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Quantum random walk approximation in Banach algebra

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QUANTUM RANDOM WALK APPROXIMATION IN BANACH ALGEBRA

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ABSTRACT. Quantum random walks in a unital Banach algebra are considered. Belton's discrete approximation scheme is extended to sesquilinear quantum stochastic cocycles through dyadic discretisation of time. We recover approximation results for Markov-regular quantum stochastic mapping cocycles, and obtain a new random walk approximation theorem for a class of isometric cocycles which includes all the unitary cocycles which induce Lévy processes on (the CQG algebra of) a separable compact quantum group.

INTRODUCTION

Quantum stochastic analysis was recently extended to Banach space by viewing processes as sesquilinear maps ([DLT]), thereby unifying the 'standard' quantum stochastic theory of operator processes on a Hilbert space ([HuP]), mapping processes on a C^*-algebra or operator space ([Eva], [LW$_1$]), see [Par], [Mey], [L], and convolution processes on a quantum group or coalgebra ([Sch], [LiS]). In the sesquilinear theory, stochastic cocycles are analysed via some elementary theory of evolutions in unital Banach algebras ([DL$_1$]).

The aim of the present paper is to extend Belton's discrete approximation scheme for quantum stochastic cocycles ([Be$_2$]) to Banach-algebra-valued sesquilinear cocycles and to apply this to various discrete approximation schemes for operator cocycles and mapping cocycles on operator spaces by appropriate choices of Banach algebra. For operator cocycles, we obtain new results, extending those of [AtP] (*cf.* [Sah] and [Be$_2$]) for isometric cocycles which are Markov-regular, equivalently have bounded stochastic generator. The class of isometric cocycle covered, namely direct sums of Markov-regular isometric cocycles, includes all those which implement the quantum Lévy processes on the CQG algebra of a separable compact quantum group.

Accordingly, the processes considered in this paper are families $(\mathsf{q}_t)_{t\geq 0}$ of sesquilinear maps $\mathcal{E}\times\mathcal{E}\to\mathcal{A}$ for a unital Banach algebra $\mathcal{A}$ and exponential domain $\mathcal{E}$ in symmetric Fock space over $L^2(\mathbb{R}_+;\mathsf{k})$. The Hilbert space k serves as the multiplicity space of the quantum noise. Natural adaptedness and regularity conditions are assumed, together with a time-homogeneous evolution property, or stochastic cocycle condition.

The plan of the paper is as follows. After a brief section of preliminaries we recall relevant results from [DLT] in Section 2. Discrete approximation of sesquilinear cocycles is treated in Section 3, and in Section 4 we apply our results to random walk approximation schemes for mapping cocycles. In the final section we consider random walk approximation for isometric operator cocycles in the Markov-regular case, and for a class of isometric cocycles in the non-Markov regular case.

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Notations. For vector spaces V, V' and W we write $\widehat{V}$ for $\mathbb{C}\oplus V$, $\widehat{v}$ for $\binom{1}{v}$ $(v \in V)$, and $SL(V', V; W)$ for the space of sesquilinear maps $V' \times V \to W$ (abbreviated to $SL(V; W)$ when $V' = V$). All sesquilinear maps are linear in their second argument. Basic examples of such maps are those of the form $|w\rangle q_T$ for $T \in L(V; V')$ and $w \in W$ where V and V' are inner product spaces:

$$q_T : V' \times V \to \mathbb{C}, \ (v', v) \mapsto \langle v', Tv\rangle \quad \text{and} \quad |w\rangle : \mathbb{C} \to W, \ \lambda \mapsto \lambda w. \tag{0.1}$$

For a subinterval J of $\mathbb{R}_+$ and $n \in \mathbb{N}$, we define the n-symplices over J as follows:

$$\Delta_J^{(n)} := \{\mathbf{t} \in J^n : t_1 < \cdots < t_n\} \quad \text{and} \quad \Delta_J^{[n]} := \{\mathbf{t} \in J^n : t_1 \leq \cdots \leq t_n\},$$

abbreviated to $\Delta^{(n)}$ and $\Delta^{[n]}$ when $J = \mathbb{R}_+$. For a vector-valued function f on $\mathbb{R}_+$ and subinterval J of $\mathbb{R}_+$, we write f_J for the function on $\mathbb{R}_+$ which agrees with f on J and vanishes outside J.

For Hilbert spaces H and h and vector $e \in \mathsf{h}$, the operator

$$I_{\mathsf{H}} \otimes |e\rangle : \mathsf{H} \to \mathsf{H} \otimes \mathsf{h}, \quad u \mapsto u \otimes e$$

is denoted by E_e, and its adjoint $I_{\mathsf{H}} \otimes \langle e|$ by E^e, with context dictating the Hilbert space H. Thus $E^e \in B(\mathsf{H} \otimes \mathsf{h}; \mathsf{H})$ and $E^e E_f = \langle e, f\rangle I_{\mathsf{H}}$.

1. Preliminaries

We need a specific tensor construction for concrete operator spaces. Let V be an operator space in $B(\mathsf{H}; \mathsf{H}')$ and set $B = B(\mathsf{h}; \mathsf{h}')$, for Hilbert spaces h and h'. The *matrix space tensor product of* V *with* B is the operator space in $B(\mathsf{H}\otimes\mathsf{h}; \mathsf{H}'\otimes\mathsf{h}') = B(\mathsf{H}; \mathsf{H}')\overline{\otimes}B$ defined as follows:

$$\mathsf{V} \otimes_{\mathsf{M}} B := \big\{T \in B(\mathsf{H}; \mathsf{H}')\overline{\otimes}B : E^{c'} T E_c \in \mathsf{V} \text{ for all } c' \in \mathsf{h}', c \in \mathsf{h}\big\}.$$

Let W be another concrete operator space, (in other words an operator space realised as a closed subspace of $B(\mathsf{K}, \mathsf{K}')$ for some Hilbert spaces K and K'). If $\phi \in CB(\mathsf{V}; \mathsf{W})$ then the map $\phi \otimes_{\mathsf{M}} \mathrm{id}_B \in CB(\mathsf{V} \otimes_{\mathsf{M}} B; \mathsf{W} \otimes_{\mathsf{M}} B)$ ([LW$_1$]) is the unique extension of $\phi \underline{\otimes} \mathrm{id}_B$. The following extended composition is very useful. For $\phi_i \in CB\big(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} B(\mathsf{h}_i; \mathsf{h}_i')\big)$ $(i = 1, 2)$,

$$\phi_1 \bullet \phi_2 := (\phi_1 \otimes_{\mathsf{M}} \mathrm{id}_{B(\mathsf{h}_2; \mathsf{h}_2')}) \circ \phi_2 \in CB\big(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} B(\mathsf{h}; \mathsf{h}')\big). \tag{1.1}$$

Here $\mathsf{h} = \mathsf{h}_1 \otimes \mathsf{h}_2$ and $\mathsf{h}' = \mathsf{h}_1' \otimes \mathsf{h}_2'$, so $B(\mathsf{h}_1; \mathsf{h}_1') \otimes_{\mathsf{M}} B(\mathsf{h}_2; \mathsf{h}_2') = B(\mathsf{h}; \mathsf{h}')$.

We need the following observation of Wills and Skalski, proved in [Be$_2$].

Lemma 1.1. *For a sequence of completely bounded maps $(\phi_n)_{n\geq 1}$ and infinite-dimensional Hilbert space h, if $\phi_n \otimes_{\mathsf{M}} \mathrm{id}_{B(\mathsf{h})} \to 0$ strongly then $\|\phi_n\|_{\mathrm{cb}} \to 0$.*

For more on matrix spaces see [LW$_1$], or [L]; an appropriate reference for operator spaces is [EfR].

We note the following elementary fact for ease of reference.

Lemma 1.2. *Let $\mathcal{A}$ be a Banach algebra, let $x_0 \in \mathcal{A}$ and let a be a step function $\mathbb{R}_+ \to \mathcal{A}$ with discontinuity set D. Then the integral equation*

$$f(t) = x_0 + \int_0^t \mathrm{d}s\, f(s)a(s) \qquad (t \geq 0).$$

and the differential equation

$$f(0) = x_0 \quad \text{and} \quad f'(s) = f(s)a(s) \quad (s \in \mathbb{R}_+ \setminus D),$$

have the same unique solution in $C\big(\mathbb{R}_+; \mathcal{A}\big)$.

2. Sesquilinear quantum stochastic calculus in Banach space

In this section we recall sesquilinear quantum stochastic theory from [DLT]. *Fix now, and for the rest of the paper*, a complex Hilbert space k referred to as the noise dimension space, a Banach space $\mathfrak{X}$ and a unital Banach algebra $\mathcal{A}$.

For a subinterval J of $\mathbb{R}_+$, let $\mathsf{K}_J := L^2(J;\mathsf{k})$ and, for $f \in \mathsf{K}_J$, write $\widehat{f}$ for the corresponding $\widehat{\mathsf{k}}$-valued function given by $\widehat{f}(s) := \widehat{f(s)}$. The space of step functions in K_J is denoted by $\mathbb{S}_J$ (we take right-continuous versions) and we denote by $\mathbb{S}_J^{\mathbb{D}}$ the subspace of step function with the set of discontinuities lies in positive dyadic rationals $\mathbb{D}_+$. The symmetric Fock space over K_J is denoted $\mathcal{F}_J$; the exponential vectors $\varepsilon(f) := ((n!)^{-1/2} f^{\otimes n})_{n\geq 0}$ $(f \in \mathsf{K}_J)$ are linearly independent and $\mathcal{E}_J := \operatorname{Lin}\{\varepsilon(f) : f \in \mathbb{S}_J\}$ and $\mathcal{E}_J^{\mathbb{D}} := \operatorname{Lin}\{\varepsilon(f) : f \in \mathbb{S}_J^{\mathbb{D}}\}$ are both dense in K_J; when $J = \mathbb{R}_+$, we drop the subscript J.

