On the existence of Johnson polynomials for $p$-groups

Mark L. Lewis and S. K. Prajapati

Indian Statistical Institute, Bangalore Centre
8th Mile Mysore Road, Bangalore, 560059 India
ON THE EXISTENCE OF JOHNSON POLYNOMIALS FOR 
p-GROUPS

MARK L. LEWIS AND S. K. PRAJAPATI

Abstract. Let $G$ be a finite $p$-group. We say that $G$ has a Johnson polynomial if there exists a polynomial $f(x) \in \mathbb{Q}[x]$ and a character $\chi \in \text{Irr}(G)$ so that $f(\chi)$ equals the total character for $G$. In this paper, we show that if $G$ has nilpotence class 2, then $G$ has a Johnson polynomial if and only if $Z(G)$ is cyclic, and we show that if $|\text{cd}(G)| = 2$, then $G$ has a Johnson polynomial if and only if $G$ has nilpotence class 2 and $Z(G)$ is cyclic.

1. Introduction

Throughout this article, all groups are finite. If $G$ is a group, we write $\text{Irr}(G)$ and $\text{cd}(G)$ for the set of all irreducible characters of $G$ and the set of character degrees of $G$, respectively. The total character of $G$, denoted by $\tau_G$, is the sum of all the irreducible characters of $G$. That is, $\tau_G = \sum_{\chi \in \text{Irr}(G)} \chi$. (Note that $\tau_G$ is the character afforded by the direct sum of all irreducible complex representations of $G$.) Since $\tau_G$ is stable under the action of the Galois group of the splitting field of $G$, it follows $\tau_G$ is integer valued.

The degree $\tau_G(1)$ seems to have remarkable connections with the structure of the group. For instance, in the case of the symmetric group $G = S_n$, the degree $\tau_G(1)$ is the number of involutions of $S_n$ ([9]), and in the case of $G = GL(n, q)$, the degree $\tau_G(1)$ is the number of symmetric matrices in $GL(n, q)$ ([5]). The degree of the total character has also been used in a number of recent papers to determine other properties of the group such as whether the group is solvable, supersolvable, nilpotent, etc. (See [8], [12], [13], [18].) It appears that the degree of the total character was first computed in [1], and in [11], they compute the degree of the total character for nonsolvable groups. A lower bound for the degree of the total character is found in [6].

We now want to consider values of the total character beyond its degree. In [3], S. M. Gagola, Jr. and the first author classified all the finite solvable groups for which $\tau_G$ equals $\chi^2$, for some character $\chi \in \text{Irr}(G)$. Motivated by this, K. W. Johnson raised the following question: does there exist an irreducible character $\chi$ of $G$ and a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau_G$? (see [15]).

2010 Mathematics Subject Classification. 20C15.
Key words and phrases. $p$-groups, Nilpotence Class, Total Character.
Our aim in this paper is to consider a weaker version of Johnson’s question: we examine the existence of a polynomial \( f(x) \in \mathbb{Q}[x] \) and a character \( \chi \in \text{Irr}(G) \) such that \( f(\chi) = \tau_G \). We call such a polynomial \( f(x) \in \mathbb{Q}[x] \), if it exists, a Johnson polynomial of \( G \). This problem has been studied for dihedral groups \( D_{2n} \) in [15], where it is proved that \( D_{2n} \) has a Johnson polynomial if and only if \( 8 \nmid n \).

A group \( G \) is said to be a VZ-group if every irreducible character vanishes on \( G \setminus Z(G) \) (see [10]). In [16], the second author and R. Sarma investigated the existence of Johnson polynomial of VZ-groups. In particular, they proved that a VZ-group has a Johnson polynomial if and only if \( Z(G) \) is cyclic. This result was used in [16] to show that the only nonabelian \( p \)-groups of order \( p^4 \) when \( p \) is an odd prime which have a Johnson polynomial are those of nilpotence class 2 with a cyclic center.

In [17], the second author and B. Sury showed that a nilpotent generalized Camina group has a Johnson polynomial if and only if it has nilpotence class 2 and a cyclic center. This was used to show that groups of order \( p^5 \) for an odd prime \( p \) have a Johnson polynomial if and only if they have nilpotence class 2 and a cyclic center.

