Liouville property on G-spaces

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Abstract

Let G be a locally compact group and E be a G-space. An irreducible probability measure μ on G is said to have Liouville property on E if G-invariant functions on E are the only continuous bounded functions on E that satisfy the mean value property with respect to μ. We first prove that the random walk induced by μ on E is transient outside a closed set and on the closed set μ has Liouville. We mainly consider actions on vector spaces and projective spaces. We show that measures on $GL(V)$ that are supported inside a ball of radius less than $a < 1$ have Liouville property on V. We also prove that measures on $GL(\mathbb{R}^2)$ have Liouville property on the projective line. We next exhibit subgroups of $GL(V)$ so that irreducible measures on such subgroups have Liouville on the projective space $\mathbb{P}(V)$ of V. We also prove irreducible measures on $SL(V)$ have Liouville property on $\mathbb{P}(\mathcal{SL}(V))$ where $\mathcal{SL}(V)$ is the Lie algebra of $SL(V)$.

1 Introduction

Let G be a locally compact group and μ be a regular Borel probability measure on G. We consider a locally compact space E on which the group G acts by homeomorphisms. A bounded measurable function on E that satisfies the mean value property with respect to μ is called a μ-harmonic function and $H_\mu(E)$ is the space of all bounded μ-harmonic functions on E. In case of left or right action of G on itself, the space of harmonic functions was introduced by Fursternberg [5] and studied by others. An earlier work of Blackwell, Choquet and Deny on abelian groups showed that constant functions are the only continuous bounded μ-harmonic functions on abelian groups G for the left or right action of G on itself - such a result is known as Choquet-Deny theorem or Liouville property (cf. [10] for recent developments in Choquet-Deny

results on groups). Here, we consider Liouville property for group actions. In this situation we say that μ has Liouville property on a G-space E if G_μ-invariant functions are the only continuous functions in $H_\mu(E)$ where G_μ is the closed subgroup generated by the support of μ.

By considering the adjoint $\check{\mu}$ of μ, it may be seen that $\check{\mu}$ has Liouville on G implies μ has Liouville on any G-space E: recall $\check{\mu}$ is defined by $\int f(g)d\check{\mu}(g) = \int f(g^{-1})d\mu(g)$. However we provide examples in Section 7 to show that there are measures on $GL(V)$ that have Liouville property on V but neither the measure nor its adjoint has Liouville property on $GL(V)$.

It may be easily observed that there is a S_μ-invariant closed (possibly empty) subset L_μ of E such that any continuous function in $H_\mu(E)$ is S_μ-invariant when restricted to L_μ: S_μ is the closed semigroup generated by the support of μ.

Recently, R. Feres and E. Ronshausen [4] studied Liouville property for group actions and it was shown that if Γ is a countable group acting on the circle S^1 or $[0, 1]$, then any irreducible (that is, $S_\mu = G$) symmetric measure on Γ has Liouville property on S^1 or $[0, 1]$.

It is also shown in [4] that the random walk generated by irreducible μ is transient on $E \setminus L_\mu$ for actions of countable groups on compact spaces using boundaries of countable groups, we extend this result to all type of actions using results about harmonic functions on groups (see Proposition 2.1).

We mainly consider actions on vector spaces and on the corresponding projective spaces. For a finite-dimensional vector space V over $\mathbb{R}$, and a linear transformation α of V, $||\alpha||$ denotes the operator norm of α. We first observe the following result which serves many interesting examples.

Theorem 1.1 *Let V be a finite-dimensional vector space over $\mathbb{R}$ and G be a subgroup of $GL(V)$ such that $\{g \in G \mid ||g|| < 1\}$ is a non-empty open set. Suppose μ is an adapted probability measure on G such that $\mu(\{g \in G \mid ||g|| \leq a\}) = 1$ for some $0 < a < 1$. Then μ has Liouville property on V.*

For a G-space E, we say that G has Liouville property on E if any irreducible probability measure on G has Liouville on G.

We next consider projective linear actions on projective spaces. For a vector space V, $\mathbb{P}(V)$ denotes the corresponding projective space. We first prove Liouville property for actions on the projective line $\mathbb{P}^1 = \mathbb{P}(\mathbb{R}^2)$: it may be recalled that S^1 and the projective line $\mathbb{P}^1 = \mathbb{P}(\mathbb{R}^2)$ are homeomorphic and [4] proved Liouville property for (symmetric measures on) countable group

actions on S^1.

