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Ovoidal packings of $PG(3, q)$ for even q

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Abstract

We show that any set of n pairwise disjoint ovals in a finite projective plane of even order has a unique common tangent. As a consequence, any set of $q+1$ pairwise disjoint ovoids in $PG(3, q)$, q even, has exactly q^2+1 common tangent lines, constituting a regular spread. Also, if $q-1$ ovoids in $PG(3, q)$ intersect pairwise exactly in two given points $x \neq y$ and share two tangent planes π_x, π_y at these two points, then these ovoids share exactly $(q+1)^2$ common tangent lines, and they consist of the transversals to the pair xy, $\pi_x \cap \pi_y$ of skew lines. There is a similar (but more complicated) result for the common tangent lines to q ovoids in $PG(3, q)$ which are mutually tangent at a common point and share a common tangent plane through this point. It is also shown that the common tangent lines to any pair of disjoint ovoids of $PG(3, q)$, q even, form a regular spread.

1 Introduction

Let us recall that an *oval* in a finite projective plane of order n is a set of $n+1$ points no three of which are collinear. It follows that an

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oval has exactly $n+1$ tangent lines, one such through each point of the oval. When A is an oval in a projective plane π of odd order n, then the set A^* of the tangent lines to A form an oval in the dual projective plane π^*; we have $A^{**} = A$. However, when n is even, any oval A in a plane of order n has a unique point $x \notin A$ such that the tangent lines to A are precisely the lines through x. The point x is called the *nucleus* of A.

When q is an odd prime power, a famous theorem due to Segre says that the ovals in the Desarguesian plane $PG(2,q)$ are precisely the conics (i.e., the set of all points, the homogeneous co-ordinates of which satisfy homogeneous quadratic equation). However, when $q = 2^e$ is large, classification of ovals in $PG(2,q)$ is a hopelessly difficult problem.

For prime powers q, an *ovoid* in the three-dimensional projective space $PG(3,q)$ is a set of points which meet every plane in a point or an oval. A trivial count shows that there are exactly q^2+1 points on it, and it has exactly q^2+1 tangent planes, one at each of its point. Indeed, the set $\mathcal{O}^*$ of tangent planes to $\mathcal{O}$ form an ovoid in the dual of $PG(3,q)$. When $q > 2$, the ovoids in $PG(3,q)$ are precisely the point sets of $PG(3,q)$ of the largest possible size no three of which are collinear.

When q is an odd prime power, there is a beautiful three-dimensional analogue, due to Barlotti and Panella (see [F], 1.4.50) of Segre's theorem. In this case, the ovoids in $PG(3,q)$ are precisely the elliptic quadrics (i.e., non-degenerate quadrics of Witt index one).

The situation with ovoids is more interesting for $q = 2^e$. When $e = 1$ or e is even, the elliptic quadrics are the only known ovoids in $PG(3,q)$. When $e > 1$ is odd, apart from the elliptic quadrics, there is only one more isomorphism class of known ovoids, namely the Tits ovoids. Despite serious attempts by many researchers, no other examples of ovoids have turned up. Thus we have the following conjecture, which is perhaps the most tantalising open problem in finite geometry.

Conjecture 1.1 *Up to isomorphism, the elliptic quadrics and Tits ovoids are the only ovoids in $PG(3,q)$.*

This conjecture gains extra importance due to a related result of Dembowski. Recall that *a finite inversive plane* of order $n > 1$ is nothing but a Steiner 3-design with parameters $(n^2+1, n+1, 1)$. In other words, it is an one-point extension of an affine plane of order n.

A well-known but apparently inaccessible conjecture asserts that the order of any finite affine plane (equivalently, a finite projective plane) is a prime power. The corresponding problem for inversive planes may be more accessible. Indeed, the only known finite inversive planes are the "egg-like" ones, i.e., those which arise as the non-trivial plane sections of ovoids in $PG(3,q)$. We still do not know if all finite inversive planes are egg-like. But this is a theorem of Demboswki in the even order case. If the order of a finite inversive plane is even, then indeed it is a power of 2, and the points and blocks of the inversive plane are (up to isomorphism) the points and non-trivial plane sections of an ovoid in $PG(3,q)$. Thus, the classification of inversive planes of even order is equivalent to the classification of ovoids in $PG(3,q)$, q even.

It is our hope that if we could gain a good understanding of how the ovoids (known or unknown) must intersect, then it would be possible to see that there is no "space" in $PG(3,q)$ for any unknown ovoid. This motivates our previous papers [1], [2] as well as the current paper.

