \sum_{k \in \mathbb{N}^n} a_k z^k \in S,
\]

we have

\[
f = P_S f = P_S \left( \sum_{k \in \mathbb{N}^n} M_z^k a_k \right) = \sum_{k \in \mathbb{N}^n} M_{z_2}^{k_2} \cdots M_{z_n}^{k_n} P_S (a_k z_1^{k_1}),
\]
where $a_k \in \mathbb{C}$ for all $k \in \mathbb{N}$. But $P_S(a_k z_1^k) = P_S(P_{H^2(\mathbb{D})} \otimes P_C \otimes \cdots \otimes P_C)(a_k z_1^k) \in \tilde{S}_1$ and hence $f \in S_1 \otimes H^2(\mathbb{D}^{n-1})$. That is, $S \subseteq S_1 \otimes H^2(\mathbb{D}^{n-1})$. On the other hand, since $\tilde{S}_1 \subseteq S$ and that $S$ is a $(M_2, \ldots, M_n)$-reducing subspace, we see that $S = S_1 \otimes H^2(\mathbb{D}^{n-1})$.

The converse part is immediate. This concludes the proof of the proposition.

The following result will be used in the final section.

**Lemma 2.5.** Let $\{P_i\}_{i=1}^n$ be a collection of commuting orthogonal projections on a Hilbert space $\mathcal{H}$. Then

$$\mathcal{L} := \sum_{i=1}^n \text{ran} P_i,$$

is closed and the orthogonal projection of $\mathcal{H}$ onto $\mathcal{L}$ is given by

$$P_{\mathcal{L}} = P_1(I - P_2) \cdots (I - P_n) + P_2(I - P_3) \cdots (I - P_n) + \cdots + P_n(I - P_1) + P_n.$$

Moreover,

$$P_{\mathcal{L}} = I - \prod_{i=1}^n (I - P_i).$$

**Proof.** We let

$$O_i = P_i(I - P_{i+1}) \cdots (I - P_{n-1})(I - P_n),$$

so that

$$O_i = \prod_{j=i+1}^n (I - P_j) - \prod_{j=i+1}^n (I - P_j),$$

for all $i = 1, \ldots, n - 1$ and $O_n = P_n$. By the assumptions, $\{O_i\}_{i=1}^n$ is a family of orthogonal projections with orthogonal ranges. We claim that

$$\mathcal{L} = \text{ran} O_1 \oplus \cdots \oplus \text{ran} O_n.$$

From the definition we see that $\mathcal{L} \supseteq \text{ran} O_1 \oplus \cdots \oplus \text{ran} O_n$. To prove the reverse inclusion, first we observe that

$$\sum_{i=1}^n O_i = I - \prod_{i=1}^n (I - P_i).$$

Now let $f = f_1 + \cdots + f_n \in \mathcal{L}$ where $f_i \in \text{ran} P_i$ for all $i = 1, \ldots, n$. Then

$$\left(\sum_{i=1}^n O_i\right) f = (I - \prod_{i=1}^n (I - P_i)) f = f - \prod_{i=1}^n (I - P_i) f$$

$$= f - \sum_{j=1}^n \prod_{i=1}^n (I - P_i) f_j = f - \sum_{j=1}^n 0$$

$$= f,$$

and hence the equality follows. This implies that $\mathcal{L}$ is a closed subspace and

$$P_{\mathcal{L}} = \sum_{i=1}^n O_i = I - \prod_{i=1}^n (I - P_i).$$
This completes the proof of the lemma.

3. Quotient Modules

In this section we prove the central result of this paper that a doubly commuting quotient module of \( H^2(\mathbb{D}^n) \) is precisely the Hilbert tensor product of \( n \) quotient modules of the Hardy module \( H^2(\mathbb{D}) \).

We begin by generalizing the fact that a closed subspace \( M \) of \( H^2(\mathbb{D}) \) is \( M_1 \)-reducing if and only if

\[
M = H^2(\mathbb{D}) \otimes \mathcal{E},
\]

for some closed subspace \( \mathcal{E} \subseteq H^2(\mathbb{D}^{n-1}) \).