Theorem 1.2 *Let G be a closed subgroup of $GL(\mathbb{R}^2)$. Then G has Liouville property on $\mathbb{P}^1$.*

We next look at locally compact subgroups of $GL(V)$.

Theorem 1.3 *Let G be locally compact subgroup of $GL(V)$. Suppose G has an unipotent subgroup U such that U has only one linearly independent invariant vector in V. Then G has Liouville property on $\mathbb{P}(V)$.*

Example 1.1 The following Lie subgroups of $GL(V)$ have unipotent subgroups that have only one linearly independent invariant vector in V.

1. G is the group of invertible upper triangular matrices and U may be taken as the group of upper triangular matrices with one on the diagonal.

2. the $(2n + 1)$-dimensional Heisenberg group and any of its extensions. For instance, $SL_2(\mathbb{R}) \ltimes H_1$ with

$$H_1 = \{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, x, b \in \mathbb{R}\}$$

   as the unipotent group.

As an illustration and application of the method involved in the proof of Theorem 1.3, we consider the conjugate action of $SL(V)$ on its Lie algebra $\mathcal{SL}(V)$. It may be noted that the Lie algebra $\mathcal{SL}(V)$ of $SL(V)$ is the space of all trace zero matrices in $GL(V)$ and the adjoint action is given by $\mathrm{Ad}\,(g)(x) = gxg^{-1}$ for all $g \in SL(V)$ and $x \in \mathcal{SL}(V)$.

Theorem 1.4 *Let V be a finite-dimensional vector space over $\mathbb{C}$. For the adjoint action of $SL(V)$ on the Lie algebra $\mathcal{SL}(V)$, $SL(V)$ has Liouville property on $\mathbb{P}(\mathcal{SL}(V))$.*

As a by-product we obtain that actions such as distal, proximal and minimal have Liouville property (see Corollary 2.1). We make a few miscellaneous remarks on amenability of the group G and Liouville property of G-spaces and regarding equicontinuous Markov operator associated to the random walk on E generated by μ.

2 Preliminaries

Let G be a locally compact group and μ be a (regular Borel) probability measure on G.

Let G_μ (resp. S_μ) denote the closed subgroup (resp. semigroup) generated by the support of μ. The measure μ is called adapted (resp. irreducible) if $G_\mu = G$ (resp. $S_\mu = G$).

A G-space E is a locally compact space with a continuous G-action.

A bounded function f on E is called μ-harmonic if $Pf(x) = \int f(gx)d\mu(g) = f(x)$ for all $x \in E$.

Let $H_\mu(E)$ denote the space of bounded μ-harmonic functions on E and $C_b(E)$ denote the space of bounded continuous functions on E.

We say that the measure μ has Liouville property (or Choquet-Deny) on E if any continuous bounded μ-harmonic function on E is G_μ-invariant. We say that (the action of) G has Liouville property on E if every irreducible probability measure on G has Liouville property on E.

For right-translation action of G on itself, we simply say that μ has Choquet-Deny if any continuous bounded right μ-harmonic function is constant on the cosets of G_μ. It is easy to verify that if μ has Choquet-Deny, then $\check{\mu}$ has Liouville property on any G-space: $\check{\mu}$ is given by $\check{\mu}(E) = \mu(E^{-1})$ where $E^{-1} = \{x^{-1} \mid x \in E\}$.

2.1 Recurrence and Transience

Let (X_n) (resp. (Z_n)) be the canonical left (resp. right) random walk on G defined by μ. Then for $x \in E$, $(X_n x)$ defines a random walk on E starting at x.

For a given $x \in E$, we consider the following recurrence property R_x and transition property T_x introduced in [7] for $(X_n x)$:

R_x: There exists a compact set $K \subset E$ such that a.e. $X_n x \in K$ infinitely often.

T_x: For any compact set $K \subset E$, a.e. ω, there exists $n(w) \in \mathbb{N}$ with $X_n(\omega)x \notin K$ for $n \geq n(\omega)$.

Several situations for the validity of R_x or T_x are given in [7].

For the action of G on E and a probability measure μ on G, we have a closed S_μ-invariant subset L of E such that any continuous $f \in H_\mu(E)$, f is S_μ-invariant on L. It is shown in [4] that the random walk is transient

on $E \setminus L$ using Poisson boundary, here we first extend this result to general actions using a well-known result about harmonic functions.