A family of maps $\mathfrak{q} = (\mathfrak{q}_t)_{t\geq 0}$ in $SL(\mathcal{E};\mathfrak{X})$ is an $\mathfrak{X}$*-valued sesquilinear process, or SL process in* $\mathfrak{X}$ if, for all $g', g \in \mathbb{S}$ and $t \in \mathbb{R}_+$,

(i) $\mathfrak{q}_t(\varepsilon(g'),\varepsilon(g)) = \mathfrak{q}_t(\varepsilon(g'_{[0,t[}),\varepsilon(g_{[0,t[}))\langle\varepsilon(g'_{[t,\infty[}),\varepsilon(g_{[t,\infty[})\rangle$.

It is a *continuous* SL process in $\mathfrak{X}$ if, for all $\varepsilon,\varepsilon' \in \mathcal{E}$,

(ii) $s \mapsto \mathfrak{q}_s(\varepsilon',\varepsilon)$ is continuous.

We denote the linear space of SL processes in $\mathfrak{X}$ by $SL\mathbb{P}(\mathfrak{X},\mathsf{k})$, and the subspace of continuous SL processes by $SL\mathbb{P}_{\mathrm{c}}(\mathfrak{X},\mathsf{k})$, and, for $\mathfrak{q} \in SL\mathbb{P}(\mathfrak{X},\mathsf{k})$, define

$$\mathfrak{q}_t^{g',g} := \mathfrak{q}_t(\varepsilon(g'_{[0,t[}),\varepsilon(g_{[0,t[})) \qquad (g', g \in \mathbb{S}_{\mathrm{loc}}, t \in \mathbb{R}_+),$$

where $\mathbb{S}_{\mathrm{loc}} \subset L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathsf{k})$ denotes the space of (right-continuous) step functions. Thus $\mathfrak{q} \in SL\mathbb{P}_{\mathrm{c}}(\mathfrak{X},\mathsf{k})$ if and only if $\mathfrak{q} \in SL\mathbb{P}(\mathfrak{X},\mathsf{k})$ and $\mathfrak{q}^{g',g} \in C(\mathbb{R}_+;\mathfrak{X})$ for all $g', g \in \mathbb{S}_{\mathrm{loc}}$.

Multiple quantum Wiener integrals are the key to construct the solution of a sesquilinear differential equation; these are defined as follows. For $n \in \mathbb{N}$, $\upsilon_n \in SL(\widehat{\mathsf{k}}^{\underline{\otimes} n};\mathfrak{X})$ and $t \geq 0$, define a map $\Lambda_t^n(\upsilon_n) \in SL(\mathcal{E};\mathfrak{X})$ by sesquilinear extension of the prescription

$$\Lambda_t^n(\upsilon_n)(\varepsilon(g'),\varepsilon(g)) := \exp\langle g', g\rangle \int_{\Delta^{[n]}_{[0,t[}} \mathrm{d}\mathbf{s}\ \upsilon_n\big(\widehat{g'}^{\otimes n}(\mathbf{s}), \widehat{g}^{\otimes n}(\mathbf{s})\big) \quad (g', g \in \mathbb{S}),$$

for the convention

$$\widehat{h}^{\otimes n}(\mathbf{s}) := \widehat{h}(s_1) \otimes \cdots \otimes \widehat{h}(s_n), \qquad (\mathbf{s} \in \Delta^{[n]}).$$

For $\upsilon_0 \in SL(\mathbb{C};\mathfrak{X})$, $\Lambda^0_\cdot(\upsilon_0)$ is the constant SL process $|\upsilon_0(1,1)\rangle\, q_I$.

The following estimate is evident:

$$\big\|\Lambda_t^n(\upsilon_n)(\varepsilon(g'),\varepsilon(g))\big\| \leq |\exp\langle g', g\rangle|\, C_n^{\upsilon_n}(g', g)\frac{t^n}{n!} \quad (t \in \mathbb{R}_+)$$

where

$$C_n^{\upsilon_n}(g', g) := \max\Big\{\big\|\upsilon_n\big(\widehat{c(1)} \otimes \cdots \otimes \widehat{c(n)}, \widehat{d(1)} \otimes \cdots \otimes \widehat{d(n)}\big)\big\| : \\ c(1),\dots,c(n) \in \operatorname{Ran} g',\ d(1),\dots,d(n) \in \operatorname{Ran} g\Big\}. \quad (2.1)$$

This leads us to define $SL\mathbb{W}(\mathfrak{X},\mathsf{k})$, the space of time-independent *Wiener integrands*, is the space of sequences $\mathcal{U} = (\upsilon_n)_{n\geq 0}$, in which $\upsilon_n \in SL(\widehat{\mathsf{k}}^{\underline{\otimes} n};\mathfrak{X})$ for each $n \in \mathbb{Z}_+$ and

$$\forall_{g',g\in\mathbb{S}}\ \forall_{\alpha\in\mathbb{R}_+}\ \sum_{n\geq 0} \frac{\alpha^n}{n!} C_n^{\upsilon_n}(g', g) < \infty. \quad (2.2)$$

Thus for $\mathcal{U} = (\upsilon_n)_{n\geq 0} \in SL\mathbb{W}(\mathfrak{X},\mathsf{k})$, $\Lambda_t(\mathcal{U}) := \text{p.w.}\sum_{n\geq 0}\Lambda_t^n(\upsilon_n) \qquad (t \geq 0)$ defines an SL process $\Lambda(\mathcal{U})$ in $\mathfrak{X}$.

Now the construction of the solution of a sesquilinear quantum stochastic differential equation, which essentially shadows the Picard iteration method, is as follows. For $\nu \in SL(\widehat{\mathsf{k}};\mathcal{A})$ define $\nu^{\otimes} = \big(\nu^{\underline{\otimes} n}\big)_{n\geq 0}$ by $\nu^{\underline{\otimes} 0} := |1_{\mathcal{A}}\rangle\, q_I$ and, for $n \in \mathbb{N}$, $\nu^{\underline{\otimes} n} : \widehat{\mathsf{k}}^{\underline{\otimes} n} \times \widehat{\mathsf{k}}^{\underline{\otimes} n} \to \mathcal{A}$ is the 'sesquilinearisation' of the map

$$\widehat{\mathsf{k}}^n \times \widehat{\mathsf{k}}^n \to \mathcal{A}, \quad (\zeta,\eta) \mapsto \overrightarrow{\prod_{1\leq i\leq n}} \nu(\zeta_i,\eta_i).$$

Then

$$C_n^{\nu^{\underline{\otimes} n}}(g',g) \leq C^{\nu}(g',g)^n \qquad (n \in \mathbb{Z}_+, g',g \in \mathbb{S}),$$

so $\nu^{\otimes} \in \mathrm{SL}\mathbb{W}(\mathcal{A},\mathsf{k})$; set $\mathfrak{q}^{\nu} := \Lambda(\nu^{\otimes})$.

Definition. Let $\nu \in SL(\widehat{\mathsf{k}};\mathcal{A})$. Then $\mathfrak{q} \in SL\mathbb{P}_{\mathrm{c}}(\mathcal{A},\mathsf{k})$ is a solution of the *left sesquilinear quantum stochastic differential equation*

$$\mathrm{d}\mathfrak{q}_t = \mathfrak{q}_t \, \mathrm{d}\Lambda_{\nu}(t), \quad \mathfrak{q}_0 = |1_{\mathcal{A}}\rangle q_I \tag{2.3}$$

if, for all $g',g \in \mathbb{S}$ and $t \in \mathbb{R}_+$,

$$\mathfrak{q}_t(\varepsilon(g'),\varepsilon(g)) = \langle \varepsilon(g'),\varepsilon(g)\rangle 1_{\mathcal{A}} + \int_0^t \mathrm{d}s \; \mathfrak{q}_s(\varepsilon(g'),\varepsilon(g))\nu(\widehat{g'}(s),\widehat{g}(s)).$$

Theorem 2.1 ([DLT], Theorem 5.1). *Let $\nu \in SL(\widehat{\mathsf{k}};\mathcal{A})$. Then $\mathfrak{q}^{\nu}$ is the unique solution of the sesquilinear quantum stochastic differential equation* (2.3).

The notion of sesquilinear quantum stochastic cocycle and its relation with sesquilinear quantum stochastic differential equations is discussed next.

Definition. A process $\mathfrak{q} \in SL\mathbb{P}(\mathcal{A},\mathsf{k})$ is a (*left*) *sesquilinear stochastic cocycle* in $\mathcal{A}$ if it satisfies

$$\mathfrak{q}_0^{g',g} = 1_{\mathcal{A}} \text{ and } \mathfrak{q}_{s+t}^{g',g} = \mathfrak{q}_s^{g',g}\,\mathfrak{q}_t^{L_s g', L_s g} \qquad (g',g \in \mathbb{S}_{\mathrm{loc}}, s,t \in \mathbb{R}_+),$$

where $(L_t)_{t\geq 0}$ is the semigroup of left shift on $L^2(\mathbb{R}_+;\mathsf{k})$. Moreover if $\mathfrak{q} \in SL\mathbb{P}_{\mathrm{c}}(\mathcal{A},\mathsf{k})$ is a sesquilinear stochastic cocycle in $\mathcal{A}$ then $\mathfrak{q}$ is said to be *Markov regular*.

We denote the classes of left SL cocycles and Markov-regular left SL cocycles by $SL\mathbb{S}\mathbb{C}(\mathcal{A},\mathsf{k})$ and $SL\mathbb{S}\mathbb{C}_{\mathrm{c}}(\mathcal{A},\mathsf{k})$ respectively. Note that, for $c',c \in \mathsf{k}$, $\mathfrak{q}^{c',c}$ is a semigroup in $\mathcal{A}$; we refer to $\{\mathfrak{q}^{c',c} : c',c \in \mathsf{k}\}$ as the family of *associated semigroups* of $\mathfrak{q}$. Then the cocycle $\mathfrak{q}$ is Markov regular if and only if each of its associated semigroups is norm continuous.