This seems to build evidence that if \( G \) is a \( p \)-group, then \( G \) has a Johnson polynomial if and only if \( G \) has nilpotence class 2 and a cyclic center. In this note, we show that one direction is true. In particular, we show that if \( G \) has nilpotence class 2 and a cyclic center, then \( G \) has a Johnson polynomial. We also show that these are the only groups.

**Theorem 1.1.** Let \( G \) be a \( p \)-group of nilpotence class 2. Then \( G \) has a Johnson polynomial if and only if \( Z(G) \) is cyclic.

We also will present two results that give further evidence that the only \( p \)-groups having a Johnson polynomial have nilpotence class 2. First, we show this is true if \( G \) is a \( p \)-group and \( |\text{cd}(G)| = 2 \).

**Theorem 1.2.** Let \( G \) be a \( p \)-group for some prime \( p \). If \( |\text{cd}(G)| = 2 \), then \( G \) has a Johnson polynomial if and only if \( G \) has nilpotence class 2 and a cyclic center.

For the other result, we need a definition. Following [14], we say that a group \( G \) is a generalized VZ-group and we write \( GVZ \)-group if \( \chi \) vanishes on \( G \setminus Z(\chi) \) for all \( \chi \in \text{Irr}(G) \). It is not difficult to show that every group of nilpotence class 2 is a GVZ-group. We show next the only GVZ-groups with a Johnson polynomial are those of nilpotence class 2 whose center is cyclic. Also, it is not difficult to see that VZ-groups and generalized Camina groups are GVZ-groups. Hence, this next theorem is a generalization of Theorem 1.1 and of the results in [16] and [17].

**Theorem 1.3.** Let \( G \) be a \( p \)-group. Then \( G \) is a GVZ-group with a Johnson polynomial if and only if \( G \) has nilpotence class at most 2 and \( Z(G) \) is cyclic.
2. Preliminaries

If $N$ is a normal subgroup of $G$, then we use by $\text{Irr}(G \mid N)$ to denote the characters in $\text{Irr}(G)$ whose kernels do not contain $N$. The following standard result is used a number of times, so we state it explicitly.

**Lemma 2.1.** [7, Theorem 2.32]

1. If $G$ has a faithful irreducible character, then $Z(G)$ is cyclic.
2. If $G$ is a $p$-group and $Z(G)$ is cyclic, then $G$ has a faithful irreducible character.

We next recall a basic, easy result.

**Lemma 2.2.** ([16]) Let $G$ be a non-abelian group. Suppose $f(x) \in \mathbb{Q}[x]$ is a Johnson polynomial of $G$ and $\chi \in \text{Irr}(G)$ is such that $f(\chi) = G$. Then $\chi$ is a nonlinear faithful character.

This next lemma essentially codifies the fact that $G = G'$ is the regular character of $G/G'$.

**Lemma 2.3.** Let $G$ be a non-abelian group. Then $\sum_{\chi \in \text{Lin}(G)} \chi(g) = 0$ for each $g \in G \setminus G'$.

In this article, whenever we prove a certain group $G$ does not possess a Johnson polynomial, we use the following simple observation.

**Lemma 2.4.** Let $\chi$ be an irreducible character of $G$. If $g_1, g_2 \in G$ are such that $\chi(g_1) = \chi(g_2)$ but $\tau_G(g_1) \neq \tau_G(g_2)$, then there does not exist a polynomial $f(x) \in \mathbb{C}[x]$ such that $f(\chi) = \tau_G$.

We now gather some facts regarding the total character. This first fact was motivated by the proof of Theorem 3.5 of [3].

**Lemma 2.5.** Let $G$ be a group. If $g \in G \setminus G'$, then $\tau_G(g) = 0$.

**Proof.** Fix an element $g \in G \setminus G'$. Then there exists a linear character $\lambda \in \text{Irr}(G/G')$ so that $\lambda(g) \neq 1$. Observe that multiplication by $\lambda$ just permutes $\text{Irr}(G)$, so $\lambda \tau_G = \sum_{\chi \in \text{Irr}(G)} \lambda \chi = \sum_{\chi \in \text{Irr}(G)} \chi = \tau_G$. We obtain $(\lambda - 1) \tau_G = \lambda \tau_G - \tau_G = 0$. It follows that $(\lambda(g) - 1) \tau_G(g) = 0$. Since $\lambda(g) - 1 \neq 0$, we conclude that $\tau_G(g) = 0$.