Proposition 2.1 *There is a S_μ-invariant closed set L such that f is S_μ-invariant on L for any continuous $f \in H_\mu(E)$ and the limit points of almost all paths $(X_n x)$ are in L for any $x \in E$. The action of G on $E \setminus L$ satisfies T_x for each $x \in E \setminus L$. In other words, if R_x is valid for all $x \in E$, then any bounded continuous μ-harmonic function on E is G_μ-invariant.*

The following lemma easily follows from a similar well-known result about μ-harmonic functions on groups, we include the proof for clarity and completeness.

Lemma 2.1 *For any any $f \in H_\mu(E)$ and $x \in E$, we have for μ-almost every $g \in G$, $\lim f(gX_n x) - f(X_n x) = 0$ a.e. ω.*

Proof Let $F(g) = f(g^{-1}x)$. Then F is a (right) $\check{\mu}$-harmonic function on G. Let (X_n) be the left random walk induced by μ. Then $(\tilde{X}_n = X_n^{-1})$ is the right random walk induced by $\check{\mu}$. It could be easily verified that $F(\tilde{X}_n)$ is a martingale, that is $E(F(\tilde{X}_{n+1})|\mathcal{A}_n) = F(\tilde{X}_n)$ where $\mathcal{A}_n$ is the σ-algebra generated by $F(\tilde{X}_1), \cdots, F(\tilde{X}_n)$. Since $F(\tilde{X}_n)$ is bounded,

$$E((F(\tilde{X}_{n+1}) - F(\tilde{X}_n))^2)$$
$$= E(E((F(\tilde{X}_{n+1}) - F(\tilde{X}_n))^2|\mathcal{A}_n))$$
$$= E(E(F(\tilde{X}_{n+1})^2|\mathcal{A}_n)) - 2E(F(\tilde{X}_n)E(F(\tilde{X}_{n+1})|\mathcal{A}_n)) + E(F(\tilde{X}_n)^2)$$
$$= E(F(\tilde{X}_{n+1})^2) - E(F(\tilde{X}_n)^2)$$

for all n. This implies that $\sum_{k=0}^{n} E((F(\tilde{X}_{k+1}) - F(\tilde{X}_k))^2) = E(F(\tilde{X}_{n+1}^2)) - E(F(\tilde{X}_0^2))$. Since $E(F(\tilde{X}_n)^2)$ is a uniformly bounded, by martingale convergence theorem $F(\tilde{X}_n)$ converges almost surely and in L^2, hence $E(F(\tilde{X}_{n+1}^2))$ converges. This implies that $\sum E((F(\tilde{X}_{n+1}) - F(\tilde{X}_n))^2) < \infty$ and hence $\sum E((F(\tilde{X}_n g^{-1}) - F(\tilde{X}_n))^2) < \infty$ for μ-almost every g. This implies that $\sum (F(\tilde{X}_n g^{-1}) - F(\tilde{X}_n))^2 < \infty$ a.e. ω. Thus, $F(\tilde{X}_n g^{-1}) - F(\tilde{X}_n) \to 0$ a.e. ω.

Proof of Proposition 2.1 Let L be the largest subset of E consisting of all $x \in E$ such that $f(x) = f(gx)$ for all $g \in S_\mu$ and all continuous $f \in H_\mu(E)$. Then L is a closed S_μ-invariant subset of E.

If X_n is the left random walk on G defined by μ and $f \in H_\mu(E)$, then by Lemma 2.1, for a fixed $x \in E$,

$$\lim f(gX_n x) - f(X_n x) = 0 \tag{1}$$

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almost all ω and μ-a.e. g. If y is a limit point of $(X_n x)$ for a path $(X_n x)$ which satisfies (1), then $f(gy) = f(y)$ for all $g \in S_\mu$ and all continuous $f \in H_\mu(E)$. This implies that the limit points of almost all paths are in L.

Let $x \in E \setminus L$ and K be any compact subset of $E \setminus L$. If $X_n(\omega)x \in K$ infinitely often, then $X_n(\omega)x$ has a limit point $y \in K$. By the first part such ω form a null set. Hence there is an almost finite random integer N_x such that $X_n(\omega)x \notin K$ for all $n \geq N(\omega)$.

2.2   A sufficient condition

We now provide a sufficient condition useful to prove the main results.

Following result is a well-known useful basic result in harmonic functions, as it is a standard result which is often needed we state it without proof.