**Proposition 3.1.** Let \( Q_1 \) be a quotient module of \( H^2(\mathbb{D}) \) and \( M \) be a closed subspace of

\[
Q = Q_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D}) \subseteq H^2(\mathbb{D}^n).
\]

Then \( M \) is a \( P_Q M_{z_1}|Q \)-reducing subspace of \( Q \) if and only if

\[
M = Q_1 \otimes \mathcal{E},
\]

for some closed subspace \( \mathcal{E} \) of \( H^2(\mathbb{D}^{n-1}) \).

**Proof.** Let \( M \) be a \( P_Q M_{z_1}|Q \)-reducing subspace of \( Q \). Then

\[
(P_Q M_{z_1}|Q) P_M = P_M (P_Q M_{z_1}|Q),
\]

or equivalently,

\[
(P_Q M_{z_1}|Q) I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}) P_M = P_M (P_Q M_{z_1}|Q) I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}).
\]

Now

\[
I_Q - (P_Q M_{z_1}|Q) (P_Q M_{z_1}|Q)^* = (P_Q M_{z_1}|Q) I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}.
\]

Further (3.1) yields

\[
P_M ((P_Q M_{z_1}|Q) I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}) = ((P_Q M_{z_1}|Q) I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}) P_M,
\]

and therefore

\[
P_M ((P_Q M_{z_1}|Q) I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})})
\]

is the orthogonal projection onto

\[
\mathcal{L} := \mathcal{M} \cap \text{ran} ((P_Q M_{z_1}|Q) I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}) = \mathcal{M} \cap (\mathcal{L}_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})),
\]

where

\[
\mathcal{L}_1 = \text{ran} (P_Q M_{z_1}|Q) \subseteq Q_1.
\]

Since \( \mathcal{L} \subseteq \mathcal{L}_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D}) \) and \( \dim \mathcal{L}_1 = 1 \) (otherwise, by Lemma 2.3 that \( \mathcal{L}_1 = \{0\} \) is equivalent to \( Q_1 = \{0\} \)) we obtain

\[
\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{E},
\]

for some closed subspace \( \mathcal{E} \subseteq H^2(\mathbb{D}^{n-1}) \). More precisely

\[
P_M ((P_Q M_{z_1}|Q) I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}) = P_{\mathcal{L}} = P_{\mathcal{L}_1 \otimes \mathcal{E}}.
\]
We claim that 
\[ \mathcal{M} = \bigvee_{l=0}^{\infty} P_{Q_l^0} M_{z_1}. \]
Since \( \mathcal{M} \) is \( P_{Q_l^0} \)-reducing subspace and \( \mathcal{M} \supseteq \mathcal{L} \), it follows that 
\[ \mathcal{M} \supseteq \bigvee_{l=0}^{\infty} P_{Q_l^0} M_{z_1}. \]

To prove the reverse inclusion, we let 
\[ f \in \mathcal{M} \subseteq (P_{Q_1^1} \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})})(H^2(\mathbb{D}) \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})), \]
so that 
\[ f = P_{Q_1^1} \sum_{k \in \mathbb{N}^n} a_k z^k, \]
where \( a_k \in \mathbb{C} \) for all \( k \in \mathbb{N}^n \). Also 
\[ f = P_M f = P_M P_{Q_1} \sum_{k \in \mathbb{N}^n} a_k z^k. \]

Observe now that for all \( k \in \mathbb{N}^n \)
\[ P_M P_{Q_1} z^k = (P_M P_{Q_1} M_{z_1}^{k_1})(z_2^{k_2} \cdots z_n^{k_n}) = (P_M P_{Q_1} M_{z_1}^{k_1} P_{Q_1})(z_2^{k_2} \cdots z_n^{k_n}) \]
and hence 
\[ P_M P_{Q_1} M_{z_1}^{k_1} P_{Q_1}(z_2^{k_2} \cdots z_n^{k_n}) \]
by (3.1). Now by applying Corollary 2.2, we obtain that \( P_{Q_1} 1 \in \mathcal{L}_1 \) and hence \( P_M(P_{Q_1} 1 \otimes z_2^{k_2} \cdots z_n^{k_n}) \in \mathcal{L}_1 \otimes \mathcal{E} = \mathcal{L} \). Therefore, we infer 
\[ P_M P_{Q_1} z^k \in \bigvee_{l=0}^{\infty} P_{Q_l^0} M_{z_1}, \]
for all \( k \in \mathbb{N}^n \) and hence 
\[ f \in \bigvee_{l=0}^{\infty} P_{Q_l^0} M_{z_1}. \]

Finally, \( \mathcal{L} = \mathcal{L}_1 \otimes \mathcal{E} \) yields 
\[ \mathcal{M} = \bigvee_{l=0}^{\infty} P_{Q_l^0} M_{z_1} \mathcal{L} = (\bigvee_{l=0}^{\infty} P_{Q_l^0} M_{z_1} \mathcal{L}_1) \otimes \mathcal{E}, \]
and therefore by Lemma 2.3
\[ \mathcal{M} = Q_1 \otimes \mathcal{E}. \]

The converse part is trivial. This finishes the proof.