Sesquilinear cocycles are constructed by solving sesquilinear quantum stochastic differential equations and, under certain regularity conditions the converse holds. We summarise this in the following theorem, where we write B_{conj} for bounded conjugate-linear.

Theorem 2.2 ([DLT], Theorems 6.2 and 6.3). *Let $\nu \in SL(\widehat{\mathsf{k}};\mathcal{A})$. Then $\mathfrak{q}^{\nu} \in SL\mathbb{S}\mathbb{C}_{\mathrm{c}}(\mathcal{A},\mathsf{k})$ and its associated semigroup generators are given by*

$$\beta_{c',c} = \nu(\widehat{c'},\widehat{c}) + \langle c',c\rangle 1_{\mathcal{A}} \qquad (c',c \in \mathsf{k}). \tag{2.4}$$

Conversely, let $\mathfrak{q} \in SL\mathbb{S}\mathbb{C}_{\mathrm{c}}(\mathcal{A},\mathsf{k})$, and suppose that there are separating families of maps $(\varphi_i \in B(\mathcal{A};\mathfrak{X}_i))_{i\in\mathcal{I}}$ and $(\varphi'_{i'} \in B(\mathcal{A};\mathfrak{X}'_{i'}))_{i'\in\mathcal{I}'}$ for Banach spaces $\mathfrak{X}_i$ and $\mathfrak{X}'_{i'}$ such that, for all $\varepsilon',\varepsilon \in \mathcal{E}$, $t \in \mathbb{R}_+$, $i \in \mathcal{I}$ and $i' \in \mathcal{I}'$,

(i) *$\varphi_i \circ \mathfrak{q}_t(\varepsilon',\cdot) \in B(\mathcal{E};\mathfrak{X}_i)$ and $\varphi'_{i'} \circ \mathfrak{q}_t(\cdot,\varepsilon) \in B_{\mathrm{conj}}(\mathcal{E};\mathfrak{X}'_{i'})$;*
(ii) *the maps $s \mapsto \varphi_i \circ \mathfrak{q}_s(\varepsilon',\cdot)$ and $s \mapsto \varphi'_{i'} \circ \mathfrak{q}_s(\cdot,\varepsilon)$ are continuous at 0.*

Then $\mathfrak{q} = \mathfrak{q}^{\nu}$ for a unique map $\nu \in SL(\widehat{\mathsf{k}};\mathcal{A})$.

Now we consider the vacuum-adapted analogue of sesquilinear processes and establish their relationship with standard sesquilinear processes. These are used in the next section to obtain random walk approximation first for vacuum-adapted sesquilinear cocycles.

Let $SL\mathbb{P}^\Omega(\mathfrak{X},\mathsf{k})$ ($SL\mathbb{P}^\Omega_{\mathrm{c}}(\mathfrak{X},\mathsf{k})$) denote the vacuum-adapted analogue of the spaces of (continuous) SL processes in $\mathfrak{X}$ that we have been considering, namely the classes of families $(\mathfrak{q}_t)_{t\geq 0}$ in $SL(\widehat{\mathsf{k}};\mathfrak{X})$ satisfying

(i)$_\Omega$ $\mathfrak{q}_t(\varepsilon(g'),\varepsilon(g)) = \mathfrak{q}_t(\varepsilon(g'_{[0,t[}),\varepsilon(g_{[0,t[}))$ for all $g, g' \in \mathbb{S}$ and $t \in \mathbb{R}_+$,

and for continuous processes,

(ii)$_\Omega$ $s \mapsto \mathfrak{q}_s(\varepsilon',\varepsilon)$ is continuous, for all $\varepsilon,\varepsilon' \in \mathcal{E}$.

Adaptedness switching

$$\mathfrak{q} \mapsto \mathfrak{q}^\Omega \quad \text{where} \quad \mathfrak{q}^\Omega_t(\varepsilon(g'),\varepsilon(g)) := \mathfrak{q}_t(\varepsilon(g'_{[0,t[}),\varepsilon(g_{[0,t[})),$$

gives a linear isomorphism $SL\mathbb{P}(\mathfrak{X},\mathsf{k}) \to SL\mathbb{P}^\Omega(\mathfrak{X},\mathsf{k})$, which restricts to an isomorphism $SL\mathbb{P}_{\mathrm{c}}(\mathfrak{X},\mathsf{k}) \to SL\mathbb{P}^\Omega_{\mathrm{c}}(\mathfrak{X},\mathsf{k})$. Vacuum-adapted multiple quantum Wiener integrals are defined by sesquilinear extension of the prescription

$$\Lambda^\Omega_t(\mathcal{U})(\varepsilon(g'),\varepsilon(g)) := \sum_{m\geq 0}\int_{\Delta^{[m]}_{[0,t[}} d\mathbf{s}\; \upsilon_m\big(\widehat{g'}^{\otimes m}(\mathbf{s}),\widehat{g}^{\otimes m}(\mathbf{s})\big) \quad (g',g\in\mathbb{S}),$$

for $\mathcal{U} = \big(\upsilon_m\big)_{m\geq 0} \in SL\mathbb{W}(\mathfrak{X},\mathsf{k})$, yielding a process $\Lambda^\Omega(\mathcal{U}) \in SL\mathbb{P}^\Omega_{\mathrm{c}}(\mathfrak{X},\mathsf{k})$. In particular, for $\lambda \in SL(\widehat{\mathsf{k}};\mathcal{A})$, we may define $\mathfrak{q}^{\Omega,\lambda} := \Lambda^\Omega\big(\lambda^\otimes\big) \in SL\mathbb{P}^\Omega_{\mathrm{c}}(\mathcal{A},\mathsf{k})$. Thus, for $g', g \in \mathbb{S}$,

$$\mathfrak{q}^{\Omega,\lambda}_t(\varepsilon(g'),\varepsilon(g)) = \sum_{m\geq 0}\int_{\Delta^{[m]}_{[0,t[}} d\mathbf{s}\; \lambda^{\otimes m}\big(\widehat{g'}^{\otimes m}(\mathbf{s}),\widehat{g}^{\otimes m}(\mathbf{s})\big) \tag{2.5}$$

which is continuous in t, and satisfies $\mathfrak{q}^{\Omega,\lambda}_0(\varepsilon(g'),\varepsilon(g)) = 1_\mathcal{A}$ and

$$\frac{d}{dt}\mathfrak{q}^{\Omega,\lambda}_t(\varepsilon(g'),\varepsilon(g)) = \mathfrak{q}^{\Omega,\lambda}_t(\varepsilon(g'),\varepsilon(g))\lambda\big(\widehat{g'}(t),\widehat{g}(t)\big) \qquad t \in \mathbb{R}_+ \setminus (\operatorname{Disc} g' \cup \operatorname{Disc} g). \tag{2.6}$$

Proposition 2.3. *Let $\nu \in SL(\widehat{\mathsf{k}};\mathcal{A})$. Then*

$$(\mathfrak{q}^\nu)^\Omega = \mathfrak{q}^{\Omega,\nu+\delta}.$$

Proof. Let $g', g \in \mathbb{S}$ and let $D = \operatorname{Disc} g' \cup \operatorname{Disc} g$. It follows from adaptedness, (2.6) and Theorem 2.1, that the functions

$$F_1 : t \mapsto \big(\mathfrak{q}^\nu\big)^\Omega_t\big(\varepsilon(g'_{[0,t[}),\varepsilon(g_{[0,t[})\big) \quad\text{and}\quad F_2 : t \mapsto \big(\mathfrak{q}^{\Omega,\nu+\delta}_t\big)\big(\varepsilon(g'_{[0,t[}),\varepsilon(g_{[0,t[})\big)$$

satisfy the hypotheses of Lemma 1.2. The result follows. □

3. Discrete approximation

In this section we show how sesquilinear quantum stochastic cocycles may be approximated by quantum random walks, by shadowing Belton's approach to discrete approximation ([Be$_2$]). The notations

$$P_\Omega \in B(\mathcal{F}),\ \widetilde{\varepsilon}(g) \in \mathcal{F}_J,\ \text{and } \Delta \in B(\widehat{\mathsf{k}})$$

(where J is a subinterval of $\mathbb{R}_+$), stand for the *vacuum projection* $P_{\mathbb{C}\Omega}$ where $\Omega = \varepsilon(0)$, the *truncated exponential vector* $(1, g, 0, 0, \cdots)$ for $g \in \mathsf{K}_J$, and the *quantum Itô projection* $P_{\{0\}\oplus\mathsf{k}}$. (Below J will be a vanishingly small interval.)