Lemma 2.2 *Let f be a continuous bounded μ-harmonic function on E. Then the sets $\{x \in E \mid f(x) = \inf_{a \in E} f(a)\}$ and $\{x \in E \mid f(x) = \sup_{a \in E} f(a)\}$ are S_μ-invariant closed sets (possibly empty).*

Liouville property for μ rests in showing that the two S_μ-invariant sets in the above Lemma 2.2 intersect (instead of E, one considers $\overline{G_\mu x}$). This motivates us to provide the following sufficient condition for Liouville property-it is a transparent version of Corollary 2.2 of [4].

Proposition 2.2 *If f is a continuous μ-harmonic function on E such that f has a minimum and maximum on E and S_μ-invariant sets overlap, then f is constant. In particular, if for any $x \in E$, $\overline{Gx}$ is compact and S_μ-invariant subsets of $\overline{Gx}$ overlap, then G has Liouville property on E.*

Remark 2.1 The condition that S_μ-invariant subsets of $\overline{Gx}$ overlap is not a necessary condition. Example 6.1 shows that there are actions having Liouville property but have orbits violating this condition. In section 5, we provide a class of measures/actions for which the above sufficient condition is necessary.

Proof   Let f be a continuous μ-harmonic function having maximum and minimum on E. Let $E_s = \{x \in E \mid f(x) = \sup_{y \in E} f(y)\}$ and $E_i = \{x \in E \mid f(x) = \inf_{y \in E} f(y)\}$. Then E_s and E_i are nonempty closed subsets in E. By Lemma 2.2, both E_s and E_i are S_μ-invariant closed subsets of E. If S_μ-invariant sets pairwise overlap, then there is a $x \in E_s \cap E_i$ and hence f

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is constant on E. Apply the first result to each orbit closure to obtain the second result.

As a consequence we get the Liouville property for proximal actions, minimal actions and distal actions on compact spaces. Recall that a semigroup S acting a compact space E is called proximal (resp. distal) if for any two distinct points $x, y \in E$, the closure of $\{(gx, gy) \mid g \in S\}$ meets (resp. does not meet) the diagonal in $E \times E$.

Corollary 2.1 *Let G be a locally compact group acting on a compact space E and μ be a probability measure on G.*

1 *If the action of S_μ on E is orbitwise proximal, that is proximal on closure of any orbit, then μ has Liouville property on E.*

2 *If the orbit closures $\overline{S_\mu x}$ are minimal, then μ has Liouville property on E.*

3 *If the action of S_μ on E is distal, then μ has Liouville property on E.*

Proof We assume that $E = \overline{S_\mu a}$ for some $a \in E$.

If the action is orbitwise proximal, then S_μ is proximal on E. Since E is compact, $\overline{S_\mu x} \cap \overline{S_\mu y} \neq \emptyset$ for any $x, y \in E$. This implies that any two S_μ-invariant sets overlap. Now the result follows from Proposition 2.2.

If the orbit closures are minimal, then the result easily follows from Proposition 2.2.

If the action is distal, then the orbit closures are minimal, hence the result follows from 2.

3 Actions on vector spaces

We now look at actions on vector spaces. Given a sequence (μ_n) of probability measures on a locally compact space E, we say that $\mu_n \to \mu$ in the weak* topology for a probability measure μ on E if $\mu_n(f) \to \mu(f)$ for any continuous bounded function f on E. It may be noted that $\mu_n \to \mu$ in the weak* topology if and only if $\mu_n(f) \to \mu(f)$ for any continuous function f with compact support on E (cf. 1.1.9 of [8]).

Proof of Theorem 1.1 Since $\mu(\{g \in G \mid ||g|| \leq a\}) = 1$ for some $0 < a < 1$, $\mu^n(\{g \in G \mid ||g|| \leq a^n\}) = 1$ for $n \geq 1$. Let $v \in V$. Then $||gv|| \leq a^n||v||$ for all g in the support of μ^n. Let $\epsilon > 0$ and ψ be any continuous function with compact support on V. Then using uniform continuity of ψ, we get that $|\psi(gv) - \psi(0)| < \epsilon$ for all g in the support of μ^n, for large n. This implies that $\mu^n * \delta_v(\psi) = \int \psi(gv)d\mu^n(g) \to \psi(0) = \delta_0(\psi)$, hence $\mu^n * \delta_v \to \delta_0$ in the weak* topology.

Let f be a continuous bounded μ-harmonic function on V. Then $f(v) = \mu^n * \delta_v(f) \to f(0)$ for any $v \in V$, hence f is constant. Thus, μ has Liouville property on V.