We are now ready to prove the main result of this section.

**Theorem 3.2.** Let \( Q \) be a quotient module of \( H^2(\mathbb{D}^n) \). Then \( Q \) is doubly commuting if and only if there exists quotient modules \( Q_1, \ldots, Q_n \) of \( H^2(\mathbb{D}) \) such that
\[ Q = Q_1 \otimes \cdots \otimes Q_n. \]
Proof. Let $Q$ be a doubly commuting quotient module of $H^2(\mathbb{D}^n)$. Define
\[
\tilde{Q}_1 = \overline{\text{span}}\{z_2^{l_2} \cdots z_n^{l_n} Q : l_2, \ldots, l_n \in \mathbb{N}\}.
\]
Then $\tilde{Q}_1$ is a joint $(M_{z_2}, \ldots, M_{z_n})$-reducing subspace of $H^2(\mathbb{D}^n)$. But Proposition 2.4 now allows us to conclude that
\[
\tilde{Q}_1 = Q_1 \otimes (H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D}))^{(n-1)\text{times}},
\]
for some closed subspace $Q_1$ of $H^2(\mathbb{D})$. Since $\tilde{Q}_1$ is $M_{z_i}^*$-invariant subspace, that $Q_1$ is a $M_{z_i}^*$-invariant subspace of $H^2(\mathbb{D})$, that is, $\tilde{Q}_1$ is a quotient module of $H^2(\mathbb{D})$. Note that $Q \subseteq \tilde{Q}_1$. We claim that $Q$ is a $M_{z_1}^*|_{\tilde{Q}_1}$-reducing subspace of $\tilde{Q}_1$, that is,
\[
P_Q(M_{z_1}^*|_{\tilde{Q}_1}) = (M_{z_1}^*|_{\tilde{Q}_1})P_Q.
\]
In order to prove the claim we first observe that for all $l \geq 0$ and $2 \leq i \leq n$,
\[
C_{z_1}^* C_{z_i}^l = C_{z_i}^l C_{z_1}^*,
\]
and hence
\[
C_{z_1}^* C_{z_2}^{l_2} \cdots C_{z_n}^{l_n} = C_{z_2}^{l_2} \cdots C_{z_n}^{l_n} C_{z_1}^*;
\]
for all $l_2, \ldots, l_n \geq 0$, that is,
\[
M_{z_1}^* P_Q M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_Q = P_Q M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} M_{z_1}^* P_Q,
\]
or,
\[
M_{z_1}^* P_Q M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_Q = P_Q M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_Q.
\]
From this it follows that for all $f \in Q$ and $l_2, \ldots, l_n \geq 0$,
\[
(P_Q M_{z_1}^*|_{\tilde{Q}_1})(z_2^{l_2} \cdots z_n^{l_n} f) = P_Q M_{z_1}^* (z_2^{l_2} \cdots z_n^{l_n} f) = (P_Q M_{z_1}^* M_{z_2}^{l_2} \cdots M_{z_n}^{l_n}) f
\]
\[
= (P_Q M_{z_1}^* M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_Q) f = M_{z_1}^* P_Q M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_Q f
\]
\[
= M_{z_1}^* P_Q M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} f = (P_{\tilde{Q}_1} P_Q)(z_2^{l_2} \cdots z_n^{l_n} f).
\]
Also by $P_{\tilde{Q}_1} \subseteq \tilde{Q}_1$ we have
\[
P_Q P_{\tilde{Q}_1} = P_{\tilde{Q}_1} P_Q P_{\tilde{Q}_1}.
\]
This yields
\[
(P_Q M_{z_1}^*|_{\tilde{Q}_1})(z_2^{l_2} \cdots z_n^{l_n} f) = (P_{\tilde{Q}_1} P_Q)(z_2^{l_2} \cdots z_n^{l_n} f)
\]
\[
= M_{z_1}^* P_{\tilde{Q}_1} P_Q (z_2^{l_2} \cdots z_n^{l_n} f)
\]
\[
= M_{z_1}^* P_{\tilde{Q}_1} P_Q (z_2^{l_2} \cdots z_n^{l_n} f)
\]
\[
= (M_{z_1}^*|_{\tilde{Q}_1})(z_2^{l_2} \cdots z_n^{l_n} f),
\]
for all $f \in Q$ and $l_2, \ldots, l_n \geq 0$, and therefore
\[
P_Q(M_{z_1}^*|_{\tilde{Q}_1}) = (M_{z_1}^*|_{\tilde{Q}_1})P_Q.
\]
Hence $Q$ is a $M^*|Q_1$-reducing subspace of $\tilde{Q}_1 = Q_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$. Applying Proposition 3.1, we obtain a closed subspace $E_1$ of $H^2(\mathbb{D}^{n-1})$ such that 
\[ Q = Q_1 \otimes E_1. \]
Moreover, since 
\[ \bigvee_{l=0}^{\infty} z_l^1 Q = \bigvee_{l=0}^{\infty} z_l^1 (Q_1 \otimes E_1) = H^2(\mathbb{D}) \otimes E_1, \]
and $\bigvee_{l=0}^{\infty} z_l^1 Q$ is a doubly commuting quotient module of $H^2(\mathbb{D}^n)$, we have that $E_1 \subseteq H^2(\mathbb{D}^{n-1})$ a doubly commuting quotient module of $H^2(\mathbb{D}^{n-1})$.