The discrete counterpart to the symmetric Fock space $\mathcal{F}$ is the countable tensor product $\Upsilon := \bigotimes_{n=0}^\infty \widehat{\mathsf{k}}_n$, with respect to the stabilising sequence $(\omega_0,\omega_1,\cdots)$, in which $\widehat{\mathsf{k}}_n = \widehat{\mathsf{k}}$ and $\omega_n = \binom{1}{0} \in \widehat{\mathsf{k}}_n$ for each n. Meyer and Journé referred to this as *toy Fock space* ([Mey]). Set $\Upsilon_{0)} := \mathbb{C}$, for $n \in \mathbb{Z}_+$ set $\Upsilon_{[n,\infty[} := \bigotimes_{i=n}^{\infty} \widehat{\mathsf{k}}_i$ (with respect to the stabilising sequence $(\omega_n, \omega_{n+1}, \cdots)$),

$$\omega_{[n,\infty[} := \bigotimes_{i=n}^{\infty} \omega_i \in \Upsilon_{[n,\infty[}$$

and $\Delta^{\perp}_{[n,\infty[} = P_{\mathbb{C}\omega_{[n,\infty[}} \in B(\Upsilon)$. Also, for $m \in \{0, 1, \cdots, n-1\}$, set

$$\omega_{[m,n[} := \bigotimes_{i=m}^{n-1} \omega_i, \ \Upsilon_{[m,n[} := \bigotimes_{i=m}^{n-1} \widehat{\mathsf{k}}_i \ \text{ and } \ \Upsilon^{\circ}_{[m,n[} := \underline{\bigotimes}_{i=m}^{n-1} \widehat{\mathsf{k}}_i.$$

Thus, for $l, m \in \mathbb{Z}_+$ with $0 \le l < m < \infty$,

$$\Upsilon = \Upsilon_{[0,l[} \otimes \Upsilon_{[l,m[} \otimes \Upsilon_{[m,\infty[}.$$

For $h > 0$ and $n \in \mathbb{N}$, let $J^{(h)} \in B(\Upsilon; \mathcal{F})$ and $J^{(h)}_n \in B(\Upsilon_{[0,n[}; \mathcal{F}_{[0,hn[})$ be the isometries determined by the inner product preserving prescriptions

$$J^{(h)}\Big(\bigotimes_{j=0}^{\infty} \widehat{c(j)}\Big) = \bigotimes_{j=0}^{\infty} \widetilde{\varepsilon}\Big(h^{-1/2} c(j)_{[jh,(j+1)h[}\Big) \qquad \text{for } \big(c(j)\big)_{j\ge 0} \in c_{00}(\mathsf{k});$$

$$J^{(h)}_n\Big(\bigotimes_{j=0}^{n-1} \widehat{c(j)}\Big) = \bigotimes_{j=0}^{n-1} \widetilde{\varepsilon}\Big(h^{-1/2} c(j)_{[jh,(j+1)h[}\Big), \qquad \text{for } c(0), \cdots, c(n-1) \in \mathsf{k}.$$

There are induced maps

$$\Xi^{(h)}_n, {}^{\Omega}\Xi^{(h)}_n : SL(\Upsilon^{\circ}_{[0,n[}; \mathcal{A}) \to SL(\mathcal{E}_{[0,nh[} \underline{\otimes} \mathcal{F}_{[nh,\infty[}; \mathcal{A}) \subset SL(\mathcal{E}; \mathcal{A}) \tag{3.1}$$

determined by

$$\Xi^{(h)}_n(\upsilon_n)(\zeta' \otimes \eta', \zeta \otimes \eta) = \upsilon_n(J^{(h)\,*}_n \zeta', J^{(h)\,*}_n \zeta)\langle \eta', \eta\rangle, \quad \text{respectively,}$$
$${}^{\Omega}\Xi^{(h)}_n(\upsilon_n)(\zeta' \otimes \eta', \zeta \otimes \eta) = \upsilon_n(J^{(h)\,*}_n \zeta', J^{(h)\,*}_n \zeta)\langle \eta', P_{\Omega}\eta\rangle$$

$(\zeta', \zeta \in \mathcal{E}_{[0,nh[}, \eta', \eta \in \mathcal{F}_{[nh,\infty[})$. For $n \in \mathbb{Z}_+$, let $P^{n;h} \in B(\mathsf{K})$ be the orthogonal projection with range $\mathsf{k} \otimes 1_{[nh,(n+1)h[}$ (under the identification $\mathsf{K} = \mathsf{k} \otimes L^2(\mathbb{R}_+)$), and let P^h be the orthogonal projection whose range consists of the subspace of functions which are constant on all intervals of the form $[mh, (m+1)h[$ $(m \in \mathbb{Z}_+)$. In terms of the notation

$$g[n; h] := h^{-1} \int_{nh}^{(n+1)h} \mathrm{d}t\, g(t) \qquad (g \in \mathsf{K},\ n \in \mathbb{Z}_+),$$

for the average of g over the interval $[nh, (n+1)h]$, we have the identities

$$P^{n;h} g = g[n; h]_{[nh,(n+1)h[}, \quad P^h = \text{st.}\sum_{n=0}^{\infty} P^{n;h},$$

$$J^*_h \varepsilon(g) = \bigotimes_{n=0}^{\infty} \widehat{\sqrt{h} g[n; h]} \ \text{ and } \ J_h J^*_h \varepsilon(g) = \bigotimes_{n=0}^{\infty} \widetilde{\varepsilon}\Big(g[n; h] \otimes 1_{[nh,(n+1)h[}\Big),$$

and the following key facts (*cf.* [Be$_1$]); our proof exploits the following version of Euler's formula: for $z, w \in \mathbb{C}$,

$$w(h) \to w \text{ as } h \to 0 \implies (1 + hz)^{w(h)/h} \to e^{zw} \text{ as } h \to 0.$$

Lemma 3.1. *As $h \to 0$, $P^h \to I_{\mathsf{K}}$ and $J_h J^*_h \to I_{\mathcal{F}}$ in the strong operator topology.*

Proof. For contraction operators, strong operator convergence to an isometry follows from weak operator convergence with respect to a total family of vectors. Accordingly, let $c \in \mathsf{k}$, $0 \le r \le t$ and $g', g \in \mathbb{S}$. The first part follows from the inequality $\|(I_{\mathsf{K}} - P^h)c_{[r,t[}\| \leq 2\sqrt{h}\|c\| \quad (h > 0)$. For the second part, set $D := \{T_1 < \cdots < T_N\} = \operatorname{Disc} g' \cup \operatorname{Disc} g$ and $T_0 = 0$, then for $0 < h < \operatorname{mesh} D$,

$$\langle J_h^* \varepsilon(g'), J_h^* \varepsilon(g) \rangle = e_0(h)\big(1 + h a_1(h)\big) \cdots e_{N-1}(h)\big(1 + h a_N(h)\big)$$

where, for $k = 0, \cdots, N-1$ and $j = 1, \cdots, N$,

$$e_k(h) := \big(1 + h\langle g'(T_k), g(T_k)\rangle\big)^{\lfloor T_{k+1}/h \rfloor - \lfloor 1 + T_k/h \rfloor} \quad \text{and}$$
$$a_j(h) := \langle g'[\lfloor T_j/h \rfloor; h], g[\lfloor T_j/h \rfloor; h] \rangle.$$

In view of the uniform bound $|a_j(h)| \leq \|g'\|_\infty \|g\|_\infty$ and the fact that $h\big(\lfloor T_{k+1}/h \rfloor - \lfloor 1 + T_k/h \rfloor\big) \to (T_{k+1} - T_k)$ as $h \to 0$, the result follows from Euler's formula:

$$\langle J_h^* \varepsilon(g'), J_h^* \varepsilon(g) \rangle \to \prod_{k=0}^{N-1} e^{(T_{k+1} - T_k)\langle g'(T_k), g(T_k)\rangle} = e^{\langle g', g \rangle} = \langle \varepsilon(g'), \varepsilon(g) \rangle$$

as $h \to 0$. □

In order to establish a combinatorial identity for tensor powers of a sesquilinear map $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$, we need some positioning notation for tensor components (*cf.* [LW₂]). For $m, n \in \mathbb{Z}_+$ with $m < n$ define

$$\delta, \delta^\perp \in SL(\widehat{\mathsf{k}}; \mathcal{A}),\ \delta^\perp_{[m,n[} \in SL(\Upsilon^\circ_{[m,n[}; \mathcal{A}) \text{ and } \delta^\perp_{[m,\infty[} \in SL(\Upsilon_{[m,\infty[}; \mathcal{A}), \text{ by}$$

$$\delta = |1_\mathcal{A}\rangle q_\Delta, \quad \delta^\perp = |1_\mathcal{A}\rangle q_{\Delta^\perp} \quad \delta^\perp_{[m,n[} := \underline{\bigotimes}_{j=m}^{n-1} \delta^\perp, \quad \text{and} \quad \delta^\perp_{[m,\infty[} := |1_\mathcal{A}\rangle q_{\Delta^\perp_{[m,\infty[}},$$

where the notation (0.1) is invoked and $\Delta^\perp_{[m,\infty[} \in B(\Upsilon_{[m,\infty[})$ denotes the orthogonal projection with range $\mathbb{C}\omega_{[m,\infty[}$. For $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$, $n \in \mathbb{Z}_+$ and $\alpha \subset \{0, \cdots, n-1\}$, define $\nu[\alpha, n] \in SL(\Upsilon^\circ_{[0,n[}; \mathcal{A})$ by

$$\nu[\alpha, n] = \gamma_0 \underline{\otimes} \cdots \underline{\otimes} \gamma_{n-1} \text{ where } \gamma_i = \begin{cases} \nu & \text{if } i \in \alpha \\ \delta^\perp & \text{if } i \notin \alpha \end{cases}$$

with the convention $\nu[\emptyset, 0] := \nu^{\underline{\otimes} 0} \in SL(\mathbb{C}; \mathcal{A})$, $(z, w) \mapsto \overline{z} w 1_\mathcal{A}$.

For $n \in \mathbb{Z}_+$ and a subset α of $\{0, \cdots, n-1\}$ (understood as $\emptyset$ if $n = 0$), we define two associated subsets of $\{0, \cdots, n\}$:

$$\overset{\bullet}{\overrightarrow{\alpha}} := \alpha \cup \{n\} \quad \text{and} \quad \overset{\circ}{\overrightarrow{\alpha}} := \alpha.$$

Lemma 3.2. *For $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$ and $n \in \mathbb{Z}_+$,*

$$(\nu + \delta^\perp)^{\underline{\otimes} n} = \sum_{\alpha \subset \{0, \cdots, n-1\}} \nu[\alpha, n].$$

Proof. Set $\widetilde{\nu} := \nu + \delta^\perp$. We prove the result by induction on n. The identity holds for $n = 0$; suppose therefore that it holds for $n = 1, \cdots, p$. Let $\beta \subset \{0, \cdots, p\}$, then for $\alpha := \beta \cap \{0, \cdots, p-1\}$ there are two mutually exclusive possibilities: either $\beta = \overset{\bullet}{\overrightarrow{\alpha}}$ or $\beta = \overset{\circ}{\overrightarrow{\alpha}}$, moreover

$$\nu[\overset{\bullet}{\overrightarrow{\alpha}}, p+1] = \nu[\alpha, p] \underline{\otimes} \nu \quad \text{and} \quad \nu[\overset{\circ}{\overrightarrow{\alpha}}, p+1] = \nu[\alpha, p] \underline{\otimes} \delta^\perp.$$

Thus

$$\begin{aligned} \sum_{\beta \subset \{0, \cdots, p\}} \nu[\beta, p+1] &= \sum_{\alpha \subset \{0, \cdots, p-1\}} \Big(\nu[\overset{\bullet}{\overrightarrow{\alpha}}, p+1] + \nu[\overset{\circ}{\overrightarrow{\alpha}}, p+1]\Big) \\ &= \sum_{\alpha \subset \{0, \cdots, p-1\}} \nu[\alpha, p] \underline{\otimes} (\nu + \delta^\perp) \;=\; \widetilde{\nu}^{\underline{\otimes} p} \underline{\otimes} \widetilde{\nu} \;=\; \widetilde{\nu}^{\underline{\otimes}(p+1)} \end{aligned}$$

and so the identity holds for $n = p + 1$, as required. □

For $\alpha \subset\subset \mathbb{Z}_+$, set

$$\nu[\alpha] := \nu[\alpha, n] \underline{\otimes} \delta^{\perp}_{[n,\infty[} \in SL(\Upsilon^{\circ}_{[0,n[} \underline{\otimes} \Upsilon_{[n,\infty[}; \mathcal{A}) \tag{3.2}$$

where $n = 1 + \max \alpha$, with the convention $\max \emptyset := -1$.