4   Actions on projective spaces

We now consider actions on projective spaces. We first prove a useful result on unipotent actions on projective spaces. A split solvable algebraic group is a solvable algebraic group whose maximal torus is a split torus: e.g. unipotent algebraic groups, $(\mathbb{R}^*)^n \ltimes \mathbb{R}^n$ where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, more generally, group of all upper triangular matrices.

Lemma 4.1 *Let V be a finite-dimensional vector space over $\mathbb{R}$ and G be a closed subgroup of $GL(V)$. If G is amenable, then action of G on any G-minimal subset of $\mathbb{P}(V)$ factors through a compact group. If the algebraic closure of G is a split solvable algebraic group, then any G-minimal subset of $\mathbb{P}(V)$ is singleton consisting of a G-fixed point.*

Proof Let E be a G-minimal subset of $\mathbb{P}(V)$. If G is amenable, then E supports a G-invariant probability measure ρ. Since E is G-minimal, support of ρ is E. Let $I_\rho = \{g \in GL(V) \mid g(x) = x \text{ for all } x \in E\}$ and $\mathcal{I}_\rho = \{g \in GL(V) \mid g(\rho) = \rho\}$. Then $G \subset \mathcal{I}_\rho$ and $\mathcal{I}_\rho/I_\rho$ is a compact algebraic group (cf. [3] and [6]). If the algebraic closure of G is a split solvable algebraic group, $G \subset I_\rho$.

We next consider actions on the projective line.

Proof of Theorem 1.2 Let $\phi \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{P}^1$ be the canonical projection. Suppose $\mathbb{R}^2$ is not G-irreducible. Then there is a nonzero vector $v \in \mathbb{R}^2$ such that for each $g \in G$, $gv = t_g v$ for some $t_g \in \mathbb{R}$. If all $g \in G$ is diagonalizable over $\mathbb{R}$, then G is abelian, hence G has Liouville on $\mathbb{P}^1$. So, assume that there

is a $g_0 \in G$ that is not diagonalizable over $\mathbb{R}$. Thus, v is the only eigenvector of $g_0 \in G$. Let G_0 be the group generated by g_0. Then $\phi(v)$ is the only G_0-invariant vector and the algebraic closure of G_0 is a split solvable algebraic group. By Lemma 4.1, any G_0-minimal subset of $\mathbb{P}^1$ consists of a G_0-fixed point. Since $\phi(v)$ is the only G_0-fixed point, we get that any G-invariant closed subset of $\mathbb{P}^1$ contains $\phi(v)$. Now the result follows from Proposition 2.2.

We now assume that $\mathbb{R}^2$ is G-irreducible. Suppose G is not a relatively compact subset of projective linear transformation of $\mathbb{P}^1$, that is, $PGL(\mathbb{R}^2)$. Then there is a sequence (g_n) in G such that (g_n) has no convergent subsequence in $PGL(\mathbb{R}^2)$. Passing to a subsequence, we may assume that $\frac{g_n}{||g_n||} \to h$ where h is a linear transformation on $\mathbb{R}^2$. Since $||h|| = 1$, $h \neq 0$. Since (g_n) has no convergent subsequence in $PGL(\mathbb{R}^2)$, h is a rank-one transformation. Let w be a nonzero vector in the image of h. Then $g_n(\phi(x)) \to \phi(w)$ for all x not in the kernel of h. If x is a nonzero vector in the kernel of h, then since $\mathbb{R}^2$ is G-irreducible, there is a $g \in G$ such that gx is not in the kernel of h. Now $\frac{g_n g}{||g_n g||} \to \frac{hg}{||hg||}$. Since $h(gx) \neq 0$ we get that $g_n g(\phi(x)) \to \phi(hg(x)) = \phi(w)$. Thus, $\phi(w) \in \overline{G\phi(x)}$ for any non-zero $x \in \mathbb{R}^2$. Now the result follows from Proposition 2.2.

We next consider actions of locally compact subgroups of $GL(V)$.

Proof of Theorem 1.3 Let U be an unipotent subgroup of G that has only one linearly independent invariant vector $v \in V$. Suppose $x \in \mathbb{P}(V)$ is U-invariant. Let $\phi\colon V \setminus \{0\} \to \mathbb{P}(V)$ be the canonical projection and $w \in V \setminus \{0\}$ be such that $\phi(w) = x$. Then for each $g \in U$ there is a $t_g \in \mathbb{R}$ such that $g(w) = t_g w$. Since $g \in U$ are unipotent, $g(w) = w$. Since U has only one linearly independent invariant vector $v \in V$, $w = tv$ for some $t \in \mathbb{R}$, that is, $\phi(v) = \phi(w)$.