If $\chi$ is a (reducible or irreducible) character of $G$, we define $V(\chi)$ to be the vanishing-off subgroup of $\chi$. That is, $V(\chi) = \{x \in G \mid \chi(x) \neq 0\}$. Thus, $V(\chi)$ is the subgroup of $G$ generated by the elements of $G$ where $\chi$ does not vanish. Notice that $V(\chi)$ is a normal subgroup of $G$. Also, if $H \leq G$, then $V(\chi) \leq H$ if and only if $\chi$ vanishes on $G \setminus H$.

**Theorem 2.6.** Let $G$ be a group. Then $V(\tau_G) = G'$.

**Proof.** We know by Lemma 2.5 that $V = V(\tau_G) \leq G'$. We suppose that $V < G'$, and we will see that this leads to a contradiction. Since $V < G'$, we see that $G/V$ is nonabelian. Let $D$ be the vector space of complex valued...
class functions of $G$ that vanish on $G \setminus V$, and observe that $\tau_G \in D$. Notice that the dimension $d$ of $D$ will equal the number of $G$-conjugacy classes contained in $V$. It follows that the number of $G$-orbits on the conjugacy classes of $V$ equals $d$. By Corollary 6.33 of [7], we deduce that $d$ equals the number of orbits in the action of $G$ on the on irreducible characters of $V$.

Let $G_1 = 1_V, \nu_2, \ldots, \nu_d$ be representatives for the orbits of the action of $G$ on $\text{Irr}(V)$. Notice that each $\Lambda_i = (\nu_i)^G$ will vanish on $G \setminus V$, and so, will lie in $D$. Also, if $i \neq j$, then $\Lambda_i$ and $\Lambda_j$ do not have any common irreducible constituents. Hence, the set $\{\Lambda_1, \ldots, \Lambda_d\}$ is a linearly independent subset of $D$. Since it has size equal to the dimension of $D$, we conclude that $\{\Lambda_1, \ldots, \Lambda_d\}$ is a basis for $D$.

We can find complex numbers $a_1, \ldots, a_d$ so that $\tau_G = \sum_{i=1}^d a_i \Lambda_i$. We know $1_G$ occurs as a constituent with multiplicity 1 in both $\tau_G$ and $\Lambda_1$. This implies that $a_1 = 1$. On the other hand, since $G/V$ is abelian, we can find a nonlinear character $\chi \in \text{Irr}(G/V)$. By Frobenius reciprocity, we know that $[\Lambda_1, \chi] = [1_V, \chi] = [1_V, \chi V] = [1_V, \chi 1_V] = \chi(1) > 1$. Thus, $\chi$ has multiplicity $\chi(1)$ as a constituent of $\Lambda_1$. We know that $\chi$ is not a constituent of $\Lambda_i$ for $i > 1$. Thus, $\chi$ has multiplicity $\chi(1)$ in $\sum_{i=1}^d a_i \Lambda_i$. On the other hand, $\chi$ has multiplicity 1 in $\tau_G$, and so, we have a contradiction. Therefore, we conclude that $V = G'$.

A word of warning here. Please note that Theorem 2.6 does not say that if $g \in G'$, then $\tau_G(g) \neq 0$. Examples show that $\tau_G$ can take the value zero on elements of $G'$. One such example is the group SmallGroup (64,8) in the small group library in Magma or Gap ([2] or [4]). Theorem 2.6 says only that there is a generating set of $G'$ with the property that $\tau_G$ does not vanish on any element of this generating set.

We close this section with one observation regarding the total character. In the many examples that we have computed, the total character took on only nonnegative integer values, and we wonder if this is always true for the total characters of $p$-groups, but we have not been able to prove this.