By the same argument as above, we conclude that 
\[ E_1 = Q_2 \otimes E_2, \]
for some doubly commuting quotient module of $H^2(\mathbb{D}^{n-2})$. Continuing this process, we have 
\[ Q = Q_1 \otimes \cdots \otimes Q_n, \]
where $Q_1, \ldots, Q_n$ are quotient modules of $H^2(\mathbb{D})$.

The converse implication follows from the fact that the module multiplication operators on 
\[ Q = Q_1 \otimes \cdots \otimes Q_n \]
are given by 
\[ \{ I_{Q_1} \otimes \cdots \otimes P_{Q_i} M_{z_i} \otimes \cdots \otimes I_{Q_n} \}_{i=1}^{n}, \]
which is certainly doubly commuting. This completes the proof.

As a corollary of the above model, we have the following result.

**Corollary 3.3.** Let $Q$ be a closed subspace of $H^2(\mathbb{D}^n)$. Then $Q$ is doubly commuting quotient module if and only if there exists $\{\Theta_i\}_{i=1}^n \subseteq H^\infty(\mathbb{D})$ such that each $\Theta_i$ is either inner or the zero function for all $1 \leq i \leq n$ and 
\[ Q = Q_{\Theta_1} \otimes \cdots \otimes Q_{\Theta_n}. \]

**Proof.** Let $Q$ be a doubly commuting quotient module of $H^2(\mathbb{D}^n)$. By Theorem 3.2, we know that 
\[ Q = Q_1 \otimes \cdots \otimes Q_n, \]
where $Q_1, \ldots, Q_n$ are quotient modules of $H^2(\mathbb{D})$. For each $i \in \{1, \ldots, n\}$, if $Q_i \not\subseteq H^2(\mathbb{D})$ then 
\[ Q_i = Q_{\Theta_i} = H^2(\mathbb{D})/\Theta_i H^2(\mathbb{D}), \]
for some inner function $\Theta_i \in H^\infty(\mathbb{D})$. Otherwise, $Q_i = H^2(\mathbb{D})$ and we define $\Theta_i \equiv 0$ on $\mathbb{D}$ so that 
\[ Q_i = H^2(\mathbb{D}) = Q_{\Theta_i} = H^2(\mathbb{D})/(0 \cdot H^2(\mathbb{D})). \]
The converse part again follows from Theorem 3.2, and the corollary is proved.

This result was obtained by Izuchi, Nakazi and Seto in [11] for the base case $n = 2$ (also see [10]).