Let E be a G-invariant closed subset of $\mathbb{P}(V)$. Then E is U-invariant and contains a U-minimal subset M. By Lemma 4.1, M consists of a fixed point of U, hence $M = \{\phi(v)\}$. Thus, any G-invariant closed subset contains $\phi(v)$. This implies by Proposition 2.2, that any continuous bounded μ-harmonic function is constant.

We now look at the conjugate action of $SL(V)$.

Proof of Theorem 5.1 Let $e_{1,n}$ be the matrix whose (i,j)-th entry is nonzero only for $i = 1$ and $j = n$ and $\phi\colon \mathcal{SL}(V) \setminus \{0\} \to \mathbb{P}(\mathcal{SL}(V))$ be

the canonical projection. We now prove that every $SL(V)$-invariant subset of $\mathbb{P}(\mathcal{SL}(V))$ contains $\phi(e_{1,n})$.

Choose subspaces $\{0\} = W_0 \subset W_1 \subset \cdots \subset W_d = V$ such that $\dim(W_i) = i$. Take

$$U = \cap_{1 \leq k < d}\{g \in SL(V) \mid g(W_k) = W_k \text{ and } g(v) - v \in W_k \text{ for all } v \in W_{k+1}\}.$$

Then U is the unipotent algebraic group of all upper triangular unipotent matrices in $SL(V)$ (with respect to a basis from the subspace W_k). It is easy to notice that W_{i+1}/W_i is the subspace of all U-invariant vectors in W_d/W_i for all $0 \leq i < d$ and can easily be seen that the center of U is $\{\exp(te_{1,n}) \mid t \in \mathbb{R}\}$.

Let $v \in \mathcal{SL}(V)$ be such that v is U-invariant, that is, $gv = vg$ for all $g \in U$. Let $v = v_s + v_n$ be the Jordan-Chevalley decomposition of v into a semisimple element v_s and a nilpotent element v_n such that $v_s v_n = v_n v_s$ and there are polynomials P and Q with $v_s = P(v)$ and $v_n = Q(v)$ (cf. Proposition 4.2 of [9]). Then since $gv = v$ for all $g \in U$, $gv_s = v_s g$ and $gv_n = v_n g$ for all $g \in U$. Since $v_n u = uv_n$ for all $u \in U$, any U-invariant vector is also v_n-invariant, hence W_1 is invariant under v_n. Since v_n is nilpotent and $\dim(W_1) = 1$, $v_n(W_1) = \{0\} = W_0$. We now claim by induction that $v_n(W_k) \subset W_{k-1}$ for all $k \geq 1$. Suppose $v_n(W_i) \subset W_{i-1}$ for some $1 \leq i < d$. Since $v_n u = uv_n$ for all $u \in U$, by considering the U-invariant one-dimensional subspace W_{i+1}/W_i of W_d/W_i, we conclude that $v_n(W_{i+1}/W_i) \subset W_{i+1}/W_i$. Since v_n is nilpotent and $\dim(W_{i+1}/W_i) = 1$, we get that $v_n(W_{i+1}) \subset W_i$. This implies that $\exp(v_n) \in U$. Since v_n commutes with every element of U, $\exp(v_n)$ is in the center of U. Thus, $v_n = te_{1,n}$ for some $t \in \mathbb{C}$. Since $uv_s = v_s u$ for all $u \in U$, it can easily be seen that v_s has only one eigenvalue and hence $v_s = tI$ for some $t \in \mathbb{C}$. Since v has trace zero, $v_s = 0$. Thus, $v = v_n = te_{1,n}$. This shows that U has only one linearly independent invariant vector $e_{1,n}$ in $\mathcal{SL}(V)$. Now the result may be proved as in Theorem 1.3.

5 Miscellaneous remarks

5.1 Amenability and Liouville property

A well-known conjecture of Furstenberg [5] states that a locally compact σ-compact group G is amenable if and only if G admits adapted probability measures having Liouville property. Several proofs of this conjectures are

available (cf. [11], [12] and [14]). In this case one could prove a similar result for Liouville property of group actions using the above conjecture.