3. Nilpotence class 2 and other GVZ-groups

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We know that if $G$ has a Johnson polynomial, then $G$ has a faithful irreducible character by Lemma 2.2, and so, by Lemma 2.1 $Z(G)$ is cyclic. Conversely, suppose that $Z = Z(G)$ is cyclic. Then by Lemma 2.1 $G$ has a faithful character $\chi \in \text{Irr}(G)$. It follows that $\chi_Z = \chi(1)\lambda$ where $\lambda \in \text{Irr}(Z)$, and observe that $\lambda$ is faithful. Let $n = |Z|$, and observe that $\text{Irr}(Z) = \{\lambda^i \mid 1 \leq i \leq n\}$. Since $(\tau_G)_Z$ is a character of $Z$, we can write $(\tau_G)_Z = \sum_{i=1}^n b_i \lambda^i$ for nonnegative integers $b_1, \ldots, b_n$. Thus, if $z \in Z$, then $\tau_G(z) = \sum_{i=1}^n b_i \lambda^i(z)$. 

Since $\chi$ is faithful, we know that $Z = Z(\chi)$, and since $G$ has nilpotence class 2, we see that $G/Z$ is abelian. By Theorem 2.31 of [7], $|G : Z| = \chi(1)^2$, and by Corollary 2.30 of [7], $\chi$ vanishes on $G \setminus Z$.

Define the polynomial $f(x) = \sum_{i=1}^{n} \frac{b_i}{\chi(1)i} x^i$. If $g \in G \setminus Z \subseteq G \setminus G'$, then $\tau_G(g) = 0$ by Lemma 2.5 and $\chi(g) = 0$, so $f(\chi(g)) = f(0) = \sum_{i=1}^{n} \frac{b_i}{\chi(1)i} 0^i = 0$. Suppose now that $z \in Z$. We have that $\tau_G(z) = \sum_{i=1}^{n} b_i \lambda(z)^i$ and $f(\chi(z)) = f(\chi(1)\lambda(z)) = \sum_{i=1}^{n} \frac{b_i}{\chi(1)i} (\chi(1)\lambda(z))^i = \sum_{i=1}^{n} b_i \lambda(z)^i$. Thus, $\tau_G(g) = f(\chi(g))$ for all $g \in G$. Therefore, $f(x)$ is a Johnson polynomial for $G$.

We now work to prove Theorem 1.3. The following lemma proves the result under a weaker hypothesis.

**Lemma 3.1.** Let $G$ be a $p$-group of nilpotence class at least 3. If every faithful character $\chi \in \text{Irr}(G)$ vanishes on $G \setminus Z(G)$, then $G$ does not have a Johnson polynomial.

**Proof.** We suppose that $G$ has a Johnson polynomial $f(x)$, and we show that this leads to a contradiction. By Lemma 2.2, an irreducible character $\chi$ associated with $f(x)$ is faithful. By hypothesis, we know that $V(\chi) = Z(G)$. Since $G$ has nilpotence class 3, we know that $G/Z(G)$ is nontrivial and nilpotent. This implies that $G'Z(G)/Z(G) = (G/Z(G))' < G/Z(G)$, and so, $G'Z(G) < G$. Thus, we can find an element $g \in G \setminus G'Z(G)$. It follows that $\chi(g) = 0$. Applying Theorem 2.6, we also have that $\tau_G(g) = 0$. We obtain $f(0) = f(\chi(g)) = f(0) = 0$. Since $\tau_G = f(\chi)$, we conclude that $G' = V(\tau_G) \leq V(\chi) = Z(G)$, and this contradicts $G$ having nilpotence class at least 3.

We obtain Theorem 1.3 as a corollary.

**Proof of Theorem 1.3.** If $G$ has nilpotence class at most 2 and $Z(G)$ is cyclic, then we apply Theorem 1.1 to see that $G$ has a Johnson polynomial. By Theorem 2.31 of [7], we have $|G : Z(\chi)| = \chi(1)^2$ for all $\chi \in \text{Irr}(G)$, and then by Corollary 2.30 of [7], $\chi$ vanishes on $G \setminus Z(\chi)$. Thus, $G$ is a GVZ-group. On the other hand, suppose $G$ is a GVZ-group and has a Johnson polynomial. By Lemma 2.2, we know that $G$ has a faithful irreducible character, and so, $Z(G)$ is cyclic by Lemma 2.1. Since $G$ is a GVZ-group, every faithful character $\chi \in \text{Irr}(G)$ vanishes on $G \setminus Z(G)$, and we may apply Lemma 3.1 to see that $G$ cannot have nilpotence class 3 or more. Therefore, $G$ has nilpotence class at most 2.