Now let $h > 0$ and, for $m \in \mathbb{N}$, set

$${}^{\langle h \rangle}\Delta^{(m)}_{[0,t[} := \bigcup_{\substack{\alpha \subset \{0,\cdots,N-1\} \\ |\alpha| = m}} [h\alpha_1, h(1+\alpha_1)[\times \cdots \times [h\alpha_m, h(1+\alpha_m)[, \quad \text{where } N = [t/h],$$

and ${}^{\langle h \rangle}\Delta^{(0)}_{[0,t[} = \Delta^{(0)}_{[0,t[}$. Note that the union is a disjoint one; it approximates the corresponding m-symplex:

$${}^{\langle h \rangle}\Delta^{(m)}_{[0,t[} \subset \Delta^{(m)}_{[0,t[} \text{ with } \big|\Delta^{(m)}_{[0,t[} \setminus {}^{\langle h \rangle}\Delta^{(m)}_{[0,t[}\big| \leq \begin{cases} hmt^{m-1}/(m-1)! & \text{if } m \leq t/h; \\ t^m/m! & \text{if } m > t/h. \end{cases}$$

Definition. Let $\mathcal{U} = \big(\upsilon_m\big)_{m \geq 0} \in SL\mathbb{W}(\mathcal{A}, \mathsf{k})$. The *step-size h, discrete multiple vacuum-adapted quantum Wiener integral* of $\mathcal{U}$ is defined by sesquilinear extension of the prescription

$${}^{\langle h \rangle}\Lambda^{\Omega}_t(\mathcal{U})(\varepsilon(g'), \varepsilon(g)) = \sum_{m=0}^{\infty} \int_{{}^{\langle h \rangle}\Delta^{(m)}_{[0,t[}} \mathrm{d}\mathbf{s}\ \upsilon_m(\widehat{g'}^{\otimes m}(\mathbf{s}), \widehat{g}^{\otimes m}(\mathbf{s})) \quad (g', g \in \mathbb{S}).$$

Recalling the bounding constants (2.1), the following estimates are now evident:

$$\|{}^{\langle h \rangle}\Lambda^{\Omega}_t(\mathcal{U})(\varepsilon(g'), \varepsilon(g))\| \leq \sum_{m=0}^{\infty} C^{\upsilon_m}_m(g', g) t^m/m! \tag{3.3}$$

$$\|\Lambda^{\Omega}_t(\mathcal{U})(\varepsilon(g'), \varepsilon(g)) - {}^{\langle h \rangle}\Lambda^{\Omega}_t(\mathcal{U})(\varepsilon(g'), \varepsilon(g))\| \leq \sum_{1 \leq m \leq [t/h]} \frac{hmt^{m-1}}{(m-1)!} C^{\upsilon_m}_m(g', g) + \sum_{m > [t/h]} \frac{t^m}{m!} C^{\upsilon_m}_m(g', g). \tag{3.4}$$

The standard *h-scaling operator* for discrete approximation is defined as follows:

$$\mathcal{S}_h := h^{-1/2}\Delta^{\perp} + \Delta = \begin{bmatrix} h^{-1/2} & 0 \\ 0 & 1_{\mathsf{k}} \end{bmatrix} \in B(\widehat{\mathsf{k}}) = B(\mathbb{C} \oplus \mathsf{k}).$$

For $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$ define its *h-scaling* by $\Sigma_h(\nu) := \nu \circ (\mathcal{S}_h \times \mathcal{S}_h) \in SL(\widehat{\mathsf{k}}; \mathcal{A})$.

Proposition 3.3. *Let $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$, and let $t > 0$, $\varepsilon \in \mathcal{E}_{P^h\mathbb{S}}$ and $\varepsilon' \in \mathcal{E}$. Then,*

$${}^{\langle h \rangle}\Lambda^{\Omega}_t(\Sigma_h(\nu)^{\otimes})(\varepsilon', \varepsilon) = \sum_{\alpha \subset \{0,\ldots,N-1\}} \nu[\alpha]\big(J^*_h\varepsilon', J^*_h\varepsilon\big) \quad \textit{where } N = \lfloor t/h \rfloor.$$

Proof. Let $g' \in \mathbb{S}$ and $g \in P^h\mathbb{S}$. First note that, for each $m \in \mathbb{Z}_+$,

$$\int_{{}^{\langle h \rangle}\Delta^{(m)}_{[0,t[}} \mathrm{d}\mathbf{s}\ \Sigma_h(\nu)^{\underline{\otimes} m}\big(\widehat{g'}^{\otimes m}(\mathbf{s}), \widehat{g}^{\otimes m}(\mathbf{s})\big) = \sum_{\substack{\alpha \subset \{0,\cdots,N-1\} \\ \#\alpha = m}} \overrightarrow{\prod_{1 \leq j \leq m}} \int_{h\alpha_j}^{h(1+\alpha_j)} \mathrm{d}s_j\ \Sigma_h(\nu)\big(\widehat{g'}(s_j), \widehat{g}(s_j)\big).$$

Now, for $\alpha = \{a_1, \cdots, a_m\} \subset \{0, \cdots, N-1\}$,

$$\int_{ha_j}^{h(1+a_j)} \mathrm{d}s_j\ \Sigma_h(\nu)\big(\widehat{g'}(s_j), \widehat{g}(s_j)\big) = \int_{ha_j}^{h(1+a_j)} \mathrm{d}s_j\ \Sigma_h(\nu)\big(\widehat{g'}(s_j), h^{-1/2}\widehat{g(a_j, h)}\big)$$
$$= \nu\big(\widehat{g(a_j, h)}, \widehat{g(a_j, h)}\big),$$

so

$$\overrightarrow{\prod_{1\leq j\leq m}} \int_{ha_j}^{h(1+a_j)} \mathrm{d}s_j \; \Sigma_h(\nu)\big(\widehat{g'}(s_j), \widehat{g}(s_j)\big) = \nu[\alpha]\big(J_h^*\varepsilon(g'), J_h^*\varepsilon(g')\big).$$

The result follows. □

Recall the definition of ${}^{\Omega}\Xi_n^{(h)}$ defined in (3.1).

Definition. Let $\gamma \in SL(\widehat{\mathsf{k}}; \mathcal{A})$. The *vacuum-embedded quantum random walk with generator γ and step size h* is the process ${}^{\langle h\rangle}\mathfrak{q}^{\Omega,\gamma} \in SL\mathbb{P}^{\Omega}(\mathcal{A}, \mathsf{k})$ defined by

$${}^{\langle h\rangle}\mathfrak{q}_t^{\Omega,\gamma} := {}^{\Omega}\Xi_{\lfloor t/h\rfloor}^{(h)}\big(\gamma^{\otimes\lfloor t/h\rfloor}\big) \qquad (t \geq 0).$$

As an immediate consequence of Lemma 3.2 and Proposition 3.3, we see that vacuum embedded random walks enjoy a discrete multiple Wiener integral decomposition.

Proposition 3.4. *Let $\lambda \in SL(\widehat{\mathsf{k}}; \mathcal{A})$, $\varepsilon' \in \mathcal{E}$ and $\varepsilon \in \mathcal{E}_{P^h\mathbb{S}}$. Then*

$${}^{\langle h\rangle}\mathfrak{q}_t^{\Omega,\lambda+\delta^{\perp}}(\varepsilon', \varepsilon) = {}^{\langle h\rangle}\Lambda_t^{\Omega}\big(\lambda_h^{\otimes}\big)(\varepsilon', \varepsilon) \qquad (t \in \mathbb{R}_+).$$

We can now establish the vacuum-embedded sesquilinear random walk approximation result. The following notations are used:

$$\mathbb{S}^{\mathbb{D}} = \{f \in \mathbb{S} : \operatorname{Disc} f \subset \mathbb{D}\}, \mathcal{E}^{\mathbb{D}} = \operatorname{Lin}\{\varepsilon(g) : g \in \mathbb{S}^{\mathbb{D}}\} \text{ and } \mathbb{D}_{>0} = \mathbb{D} \cap \left]0, \infty\right[$$

where $\mathbb{D}$ denotes the field of dyadic rationals. Thus $\mathbb{S}^{\mathbb{D}}$ and $\mathcal{E}^{\mathbb{D}}$ are dense in K and $\mathcal{F}$ respectively.