Proposition 5.1 *Let G be a locally compact σ-compact group and μ be an adapted probability measure on G. Then the following are equivalent:*

(1) $\check{\mu}$ has Choquet-Deny;

(2) μ has Liouville property on all G-spaces;

(3) μ has Liouville property on all compact G-spaces;

(4) μ has Liouville property on all compact affine G-spaces.

In particular, G is amenable if and only if G has an adapted probability measure μ such that μ has Liouville property on all compact G-spaces.

Proof It is sufficient to prove that $(4) \Rightarrow (1)$. Suppose μ has Liouville property on all compact affine G-spaces. Let $C_b(G)$ be the Banach space of all continuous bounded functions on G. Let E be the unit ball in the dual $C_b(G)^*$ of $C_b(G)$: it may be noted that the dual of $C_b(G)$ is the space of all regular bounded finitely additive measures on G. Then E is compact in the weak* topology. G acts on $C_b(G)$ by $gf(x) = f(xg)$ for all $x, g \in G$ and $f \in C_b(G)$ and the action is by isometries. By duality each $g \in G$ defines a continuous map g^* on E. Now, $g \mapsto g^{-1*}$ defines an action of G on E.

Let f be a continuous bounded $\check{\mu}$-harmonic function on G, that is,

$$\int f(xg^{-1})d\mu(g) = f(x), \quad x \in G.$$

Define the function ϕ on E by $\phi(\sigma) = <\sigma, f>$ for $\sigma \in E$. Then ϕ is a continuous bounded function on E. Now, $\int \phi(g^{-1*}\sigma)d\mu(g) = \int <g^{-1*}\sigma, f> d\mu(g) = \int <\sigma, g^{-1}f> d\mu(g) = <\sigma, \int g^{-1}fd\mu(g)> = <\sigma, f> = \phi(\sigma)$. Thus, ϕ is a μ-harmonic function on E. By assumption ϕ is constant on G-orbits, that is, $<x^{-1*}\sigma, f> = <\sigma, f>$ for all $x \in G$ and $\sigma \in E$. For any $x \in G$, since $\delta_x \in E$ we have $f(x) = \phi(\delta_x) = \phi(x^{-1}\delta_e) = \phi(\delta_e) = f(e)$. Thus, f is constant.

5.2   Equicontinuous Markov operator

If E is a G-space and μ is a probability measure on G, then we define a Markov operator P on $C_b(E)$ by

$$Pf(x) = \int f(gx)d\mu(g), \quad f \in C_b(E), \quad x \in E.$$

We say that the Markov operator P defined by μ on E is equicontinuous if for every $f \in C_b(E)$, there exists a subsequence of integers (k_n) and $F \in C_b(E)$ such that $\{\frac{1}{k_n}\sum_{i=1}^{k_n} P^i f\} \to F$ pointwise.

Remark 5.1   1. If the closed semigroup (or equivalently the closed subgroup) generated by the support of μ is compact, then P is equicontinuous on any G-space E which may be seen as follows. Since the closed semigroup generated by the support of μ is compact, $\frac{1}{n}\sum_{k=1}^{n} \mu^k$ converges to a probability measure λ in the weak*-topology, that is, $\frac{1}{n}\sum_{k=1}^{n} \mu^k(f) \to \lambda(f)$ for all continuous bounded functions f on G (cf. [13]). Thus, if E is a G-space, then for $x \in E$, $\frac{1}{n}\sum_{k=1}^{n} \mu^k * \delta_x \to \lambda * \delta_x$ in the weak* topology on E, hence $\frac{1}{n}\sum_{k=1}^{n} P^k f(x) \to \lambda * \delta_x(f)$ for all continuous bounded function f on E. Thus, P is equicontinuous on E.

  2. If V is a finite-dimensional vector space and μ is a measure on $GL(V)$ such that S_μ is strongly irreducible on V, that is, no finite union of proper subspaces is invariant under S_μ, then by Proposition 3.1 of [2] we get that P defined by μ on $\mathbb{P}(V)$ is equicontinuous.

We now characterize equicontinuous P that has Liouville property.

Theorem 5.1 *Let E be a compact G-space and μ be an adapted probability measure on G such that the corresponding Markov operator P on E is equicontinuous. Then the following are equivalent:*

(1) λ has Liouville property on E for all adapted probability measures λ on G;

(2) μ has Liouville property on E;

(3) for any $x \in E$, $\overline{S_\mu x}$ has a unique minimal subset;

(4) for any $x \in E$, S_μ-invariant subsets of $\overline{Gx}$ overlap.