Recall that $\mathcal{O}^*$ is the dual ovoid consisting of the tangent planes to a given ovoid $\mathcal{O}$. The following remarkable result is due to Bruen and Hirschfield [4]. We cannot resist the temptation to reproduce its simple and elegant proof. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two ovoids in $PG(3,q)$. Let us count the number of pairs (x,π) where $x \in \mathcal{O}_1, \pi \in \mathcal{O}_2^*$ and $x \in \pi$. Since each point on $\mathcal{O}_2$ is on a unique tangent plane to $\mathcal{O}_2$ and each point off $\mathcal{O}_2$ is on $q+1$ tangent planes to $\mathcal{O}_2$, the number of such pairs is $\#(\mathcal{O}_1 \cap \mathcal{O}_2) + (q+1)(q^2+1-\#(\mathcal{O}_1 \cap \mathcal{O}_2))$. A dual way of counting these pairs (effect the interchanges $\mathcal{O}_2^* \leftrightarrow \mathcal{O}_1$, $\mathcal{O}_1^* \leftrightarrow \mathcal{O}_2$) yields $\#(\mathcal{O}_1^* \cap \mathcal{O}_2^*) + (q+1)(q^2+1-\#(\mathcal{O}_1^* \cap \mathcal{O}_2^*))$ as their number. Equating the two answers, we get:

Theorem 1.2 *(Bruen and Hirschfield): For any two ovoids $\mathcal{O}_1$ and $\mathcal{O}_2$ in $PG(3,q)$, we have $\#(\mathcal{O}_1^* \cap \mathcal{O}_2^*) = \#(\mathcal{O}_1 \cap \mathcal{O}_2)$.*

Notice that this result is valid for any prime power q (even or odd). This result is all the more remarkable since its analogue for ovals is false. (For instance, there are disjoint ovals sharing a tangent, compare Theorem 2.1 below.) So this result cries out for a geometric explanation. In this context, it seems very natural to ask.

Question 1.3 *Is it true that any common point of two ovoids in $PG(3,q)$ lie on a common tangent plane to them?*

Notice that, if the answer to this questions was in the affirmative, then the map $x \mapsto \pi_x$, sending any common point x to the unique common tangent plane π_x through x, would provide a bijection between $\mathcal{O}_1 \cap \mathcal{O}_2$ and $\mathcal{O}_1^* \cap \mathcal{O}_2^*$, thus explaining Theorem 1.2. Unfortunately, we find that the answer to this question is in the negative, at least for $q > 2$, even. To explain our answer, let us introduce the following notation. For ovoids $\mathcal{O}_1, \mathcal{O}_2$, let $I(\mathcal{O}_1, \mathcal{O}_2)$ be the set of all pairs (x, π) where $x \in \mathcal{O}_1 \cap \mathcal{O}_2$, $\pi \in \mathcal{O}_1^* \cap \mathcal{O}_2^*$ and $x \in \pi$. In Section 3 of this paper, we show that for $q = 2^e$ and ovoids $\mathcal{O}_1, \mathcal{O}_2$ of $PG(3, q)$ which do not share all tangent lines, we have $\#I(\mathcal{O}_1, \mathcal{O}_2) \leq 2$. Therefore, if Question 1.3 had an affirmative answer, then it would follow that we have $\#(\mathcal{O}_1 \cap \mathcal{O}_2) \leq 2$ for any such pair of ovoids. But this is easily disproved when $q > 2$. So we are forced to ask the following vague question.

Revised Question 1.4 *For ovoids $\mathcal{O}_1, \mathcal{O}_2$ in $PG(3, q)$, is there a geometrically defined ("natural") bijection between $\mathcal{O}_1 \cap \mathcal{O}_2$ and $\mathcal{O}_1^* \cap \mathcal{O}_2^*$?*

We recall that a *linear complex* in $PG(3, q)$ is the set of all absolute lines with respect to a symplectic polarity of $PG(3, q)$ (i.e. a polarity for which all points are absolute). If P is the set of all points of $PG(3, q)$ and L is a linear complex in $PG(3, q)$ then the incidence system (P, L) is denoted by $W(q)$. Since the action of the collineation group $PGL(4, q)$ by conjugation is transitive on the set of all symplectic polarities, the incidence system $W(q)$ is uniquely defined, up to isomorphism. It is a generalized quadrangle of order (q, q) (Cf. [Payne + Thas]). (An...) *ovoid of* $W(q)$ is a set of $q^2 + 1$ points no two of which are collinear. It is easy to see that every ovoid of $W(q)$ is an ovoid of $PG(3, q)$.

Now consider the following construction. Let $q = 2^e$ and $\mathcal{O}$ be an ovoid of $PG(3, q)$. Consider the incidence system $W(\mathcal{O})$ whose points are the points of $PG(3, q)$ and whose lines are the tangent lines to $\mathcal{O}$. For each point x, the union of the $q + 1$ tangent lines to $\mathcal{O}$ through x is a plane π_x. (This is true only for q even and may be seen by dualizing the statement that the ovals which are the non-trivial the plane sections of $\mathcal{O}$ have their respective nudes.) Then $x \leftrightarrow \pi_x$ defines a symplectic polarity of $PG(3, q)$ whose absolute lines are precisely the lines of $W(\mathcal{O})$. Thus, for each ovoid of $\mathcal{O}$ of $PG(3, q)$, q even, $W(\mathcal{O})$ is a copy of $W(q)$ and $\mathcal{O}$ is an ovoid of this copy of $W(q)$. Thus, each ovoid of $PG(3, q)$ is an ovoid of (a unique copy of) $W(q)$. In view of this observation, the classification of the ovoids of $PG(3, q)$, q even, is equivalent to the classification of the ovoids of $W(q)$.