Theorem 3.5. *Let $\lambda \in SL(\widehat{\mathsf{k}}; \mathcal{A})$ and let $\big(h_n, \gamma_n\big)$ be a sequence in $\mathbb{D}_{>0} \times SL(\widehat{\mathsf{k}}; \mathcal{A})$ satisfying*

$$\Sigma_{h_n}(\gamma_n - \delta^{\perp}) \to \lambda \text{ pointwise on } \widehat{\mathsf{k}} \times \widehat{\mathsf{k}} \text{ and } h_n \to 0 \text{ as } n \to \infty.$$

Then, for all $\varepsilon' \in \mathcal{E}$, $\varepsilon \in \mathcal{E}^{\mathbb{D}}$ and $T \in \mathbb{R}_+$,

$$\sup_{t\in[0,T]} \big\|{}^{\langle h_n\rangle}\mathfrak{q}_t^{\Omega,\gamma_n}(\varepsilon', \varepsilon) - \mathfrak{q}_t^{\Omega,\lambda}(\varepsilon', \varepsilon)\big\| \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. Set $\varepsilon' = \varepsilon(g')$ and $\varepsilon = \varepsilon(g)$ where $g' \in \mathbb{S}$ and $g \in \mathbb{S}^{\mathbb{D}}$, let $T > 0$ and let $n \in \mathbb{N}$ be sufficiently large that $P^{h_n}g = g$. By Proposition 3.4 and the definition in (2.5), we must show that $\sup_{t\in[0,T]} \alpha(n, t) \to 0$ as $n \to \infty$, where

$$\alpha(n, t) := \big\|{}^{\langle h_n\rangle}\Lambda_t^{\Omega}\big(\lambda_n^{\otimes}\big)(\varepsilon', \varepsilon) - \Lambda_t^{\Omega}(\lambda^{\otimes})(\varepsilon', \varepsilon)\big\| \text{ and } \lambda_n := \Sigma_{h_n}(\gamma_n - \delta^{\perp}).$$

Define constants

$$C_1(n) := \max\big\{\|(\lambda_n - \lambda)(\widehat{c'}, \widehat{c})\| : c' \in \operatorname{Ran} g', c \in \operatorname{Ran} g\big\} \text{ and}$$

$$C_2(n) := \max\big\{\max\{\|\lambda(\widehat{c'}, \widehat{c})\|\|\lambda_n(\widehat{c'}, \widehat{c})\|\} : c' \in \operatorname{Ran} g', c \in \operatorname{Ran} g\big\},$$

and set $\mathcal{U} = \lambda_n^{\otimes} - \lambda^{\otimes}$. Thus $\upsilon_0 = 0$ and the bounding constants for the resulting quantum Wiener integrals satisfy $C_m^{\upsilon_m}(g', g) \leq mC_1(n)C_2(n)^{m-1}$ $(m \in \mathbb{N})$, and

$$C_1(n) \to 0 \text{ and } C_2(n) \to \max\big\{\|\lambda(\widehat{c'}, \widehat{c})\| : c' \in \operatorname{Ran} g', c \in \operatorname{Ran} g\big\} \quad \text{as } n \to \infty.$$

Therefore, using (3.3) and (3.4), and setting $C^{\lambda}(g', g) := C_1^{\lambda}(g', g)$,

$$\sup_{t\in[0,T]} \alpha(n, t) \leq \sup_{t\in[0,T]} \Big(\big\|{}^{\langle h_n\rangle}\Lambda_t^{\Omega}\big(\lambda_n^{\otimes} - \lambda^{\otimes}\big)(\varepsilon', \varepsilon)\big\| + \big\|\big({}^{\langle h_n\rangle}\Lambda_t^{\Omega} - \Lambda_t^{\Omega}\big)(\lambda^{\otimes})(\varepsilon', \varepsilon)\big\|\Big)$$

$$\leq C_1(n) \sum_{m=1}^{\infty} mC_2(n)^{m-1}\frac{T^m}{m!} + h_n \sum_{m=1}^{\infty} mC^{\lambda}(g', g)^m \frac{T^{m-1}}{(m-1)!},$$

which tends to 0 as $n \to \infty$, as required. □

Next we define identity-embeded quantum random walks and obtain the identity-embeded quantum random walk approximation result.

Definition. Let $\gamma \in SL(\widehat{\mathsf{k}}; \mathcal{A})$. The *identity-embedded quantum random walk with generator* γ *and step size* h, is the process ${}^{\langle h \rangle}\mathfrak{q}^{\gamma} \in SL\mathbb{P}(\mathcal{A}, \mathsf{k})$ defined by

$$ {}^{\langle h \rangle}\mathfrak{q}^{\gamma}_t := \Xi^{(h)}_{\lfloor t/h \rfloor}\big(\gamma^{\otimes \lfloor t/h \rfloor}\big) \qquad (t \geq 0). $$

Note that, from the definitions, for $g', g \in \mathbb{S}$, and $N = \lfloor t/h \rfloor$,

$$ {}^{\langle h \rangle}\mathfrak{q}^{\gamma}_t(\varepsilon(g'), \varepsilon(g)) = {}^{\langle h \rangle}\mathfrak{q}^{\Omega,\gamma}_t(\varepsilon(g'), \varepsilon(g)) \, \langle \varepsilon(g'_{[Nh,\infty[}), \varepsilon(g_{[Nh,\infty[}) \rangle. \tag{3.5} $$

Theorem 3.6. *Let* $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$*, and let* $\big(h_n, \gamma_n\big)$ *be a sequence in* $\mathbb{D}_{>0} \times SL(\widehat{\mathsf{k}}; \mathcal{A})$ *satisfying*

$$ \Sigma_{h_n}(\gamma_n - \iota) \to \nu \text{ pointwise on } \widehat{\mathsf{k}} \times \widehat{\mathsf{k}} \quad \text{and} \quad h_n \to 0 \quad \text{as } n \to \infty, $$

where $\iota := |1_{\mathcal{A}}\rangle q_I \in SL(\widehat{\mathsf{k}}; \mathcal{A})$*. Then*

$$ \sup_{t \in [0,T]} \big\| {}^{\langle h_n \rangle}\mathfrak{q}^{\gamma_n}_t(\varepsilon', \varepsilon) - \mathfrak{q}^{\nu}_t(\varepsilon', \varepsilon) \big\| \to 0 \quad \text{as} \quad n \to \infty \qquad (\varepsilon' \in \mathcal{E}, \varepsilon \in \mathcal{E}^{\mathbb{D}}, T \in \mathbb{R}_+). $$

Proof. Let $g' \in \mathbb{S}$, $g \in \mathbb{S}^{\mathbb{D}}$ and $T > 0$. From Theorem 2.3 we have

$$ \mathfrak{q}^{\nu}_t(\varepsilon(g'), \varepsilon(g)) = \mathfrak{q}^{\Omega,\nu+\delta}_t(\varepsilon(g'), \varepsilon(g)) \, \langle \varepsilon(g'_{[t,\infty[}), \varepsilon(g_{[t,\infty[}) \rangle \qquad (t \in \mathbb{R}_+) $$

Therefore, by (3.5), it suffices to show that

$$ \sup_{t \in [0,T]} \big\| {}^{\Omega}\Xi^{(h_n)}_{\lfloor t/h_n \rfloor}(\gamma_n)(\varepsilon(g'), \varepsilon(g)) - \mathfrak{q}^{\Omega,\nu+\delta}_t(\varepsilon(g'), \varepsilon(g)) \big\| = 0. $$

This follows from Theorem 3.5, since

$$ \Sigma_{h_n}(\gamma_n - \delta^{\perp}) = \Sigma_{h_n}(\gamma_n - \iota) + \delta \to \nu + \delta \text{ pointwise as } n \to \infty. $$

□

Remark. Given $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$ and any sequence (h_n) in $\mathbb{R}_{>0}$, letting $\gamma_n \in SL(\widehat{\mathsf{k}}; \mathcal{A})$ be the map defined by

$$ \gamma_n\Big(\begin{pmatrix}\alpha \\ c'\end{pmatrix}, \begin{pmatrix}\beta \\ c\end{pmatrix}\Big) := \Big\langle \begin{pmatrix}\alpha \\ c'\end{pmatrix}, \begin{pmatrix}\beta \\ c\end{pmatrix} \Big\rangle + \nu\Big(\begin{pmatrix}\sqrt{h_n}\alpha \\ c'\end{pmatrix}, \begin{pmatrix}\sqrt{h_n}\beta \\ c\end{pmatrix}\Big), $$

we have $\Sigma_{h_n}(\gamma_n - \iota) = \nu$ for each $n \in \mathbb{N}$; we may call the sequence (γ_n) resulting from the choice $h_n = 2^{-n}$ $(n \in \mathbb{N})$, the *dyadic random walk approximation.*

4. Random walk approximation for mapping cocycles

In this section we obtain random walk approximation results for mapping cocycles on a concrete operator space V from discrete approximation results obtained in the previous section.

Recall the matrix space tensor product and extended composition of maps defined in Section 1. Let $B(\mathfrak{h}; \mathfrak{h}')$ be the ambient full operator space of V. For a map $\psi \in L\big(\widehat{\mathsf{D}}; CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\widehat{\mathsf{k}}\rangle)\big)$, we use the following subscript notation $\psi_{\zeta} := \psi(\zeta), \zeta \in \widehat{\mathsf{D}}$. Define $\Sigma_h(\psi) \in L\big(\widehat{\mathsf{D}}; CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\widehat{\mathsf{k}}\rangle)\big)$ by $\Sigma_h(\psi)_{\zeta} := \big(I_{\mathfrak{h}'} \otimes \mathcal{S}_h\big)\psi_{\mathcal{S}_h \zeta}(\cdot)$ and, in the extended composition notation (1.1) define the *identity-embedded quantum random walk with generator* ψ *and step size* h to be the cb column-bounded process given by

$$ {}^{\langle h \rangle}k^{\psi}_{t,\varepsilon(g_{[0,t[})}(x) := (I_{\mathfrak{h}'} \otimes J^{(h)}_N)\big(\psi_{\widehat{g(0;h)}} \bullet \cdots \bullet \psi_{g(\widehat{(N-1)};h)}\big)(x) \otimes |\Omega_{[hN,\infty[}\rangle $$

$(x \in \mathsf{V}, g \in \mathbb{S})$ where $N = \lfloor t/h \rfloor$.