Proof $(1) \Rightarrow (2)$ is evident and $(4) \Rightarrow (1)$ follows from Proposition 2.2. It only remains to show that $(2) \Rightarrow (3) \Rightarrow (4)$.

If there is a $x \in E$ such that $\overline{S_\mu x}$ contains disjoint nonempty closed S_μ-minimal sets E_1 and E_2. Let f be a continuous function on E such that $f(E_1) = 1$ and $f(E_2) = 2$. Since P is equicontinuous, there exists (k_n) such that $\frac{1}{k_n} \sum_{i=1}^{k_n} P^i f$ converges to a continuous function F on E. It can easily be verified that $PF = F$ but $F(E_1) = 1$ and $F(E_2) = 2$. Since $E_1, E_2 \subset \overline{S_\mu x}$, F is not S_μ-invariant but μ-harmonic. This proves that $(2) \Rightarrow (3)$.

Let X and Y be closed S_μ-invariant subsets of $\overline{S_\mu x}$ for $x \in E$. Then X and Y contain S_μ-minimal subsets. This proves that $(3) \Rightarrow (4)$.

6 Examples

Example 6.1 Consider the following linear action of $\mathbb{R}^2$ on $\mathbb{R}^4$ given by

$$(t, s) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{t-s} & 0 \\ 0 & 0 & 0 & e^{s-t} \end{pmatrix}$$

for $s, t \in \mathbb{R}$. Then the orbit of $v = (1, 1, 1, 0)$ is $\{(1, 1, e^{t-s}, 0) \mid t, s \in \mathbb{R}\}$. Let $\mathbb{P}^3(\mathbb{R})$ be the corresponding projective space and $\pi \colon \mathbb{R}^4 \setminus \{0\} \to \mathbb{P}^3(\mathbb{R})$ be the canonical projection. We now claim that the closure of the orbit of $\pi(v)$ has two disjoint invariant sets.

Choose sequences (t_n) and (s_n) in $\mathbb{R}$ such that $t_n - s_n \to -\infty$. Then orbit closure of v contains $(1, 1, 0, 0)$.

Choose sequences (t_n) and (s_n) in $\mathbb{R}$ such that $t_n - s_n \to \infty$. Then $e^{s_n - t_n}(1, 1, e^{t_n - s_n}, 0) \to (0, 0, 1, 0)$. This implies that $\pi(0, 0, 1, 0)$ is in the closure of the orbit of $\pi(v)$.

It can easily be seen that $(1, 1, 0, 0)$ is an invariant vector in $\mathbb{R}^4$ and $\pi(0, 0, 1, 0)$ is an invariant point in $\mathbb{P}^3(\mathbb{R})$. Thus, the closure of the orbit of $\pi(v)$ has two disjoint invariant sets $\pi(1, 1, 0, 0)$ and $\pi(0, 0, 1, 0)$.

But since $\mathbb{R}^2$ is abelian, $\mathbb{R}^2$ has Liouville property and hence any action of $\mathbb{R}^2$ also has Liouville property.

The following example produces a measure on the $ax + b$-group that does not have Choquet-Deny but has Liouville for its action on $\mathbb{R}^2$.

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Example 6.2 Let $G = \{\begin{pmatrix} t^2 & a \\ 0 & t \end{pmatrix} \mid t > 0, \quad a \in \mathbb{R}\}$. Then G is a solvable group and G is the $ax + b$-group. Any measure μ on G supported on $\{\begin{pmatrix} t^2 & a \\ 0 & t \end{pmatrix} \mid 0 < t < 1/5, \quad |a| < 1/5\}$ satisfies the condition in Theorem 1.1. Hence μ has Liouville on $\mathbb{R}^2$. But μ itself is not Liouville, that is, there are non-constant continuous bounded μ-harmonic functions on G - this could be seen from section 5.1.2 of [1].

The next example provides measures μ on $GL(V)$ that has Liouville property on V but neither μ nor $\check{\mu}$ has Choquet-Deny.

Example 6.3 Assume that V has dimension at least two. Let $0 < a < 1$ and μ be a probability measure on $GL(V)$ supported on $\{g \in GL(V) \mid ||g|| \leq a\}$. Then by Theorem 1.1, μ has Liouville on V. If G_μ is nonamenable , then neither μ or $\check{\mu}$ can have Liouville property (or Choquet-Deny). It may be noted that G_μ is nonamenable if the support of μ is $\{g \in GL(V) \mid ||g|| \leq a\}$.

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