We conclude this section by recording the uniqueness of the tensor product representations of the doubly commuting quotient modules in Theorem 3.2. The same conclusion holds for a more general framework. Here, we provide a proof using the Hardy space method.
Let $Q$ be a doubly commuting quotient module of $H^2(\mathbb{D}^n)$ and

$$Q = Q_1 \otimes \cdots \otimes Q_n = R_1 \otimes \cdots \otimes R_n,$$

for quotient modules $\{Q_i\}_{i=1}^n$ and $\{R_i\}_{i=1}^n$ of $H^2(\mathbb{D})$. We claim that $Q_i = R_i$ for all $i$. In fact,

$$\tilde{Q}_1 := \bigoplus_{l_2, \ldots, l_n \geq 0} z_2^{l_2} \cdots z_n^{l_n}, Q = Q_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D}) = R_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D}),$$

and

$$\bigcap_{i=2}^n \ker M_{z_i}|Q_i = Q_1 \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C} = R_1 \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C}.$$

Consequently, $Q_1 = R_1$ and similarly, for all other $i = 2, \ldots, n$.

4. SUBMODULES

In this section we relate the Hilbert tensor product structure of the doubly commuting quotient modules to the Beurling like representations of the corresponding co-doubly commuting submodules.

To proceed further we introduce one more piece of notation. Let $\Theta_i \in H^\infty(\mathbb{D})$ be a given function indexed by $i \in \{1, \ldots, n\}$. In what follows by $\tilde{\Theta}_i \in H^\infty(\mathbb{D}^n)$ we denote the extension function defined by

$$\tilde{\Theta}_i(z) = \Theta_i(z),$$

for all $z \in \mathbb{D}^n$.

The following provides an explicit correspondence between the doubly commuting quotient modules and the co-doubly commuting submodules of $H^2(\mathbb{D}^n)$.

**THEOREM 4.1.** Let $Q$ be a quotient module of $H^2(\mathbb{D}^n)$ and $Q \neq H^2(\mathbb{D}^n)$. Then $Q$ is doubly commuting if and only if there exists inner functions $\Theta_{ij} \in H^\infty(\mathbb{D})$ for $1 \leq i_1 < \cdots < i_m \leq n$ for some integer $m \in \{1, \ldots, n\}$ such that

$$Q = H^2(\mathbb{D}^n)/[\tilde{\Theta}_1 H^2(\mathbb{D}^n) + \cdots + \tilde{\Theta}_m H^2(\mathbb{D}^n)],$$

where $\tilde{\Theta}_i(z) = \Theta_i(z_i)$ for all $z \in \mathbb{D}^n$.

**Proof.** Let $Q$ be a doubly commuting quotient module of $H^2(\mathbb{D}^n)$. Then by Theorem 3.2 we have

$$Q = Q_1 \otimes \cdots \otimes Q_n,$$

where for each $1 \leq i \leq n$, $Q_i$ is a submodule of $H^2(\mathbb{D})$. Choose $1 \leq m \leq n$ such that

$$Q_{ij} \neq H^2(\mathbb{D}),$$

for $1 \leq i_1 < \cdots < i_m \leq n$. Then

$$Q = H^2(\mathbb{D}) \otimes \cdots \otimes Q_{i_1} \otimes \cdots \otimes Q_{i_m} \otimes \cdots \otimes H^2(\mathbb{D}),$$

where $Q_{ij} \subset H^2(\mathbb{D})$ for all $1 \leq i_1 < \cdots < i_m \leq n$. Let

$$Q_{ij} = Q_{e_{ij}} = (\text{ran} M_{\Theta_{ij}})^\perp = \text{ran}(I_{H^2(\mathbb{D})} - M_{\Theta_{ij}} M_{\Theta_{ij}}^*),$$
for some inner function $\Theta_{i_j} \in H^\infty(\mathbb{D})$ for all $j = 1, \ldots, m$. Let $\tilde{\Theta}_{i_j}$ be the extension of $\Theta_{i_j}$ to $H^\infty(\mathbb{D}^n)$, that is, as a multiplier,

$$M_{\tilde{\Theta}_{i_j}} = I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})} \otimes M_{\Theta_{i_j}} \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}.$$ 