In case $\psi \in CB\big(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} B(\widehat{\mathsf{k}})\big)$, the random walk is given by

$$ {}^{\langle h \rangle}k^{\psi}_t(x) := \big(I_{\mathfrak{h}'} \otimes J^{(h)}_N\big)\psi^{\bullet N}(x)\big(I_{\mathfrak{h}} \otimes J^{(h)}_N\big)^* \otimes I_{[Nh,\infty[}, \qquad \text{where } N = \lfloor t/h \rfloor. $$

Remark. The random walk ${}^{\langle h\rangle}k^{\psi}$ is (completely) contractive if the map ψ is; similarly, it inherits (complete) positivity/*-homomorphic properties from ψ when V is a C^*-algebra; when ψ is unital, ${}^{\langle h\rangle}k^{\psi}$ is 'almost unital':

$$ {}^{\langle h\rangle}k^{\psi}_t(1) = I_{\mathfrak{h}} \otimes J^{(h)}_N J^{(h)*}_N \otimes I_{[Nh,\infty[} \text{ where } N = \lfloor t/h \rfloor. $$

For $\phi \in L\big(\widehat{\mathsf{D}}; CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\widehat{\mathsf{k}}\rangle)\big)$, the mapping cocycle which satisfies the QSDE

$$ \mathrm{d}k_t = k_t \, \mathrm{d}\Lambda_\phi(t), \quad k_0 = \iota^{\mathsf{V}}_{\mathcal{F}} $$

where $\iota^{\mathsf{V}}_{\mathcal{F}}$ is the ampliation (see [LW$_2$], or [L]), is denoted k^ϕ.

Given an element ϕ of $L\big(\widehat{\mathsf{D}}; CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\widehat{\mathsf{k}}\rangle)\big)$ define the associated sesquilinear map $\nu_\phi \in SL(\widehat{\mathsf{D}}; CB(\mathsf{V}))$ by $\nu_\phi(\eta,\zeta) = E^\eta \phi_\zeta(.) \quad (\eta,\zeta \in \widehat{\mathsf{D}})$. Then the associated sesquilinear process of k^ϕ is $\mathfrak{q}^{\nu_\phi}$ (see Proposition 5.3, [DLT]), that is $\mathfrak{q}^{\nu_\phi}(\varepsilon',\varepsilon) = E^{\varepsilon'} k^{\phi}_{t,\varepsilon}(.)$ for all $t \in \mathbb{R}_+$ and $\varepsilon, \varepsilon' \in \mathcal{E}_{\mathsf{D}}$.

Theorem 4.1. *Let $\phi, \psi_1, \psi_2, \dots \in L\big(\widehat{\mathsf{D}}; CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\widehat{\mathsf{k}}\rangle)\big)$ and $h_1, h_2, \dots \in \mathbb{D}_{>0}$ be such that*

$$ \forall_{c' \in \mathsf{k}, c \in \mathsf{D}} \quad \big\| E^{\widehat{c'}} (\Sigma_{h_n}(\psi_n - \iota) - \phi)_{\widehat{c}}(\cdot) \big\|_{\mathrm{cb}} \to 0 \quad \text{and} \quad h_n \to 0 \quad \text{as } n \to \infty. $$

Then the following hold, for all $T > 0$.

(a) *For all $\varepsilon' \in \mathcal{E}$ and $\varepsilon \in \mathcal{E}^{\mathbb{D}}_{\mathsf{D}}$,*

$$ \sup_{t \in [0,T]} \big\| E^{\varepsilon'} \big({}^{\langle h_n\rangle}k^{\psi_n}_{t,\varepsilon} - k^{\phi}_{t,\varepsilon} \big)(\cdot) \big\|_{\mathrm{cb}} \to 0 \quad \text{as } n \to \infty. $$

(b) *Suppose that the cocycle k^ϕ, as well as each map ψ_n, is completely contractive. Then, for all $\omega \in B(\mathcal{F})_*$,*

$$ \sup_{t \in [0,T]} \big\| \big(\mathrm{id}_{\mathsf{V}} \otimes_{\mathsf{M}} \omega \big) \circ \big({}^{\langle h_n\rangle}k^{\psi_n}_t - k^{\phi}_t \big) \big\|_{\mathrm{cb}} \to 0 \quad \text{as } n \to \infty. $$

(c) *Suppose that V is a C^*-algebra, the cocycle k^ϕ is *-homomorphic and each map ψ_n is completely positive and contractive. Then, for all $a \in \mathsf{V}$ and $\xi \in \mathfrak{h} \otimes \mathcal{F}$,*

$$ \sup_{t \in [0,T]} \big\| {}^{\langle h_n\rangle}k^{\psi_n}_t(a)\xi - k^{\phi}_t(a)\xi \big\| \to 0 \quad \text{as } n \to \infty. $$

Proof. (a) This is precisely the conclusion obtained from applying Theorem 3.6 to the associated sesquilinear maps and process.

(b) This follows from (a), by the remark preceding the theorem and uniform boundedness.

(c) In this case, by the operator Schwarz inequality, if $a \in \mathsf{V}$, $\xi \in \mathfrak{h} \otimes \mathcal{F}$ and $n \in \mathbb{N}$, then

$$ \|(\kappa_n(a) - \kappa(a))\xi\|^2 \le \omega_\xi \circ (\kappa_n - \kappa)(a^*a) + 2\,\mathrm{Re}\,\omega_{\kappa(a)\xi,\xi} \circ (\kappa - \kappa_n)(a) $$

for $\kappa_n := {}^{\langle h_n\rangle}k^{\psi_n}_t$ and $\kappa = k^\phi_t$, so (c) follows from (b) and uniform boundedness. $\square$

Remarks. Quantum random walk approximation for mapping cocyles is obtained in [Be$_2$]; in [DL$_2$] strengthened forms of these results are obtained, in an abstract operator space setting. For comparison, we summarise these now. Without the dyadic restriction on (h_n), Belton obtains slightly stronger conclusions under slightly stronger hypotheses. He assumes that $\phi, \psi_1, \psi_2, \dots \in CB\big(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} B(\widehat{\mathsf{k}})\big)$ and proves that if

$$ \big\| \big(\Sigma_{h_n}(\psi_n - \iota) - \phi \big) \otimes_{\mathsf{M}} \mathrm{id}_{B(\widehat{\mathsf{k}})}(X) \big\| \to 0 \text{ and } h_n \to 0 \text{ as } n \to \infty, \tag{4.1} $$

for all $X \in \mathsf{V} \otimes_{\mathsf{M}} B(\widehat{\mathsf{k}})$, then

$$\sup_{t\in[0,T]} \big\| \big({}^{\langle h_n\rangle}k^{\psi_n}_{t,\varepsilon} - k^{\phi}_{t,\varepsilon}\big)(x)\big\| \to 0 \text{ as } n\to\infty \qquad (x\in\mathsf{V}, \varepsilon\in\mathcal{E}, T>0)$$

whereas, if

$$\big\|\Sigma_{h_n}(\psi_n - \iota) - \phi\big\|_{\mathrm{cb}} \to 0 \text{ and } h_n\to 0 \text{ as } n\to\infty \tag{4.2}$$

then

$$\sup_{t\in[0,T]} \big\| {}^{\langle h_n\rangle}k^{\psi_n}_{t,\varepsilon} - k^{\phi}_{t,\varepsilon}\big\|_{\mathrm{cb}} \to 0 \text{ as } n\to\infty \qquad (\varepsilon\in\mathcal{E}, T>0),$$

and if *either* V is a C^*-algebra and each map is ψ_n is *-homomorphic with range in the spatial tensor product $\mathsf{V}\otimes B(\widehat{\mathsf{k}})$, *or* V is a von Neumann algebra and each map ψ_n is normal and *-homomorphic, then k^ϕ is *-homomorphic, moreover he constructs such a sequence $(\psi_n)_{n\geq 0}$ for maps ϕ enjoying the standard form of stochastic generator (see [L]) for generators of Markov-regular quantum stochastic flows (i.e. unital $*$-homomorphic quantum stochastic cocycles). As noted by Belton, if $\dim\mathsf{k}=\infty$ then the hypotheses (4.1) and (4.2) are actually equivalent, by Lemma 1.1.

5. Random walk approximation for isometric operator cocycles

In this section we consider isometric operator cocycles which are Markov-regular, equivalently have bounded stochastic generator, and the class of direct sums of Markov-regular isometric cocycles. Direct sums of unitary cocycles arise naturally in the theory of Lévy processes on compact quantum groups, where they implement quantum Lévy processes on the underlying Hopf*-algebra ([DaL]).

Fix a Hilbert space $\mathfrak{h}$. For $G\in B(\mathfrak{h}\otimes\widehat{\mathsf{k}})$, its h-scaling is the operator $\Sigma_h(G)\in B(\mathfrak{h}\otimes\widehat{\mathsf{k}})$ defined by $\Sigma_h(G) := (I_{\mathfrak{h}}\otimes\mathcal{S}_h)G(I_{\mathfrak{h}}\otimes\mathcal{S}_h)$. Given $G\in B(\mathfrak{h}\otimes\widehat{\mathsf{k}})$, the random walk generated by G is constructed in terms of the extended compositions of G's, defined recursively as follows:

$$G^{\bullet m+1} := (G\otimes I_{\widehat{\mathsf{k}}^{\otimes m}})(\Sigma^*\otimes I_{\widehat{\mathsf{k}}^{\otimes m}})(I_{\widehat{\mathsf{k}}}\otimes G^{\bullet m})(\Sigma\otimes I_{\widehat{\mathsf{k}}^{\otimes m}})$$

with $G^{\bullet 0} := I_{\mathfrak{h}}$, where Σ here denotes the tensor flip $\mathfrak{h}\otimes\widehat{\mathsf{k}}\to\widehat{\mathsf{k}}\otimes\mathfrak{h}$.