We now work to prove Theorem 1.2. We begin by finding a formula for counting the number of irreducible constituents of character induced from the center of the group.
Lemma 4.1. Let $p$ be a prime and let $G$ be a group satisfying $\text{cd}(G) = \{1, p^e\}$ where $e \geq 1$ is an integer. If $N \leq Z(G) \cap G'$ and $1_N \neq \theta \in \text{Irr}(N)$, then $|\text{Irr}(G \mid \theta)| = |G : N|/p^{2e}$.

Proof. If $\chi \in \text{Irr}(G \mid \theta)$, then $\chi$ is not linear, so $\chi(1) = p^e$. Since $N$ is central in $G$, we have that $\chi_N = p^e\theta$. By Frobenius Reciprocity, we have $\theta^G = p^e \sum_{\chi \in \text{Irr}(G \mid \theta)} \chi$. It follows that

$$|G : N| = \theta^G(1) = p^e \sum_{\chi \in \text{Irr}(G \mid \theta)} \chi(1) = p^{2e}|\text{Irr}(G \mid \theta)|.$$

Dividing both sides by $p^{2e}$, we conclude that $|\text{Irr}(G \mid \theta)| = |G : N|/p^{2e}$. □

We now compute the degree of the total character when the group has only two character degrees.

Lemma 4.2. Let $G$ be a $p$-group for some prime $p$ so that $\text{cd}(G) \subseteq \{1, p^e\}$ where $e \geq 1$ is an integer. Then $\tau_G(1) = \frac{p^e-1}{p^e}|G : G'| + \frac{|G|}{p^e}$.

Proof. We work by induction on $|G|$. If $G$ is abelian, then $\tau_G = \rho_G$, the regular character for $G$, and $\tau_G(1) = \rho_G(1) = |G| = \frac{p^e-1}{p^e}|G : G'| + \frac{|G|}{p^e}$. Thus, we may assume that $G$ is nonabelian. Let $r$ be the nilpotence class of $G$ so that $G_{r+1} = 1$ and $G_r > 1$. We know that $r \geq 2$. By induction, $\tau_{G/G_r}(1) = \frac{p^e-1}{p^e}|G : G'| + \frac{|G/G_r|}{p^e}$. Observe that $\tau_G = \tau_{G/G_r} + \sum_{\chi \in \text{Irr}(G/G_r)} \chi$. We see that $\sum_{\chi \in \text{Irr}(G/G_r)} \chi = \sum_{\lambda \in \text{Irr}(G_r)} \sum_{\chi \in \text{Irr}(G/G_r)} \chi(1) = \sum_{\lambda \in \text{Irr}(G_r)} \chi(1)$, by Lemma 4.1. This yields

$$\sum_{\chi \in \text{Irr}(G/G_r)} \chi(1) = \sum_{\lambda \in \text{Irr}(G_r)} \frac{|G : G_r|}{p^e} = (|G_r|-1)\frac{|G : G_r|}{p^e} = \frac{|G|-|G : G_r|}{p^e}.$$

It follows that $\tau_G(1) = \frac{p^e-1}{p^e}|G : G'| + \frac{|G/G_r|}{p^e} = \frac{p^e-1}{p^e}|G : G'| + \frac{|G|}{p^e}$, as desired.

We now everything in place to prove Theorem 1.2.

Proof of Theorem 1.2. We have $\text{cd}(G) = \{1, p^e\}$ for some positive integer $e$. We know from Theorem 1.1 that if $G$ has nilpotence class 2 and a cyclic center, then $G$ has a Johnson polynomial. Also, we know that if $G$ has a Johnson polynomial, then $G$ must have a faithful irreducible character. By Lemma 2.1, this implies that the center of $G$ is cyclic. Thus, it suffices to prove that if $G$ has nilpotence class 3 or more and $G$ has a cyclic center, then $G$ does not have a Johnson polynomial.