Hence,

$$I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}} = I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})} \otimes (I_{H^2(\mathbb{D})} - M_{\Theta_{i_j}} M^*_{\Theta_{i_j}}) \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})},$$

so that

$$\prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}})$$

$$= I_{H^2(\mathbb{D})} \otimes \cdots \otimes (I_{H^2(\mathbb{D})} - M_{\Theta_{i_1}} M^*_{\Theta_{i_1}}) \otimes \cdots \otimes (I_{H^2(\mathbb{D})} - M_{\Theta_{i_m}} M^*_{\Theta_{i_m}}) \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}.$$ 

Taking into account that $Q$ is the range of the right hand side operator, that is,

$$Q = H^2(\mathbb{D}) \otimes \cdots \otimes Q_{i_1} \otimes \cdots \otimes Q_{i_m} \otimes \cdots \otimes H^2(\mathbb{D}),$$

we deduce readily that

$$P_Q^\perp = I_{H^2(\mathbb{D}^n)} - \prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}).$$

Consequently, by Lemma 2.5 we have

$$Q^\perp = \tilde{\Theta}_{i_1} H^2(\mathbb{D}^n) + \cdots + \tilde{\Theta}_{i_m} H^2(\mathbb{D}^n),$$

or

$$Q = H^2(\mathbb{D}) / [\tilde{\Theta}_{i_1} H^2(\mathbb{D}^n) + \cdots + \tilde{\Theta}_{i_m} H^2(\mathbb{D}^n)].$$

Conversely, let

$$Q = H^2(\mathbb{D}) / [\tilde{\Theta}_{i_1} H^2(\mathbb{D}^n) + \cdots + \tilde{\Theta}_{i_m} H^2(\mathbb{D}^n)].$$

Then

$$P_Q = \prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}).$$

Also for all $s \neq t$,

$$P_Q M^*_{z_s} P_Q = \prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}) M^*_{z_s} M_{z_s} \prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}})$$

$$= P_Q M^*_{z_s} \left[ \prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}) \right]$$

$$= P_Q M^*_{z_s} \left[ \prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}) \right] M_{z_s} P_Q$$

$$= P_Q M^*_{z_s} P_Q M_{z_s} P_Q.$$

Consequently, for all $s \neq t$

$$C_{z_s} C_{z_t}^* = P_Q M^*_{z_s} M_{z_s} |_{Q} = P_Q M^*_{z_s} P_Q M_{z_s} |_{Q} = C_{z_t}^* C_{z_s},$$

and hence $Q$ is doubly commuting. This concludes the proof. 

$\blacksquare$
This result is a generalization of Theorem 3.1 of [11] by Izuchi, Nakazi and Seto on the base case $n = 2$.

To complete this section, we present the following result concerning the orthogonal projection formulae of the co-doubly commuting submodules and the doubly commuting quotient modules of $H^2(\mathbb{D}^n)$.

**Corollary 4.2.** Let $Q$ be a doubly commuting submodule of $H^2(\mathbb{D}^n)$. Then there exists an integer $m \in \{1, \ldots, n\}$ and inner functions $\Theta_{ij} \in H^\infty(\mathbb{D})$ such that

$$Q^\perp = \sum_{1 \leq i_1 < \cdots < i_m \leq n} \hat{\Theta}_{i_j} H^2(\mathbb{D}^n),$$

where $\hat{\Theta}_{i_j}(z) = \Theta_{i_j}(z_{i_j})$ for all $z \in \mathbb{D}^n$. Moreover,

$$P_Q = I_{H^2(\mathbb{D}^n)} - \prod_{j=1}^m (I_{H^2(\mathbb{D}^n)} - M_{\hat{\Theta}_{i_j}} M^*_{\hat{\Theta}_{i_j}}),$$

and

$$P_{Q^\perp} = \prod_{j=1}^m (I_{H^2(\mathbb{D}^n)} - M_{\hat{\Theta}_{i_j}} M^*_{\hat{\Theta}_{i_j}}).$$

The above result is the co-doubly commuting submodules analogue of Beurling’s theorem on submodules of $H^2(\mathbb{D})$.

We finally point out that the earlier classifications of the doubly commuting quotient modules by Izuchi, Nakazi and Seto [11] has many deep applications in the study of the submodules and the quotient modules of the Hardy module over the bidisc (cf. [8, 9]). Some of these extensions in $n$-variables ($n \geq 2$) will be addressed in future work. However, the issue of essential normality of the co-doubly commuting submodules of $H^2(\mathbb{D}^n)$ will be discussed in the forthcoming paper [15].

**References**


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