Definition. Let $G\in B(\mathfrak{h}\otimes\widehat{\mathsf{k}})$ and $h>0$. The *identity-embedded quantum random walk with generator G and step size h*, is the operator process ${}^{\langle h\rangle}X^G$ defined by

$${}^{\langle h\rangle}X^G_t := (I_{\mathfrak{h}}\otimes J^{(h)}_N)F^{\bullet N}(I_{\mathfrak{h}}\otimes J^{(h)}_N)^*\otimes I_{\mathcal{F}_{[Nh,\infty[}} \quad (t\geq 0, N=\lfloor t/h\rfloor).$$

The quantum stochastic operator cocycle generated by $F\in B(\mathfrak{h}\otimes\widehat{\mathsf{k}})$, denoted X^F, is the unique strong solution of the operator quantum stochastic differential equation

$$\mathrm{d}X_t = X_t\Lambda_F(t), \quad X_0 = I_{\mathfrak{h}\otimes\mathcal{F}}$$

which is 'strongly regular, that is satisfies $X_{t,\varepsilon} := X_tE_\varepsilon\in B(\mathfrak{h};\mathfrak{h}\otimes\mathcal{F})$ and $(X_tE_\varepsilon)_{t\geqslant 0}$ is locally uniformly bounded (see [L]). For any such F there is its associated sesquilinear map

$$\nu_F\in SL(\widehat{\mathsf{k}};B(\mathfrak{h})), \quad (\zeta,\eta)\mapsto E^\zeta FE_\eta,$$

so that $E^{\varepsilon'}X^FE_\varepsilon = \mathfrak{q}^{\nu_F}(\varepsilon',\varepsilon)$ for all $\varepsilon,\varepsilon'\in\mathcal{E}$.

Theorem 5.1. *Let $F\in B(\mathfrak{h}\otimes\widehat{\mathsf{k}})$ and let (h_m,G_m) be a sequence in $\mathbb{D}_{>0}\times B(\mathfrak{h}\otimes\widehat{\mathsf{k}})$ satisfying*

$$E^\zeta\Sigma_{h_m}(G_m - I_{\mathfrak{h}\otimes\widehat{\mathsf{k}}})E_\eta \to E^\zeta FE_\eta \text{ and } h_m\to 0 \text{ as } m\to\infty \quad (\zeta,\eta\in\widehat{\mathsf{k}}).$$

Then for all $T\geq 0, \varepsilon'\in\mathcal{E}$ and $\varepsilon\in\mathcal{E}^{\mathbb{D}}$,

$$\sup_{t\in[0,T]} \|E^{\varepsilon'}({}^{\langle h_m\rangle}X^{G_m}_{t,\varepsilon} - X^F_{t,\varepsilon})\| \to 0 \text{ as } m\to\infty.$$

Proof. Let $\nu, \gamma_n \in SL(\widehat{\mathsf{k}}; B(\mathfrak{h}))$ be the sesquilinear maps associated to F and G_n, as above. Then the hypothesis of Theorem 3.6 is satisfied for ν and γ_n. The result therefore follows from the fact that $E^{\varepsilon'\,\langle h_n\rangle}X^{G_m}_{t,\varepsilon} = {}^{\langle h_n\rangle}\mathfrak{q}^{\gamma_m}_t(\varepsilon', \varepsilon)$ for all $m, n \in \mathbb{N}$, $\varepsilon', \varepsilon \in \mathcal{E}$, and $t \in \mathbb{R}_+$.

□

Remark. In [AtP], Theorem 13, a similar approximation result is obtained under the following hypothesis on the convergence of scaled generators:

$$\sup_{\|u\|\leqslant 1} \Big\| \big(\Sigma_{h_m}(G_m - I_{\mathfrak{h}\otimes\widehat{\mathsf{k}}}) - F\big)E_u \Big\|_{HS(\widehat{\mathsf{k}};\mathfrak{h}\otimes\widehat{\mathsf{k}})} \to 0 \text{ and } \ h_m \to 0 \ \text{ as } \ m \to \infty.$$

This assumption is considerably stronger than ours when k is infinite-dimensional.

Let X^F be the Markov-regular quantum stochastic isometric cocycle with stochastic generator $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ having block matrix form

$$F = \begin{bmatrix} iH - \frac{1}{2}L^*L & -L^*W \\ L & W - I \end{bmatrix} \tag{5.1}$$

where $H \in B(\mathfrak{h})$ is bounded selfadjoint, $L \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ and $W \in B(\mathfrak{h} \otimes \mathsf{k})$ is isometric. Now define isometries associated to F as follows (cf. [Sah] Theorem 4.2, [Be$_2$] Example 6.2):

$$G^h := e^{h^{1/2}R_L} \left[\begin{smallmatrix} e^{ihH} & 0 \\ 0 & W \end{smallmatrix}\right] \tag{5.2}$$

where $R_L := \left[\begin{smallmatrix} 0 & -L^* \\ L & 0 \end{smallmatrix}\right]$. Then a straightforward calculation confirms that

$$\Sigma_h(G^h - I_{\mathfrak{h}\otimes\widehat{\mathsf{k}}}) = F + O(\sqrt{h}) \ \text{ as } \ h \to 0. \tag{5.3}$$

Theorem 5.2. *Let X be a Markov-regular isometric quantum stochastic operator cocycle. with stochastic generator $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ and let (h_n) be a sequence in $\mathbb{D}_{>0}$ converging to* 0. *Then there is a sequence of isometries (G_n) in $B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ such that for all $t \in \mathbb{R}_+$ and $\xi \in \mathfrak{h} \otimes \mathcal{F}$,*

$${}^{\langle h_n\rangle}X^{G_n}_t \xi \to X_t\xi \ \textit{as } n \to \infty.$$

Proof. By the isometry of X, it suffices to find contractions (G_n) in $B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ such that

$$\langle \xi', ({}^{\langle h_n\rangle}X^{G_n}_t - X_t)\xi\rangle \to 0 \text{ as } n \to \infty \tag{5.4}$$

for all $\xi, \xi \in \mathfrak{h}\underline{\otimes}\mathcal{E}$. Isometry for X implies that its stochastic generator F has the form (5.1). Set $G_n := G^{h_n}$ as in (5.2). Then, by (5.3), (h_n, G_n) and F satisfy the hypothesis of Theorem 5.1 and so (5.4) holds, as required.

□

When $\mathfrak{h}$ decomposes as an orthogonal sum $\bigoplus_{n\geqslant 1} \mathfrak{h}_n$, $\bigoplus_{n\geq 0} \mathbb{QS}_{\mathrm{i}}\mathbb{C}_{\mathrm{Mr}}(\mathfrak{h}_n, \mathsf{k})$ denotes the class of quantum stochastic isometric cocycles U on $\mathfrak{h}$ of the form $\bigoplus_{n\geq 1} U^n$, where U^n is a Markov-regular isometric cocycle on $\mathfrak{h}_n \otimes \mathcal{F}$, for each n. We finish by showing that this class is approximable by isometric quantum random walks. To this end, we use the notation

$$\mathfrak{h}_{\leqslant N} := \{(u_n) \in \mathfrak{h} : u_n = 0 \text{ for } n > N\}.$$

Corollary 5.3. *Let $\mathfrak{h} = \bigoplus_{n\geq 0} \mathfrak{h}_n$, let*

$$U = \bigoplus_{n\geq 0} U^n \in \bigoplus_{n\geq 0} \mathbb{QS}_{\mathrm{i}}\mathbb{C}_{\mathrm{Mr}}(\mathfrak{h}_n, \mathsf{k}),$$

and set

$$G^{(h)}_{\leqslant N} := \bigoplus_{n=1}^{N} G^{(h)}_n \oplus \bigoplus_{n>N} I_{\mathfrak{h}_n\otimes\widehat{\mathsf{k}}} \qquad (h > 0, N \in \mathbb{N}),$$

where $G_n^{(h)}$ is the isometry defined as in (5.2)*, in terms of the stochastic generator of U^n. Then, for any sequence (h_N) in $\mathbb{D}_{>0}$ converging to* 0*,*

$$ {}^{\langle h_N\rangle}X_t^{G_{\leqslant N}^{(h_N)}}\xi \to U_t\xi \text{ as } N\to\infty \qquad (\xi\in\mathfrak{h}\otimes\mathcal{F}, t\in\mathbb{R}_+). $$

Proof. Let $\xi \in \mathfrak{h}\otimes\mathcal{F}$ and $\epsilon > 0$. Choose $p\in\mathbb{N}$ and $\xi'\in\mathfrak{h}_{\leqslant p}\otimes\mathcal{F}$ such that $\|\xi-\xi'\|<\epsilon$ and set

$$ U_t^{\leqslant p} := \bigoplus_{n=1}^{p} U_t^n \oplus \bigoplus_{n>p} I_{\mathfrak{h}_n\otimes\mathcal{F}} \in B(\mathfrak{h}\otimes\mathcal{F}) \qquad (t\in\mathbb{R}_+), $$

noting that $(U_t^{\leqslant p})_{t\geqslant 0}\in \mathbb{QS}_{\mathrm{i}}\mathbb{C}_{\mathrm{Mr}}(\mathfrak{h},\mathsf{k})$. Then

$$ U_t\xi' = U_t^{\leqslant p}\xi' \ (t\in\mathbb{R}_+), $$
$$ {}^{\langle h\rangle}X_t^{G_{\leqslant N}^{(h)}}\xi' = {}^{\langle h\rangle}X_t^{G_{\leqslant p}^{(h)}}\xi' \ (N\geqslant p, h>0, t\in\mathbb{R}_+), $$

and, by Theorem 5.2,

$$ {}^{\langle h_N\rangle}X_t^{G_{\leqslant p}^{(h_N)}}\xi' \to U_t^{\leqslant p}\xi' \ \text{ as } \ N\to\infty. $$

The result therefore follows by the uniform boundedness of all the operators concerned and the arbitrariness of ϵ. □

Remark. It is not hard to see that the strong-operator convergence in these two results is again locally uniform in t.

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