Let $Z$ be the center of $G$. Write $r$ for the nilpotence class of $G$. We know that $G_{r+1} = 1$ and $1 < G_r \leq Z$. Let $Y = Z \cap G_{r-1}$. Since $r \geq 3$, we have $Y \leq G_{r-1} \leq G_2 = G'$, which is a cyclic group of order $p^e$. Thus, there is a subgroup $N$ so that $Y < N \leq G_{r-1}$ and $|N : Y| = p$. Write $M = [N, G]$ and, observe that
we know that \( T < G \) cannot have that \( \sum x \). We also know that \( e \). This implies that \( \psi(x) = p^{e-1} \rho_{N/Y}(x) \nu(x) = p^{e-1} 0 \nu(x) = 0 \). In particular, we conclude that \( \sum_{\chi \in \text{Irr}(G/M)} \chi(x) = \sum 0 = 0 \).

Finally, we compute \( \sum_{\chi \in \text{Irr}(G/M|N/M)} \chi = \sum_{\lambda \in \text{Irr}(N/M)\setminus\{1\}} \sum_{\chi \in \text{Irr}(G/\lambda)} \chi \). Fix the character \( \lambda \in \text{Irr}(N/M)\setminus\{1\} \). Since \( N/M \) is central in \( G/M \) and \( N \leq G' \), we see for each character \( \chi \in \text{Irr}(G \mid \lambda) \) that \( \chi_N = \chi(1) \lambda = p^e \lambda \). We then have \( \sum_{\chi \in \text{Irr}(G/\lambda)} \chi_N = |\text{Irr}(G \mid \lambda)| p^e \lambda = \frac{|G : N|}{p^e} p^e \lambda = \frac{|G : N|}{p^e} \lambda \) for each \( \lambda \in \text{Irr}(N/M)\setminus\{1\} \), where the last equality comes from Lemma 4.1. This yields

\[
\sum_{\chi \in \text{Irr}(G/M|N/M)} \chi_N = \frac{|G : N|}{p^e} \sum_{\lambda \in \text{Irr}(N/M)\setminus\{1\}} \lambda = \frac{|G : N|}{p^e} (\rho_{N/M} - 1_{N/M}),
\]

where \( \rho_{N/M} \) is the regular character for \( N/M \) inflated to \( N \). It follows that

\[
\sum_{\chi \in \text{Irr}(G/M|N/M)} \chi(x) = \frac{|G : N|}{p^e} (\rho_{N/M}(x) - 1_{N/M}(x)) = \frac{|G : N|}{p^e} (-1).
\]

We conclude that

\[
\tau_G(x) = \frac{p^e - 1}{p^e} |G : G'| + \frac{|G : N|}{p^e} - \frac{|G : N|}{p^e} = \frac{p^e - 1}{p^e} |G : G'| \neq 0.
\]

Suppose now that \( \psi \) is a faithful irreducible character of \( G \). It follows that \( \psi \in \text{Irr}(G \mid Y) \), and so \( \psi(x) = 0 \) for \( x \in N \setminus Y \). Since \( \text{cd}(G) = \{1, p^e\} \), we know that \( G' \) is abelian (see Corollary 12.6 of [7]). Let \( \lambda \in \text{Irr}(G') \) be a constituent of \( \psi \). We know that \( \lambda \) is linear. Since \( Z(G') = Z(\chi) \), we cannot have that \( \lambda \) is \( G \)-invariant. Let \( T \) be the stabilizer of \( \lambda \) in \( G \) so that \( T < G \). We also know that \( G \) cannot be the union of conjugates of \( T \), and so, there is an element \( g \in G \) that is not conjugate to any element of \( T \). Since \( \psi \) is induced from \( T \) (by Clifford’s theorem), it follows that \( \psi(g) = 0 \).

Note that \( G' \leq T \) implies that \( g \in G \setminus G' \), and \( \tau_G(g) = 0 \) by Lemma 2.5. On the other hand, we have shown that \( \tau_G(x) \neq 0 \). Because \( \psi(g) = \psi(x) = 0 \), we conclude that \( G \) does not have a Johnson polynomial. \( \square \)

5. ACKNOWLEDGMENT

The second author was supported by National Board for Higher Mathematics (NBHM), India.
REFERENCES


Department of Mathematical Sciences, Kent State University, Kent, OH 44242

E-mail address: lewis@math.kent.edu

Stat Math Unit, Indian Statistical Institute, 8th Mile Mysore Road, Bangalore-560059, INDIA

E-mail address: skprajapati.iitd@gmail.com