Remark 1.5 *The above construction/observation is due to Segre. The two uses of the word "ovoid" (ovoid of $PG(3, q)$ versus ovoid of $W(q)$) has apparently led to endless confusions, compounded by Segre's observation. So we clarify further. The action of $PGL(4, q)$ on the class of all ovoids of $PG(3, q)$ induce a natural partition of this class. $PGL(4, q)$ itself is transitive on the cells of this partition, while the stabilizer of each cell is a copy of the symplectic group $PSP(4, q)$, to wit, the group of collineations of the common copy of $W(q)$ defined by the ovoids in the given cell.*

In [2], we proved that any two of the known ovoids of $W(q)$ meet in $1, q+1, 2q+1, q+\sqrt{2q}+1$ or $q-\sqrt{2q}+1$ points. In [1] we combined coding theory and group action to prove that any known ovoid of $W(q)$ meets all possible ovoids of $W(q)$ oddly. This last result was generalized by Butler [3] as follows:

Theorem 1.6 *(Butler) Let $q = 2^e$ and $\mathcal{O}_1, \mathcal{O}_2$ be two ovoids of $PG(3, q)$ such that $W(\mathcal{O}_1) \neq W(\mathcal{O}_2)$. Then $\#(\mathcal{O}_1 \cap \mathcal{O}_2)$ is an odd number.*

In other words, any two ovoids in each of the cells of Remark 1.5 have pairwise odd intersection.

In the next section, we use counting arguments to show that any n pairwise disjoint ovals in a projective plane of even order n have a unique common tangent line (Lemma 2.1). As an immediate consequence of this lemma, we show that, for even q, the common tangent lines to any $q + 1$ pairwise disjoint ovoids in $PG(3, q)$ form a spread. However, we defer the proof of regularity of this spread to the final section.

It is easy to see that any three mutually skew lines in $PG(3, q)$ have exactly $q + 1$ transversals (lines meeting all three given lines). Such a set of $q + 1$ transversals is called a *regulus* in $PG(3, q)$. The transversals to the lines in any regulus form another regulus, called the *opposite* of the given regulus. Thus, any three pairwise skew lines of $PG(3, q)$ are together in a unique regulus. A *spread* of $PG(3, q)$ is a set of $q^2 + 1$ mutually skew lines (partitioning the point set). A spread Σ in $PG(3, q)$ is said to be *regular* if for any three disjoint lines $\ell_1, \ell_2, \ell_3 \in \Sigma$, the unique regulus containing ℓ_1, ℓ_2, ℓ_3 is contained in

Σ. (Thus, the regulus contained in a regular spread form the circles of an inversive plane of order q.)

The *Klein correspondence* is a bijection between the set of lines of $PG(3,q)$ and the set of points of a hyperbolic quadric $Q^+(5,q)$ (the Klein quadric) in $PG(5,q)$ which takes intersecting pairs of lines to collinear pairs of points. (Two points of $Q^+(5,q)$ are collinear if the line joining them is contained in $Q^+(5,q)$).

In Section 3, we observe that the existence of the Klein correspondence immediately implies that there are only three possibilities for the intersection of distinct linear complexes L_1, L_2 in $PG(3,q)$. One of these possibilities is a regular spread. In view of the preceding discussion, this observation has immediate implications for the set of common tangent lines between two ovoids in $PG(3,q)$ in case q is even. One consequence is the bound on the size of $I(\mathcal{O}_1, \mathcal{O}_2)$ quoted above. We also prove that, for any two disjoint ovoids $\mathcal{O}_1, \mathcal{O}_2$ of $PG(3,2^e)$, the common tangent lines to $\mathcal{O}_1$ and $\mathcal{O}_2$ form a regular spread. (However, this does not imply the main result of Section 2. The counting argument in Section 2 appears to be indispensable.) If $q-1$ ovoids of $PG(3,q)$, q even, intersect pairwise in two common points $x \neq y$ (exactly) and share then two tangent planes π_x, π_y at x and y (respectively), then the common tangent lines to these ovoids are precisely the tranversals to the pair $xy, \pi_x \cap \pi_y$ of skew lines. Also, if q ovoids $\mathcal{O}_i, 1 \leq i \leq q$, of $PG(3,q)$, q even, intersect pairwise exactly at a point x and share a common tangent plane π_x at x, then either (i) these ovoids have the same set of tangent lines (i.e., $W(\mathcal{O}_1) = \cdots = W(\mathcal{O}_2)$), or (ii) they have exactly $q+1$ common tangent lines, namely those lines through x which are contained in π_x, or (iii) there is a unique line ℓ_0 with $x \in \ell_0 \subseteq \pi_r$ such that the q ovoids have exactly q^2+q+1 common tangent lines, namely the tangent lines of $\mathcal{O}_1$ which intersect ℓ_0.

For the unproved general results on finite projective geometries quoted here, the reader may consult [5].

2 Counting arguments

We begin with

Lemma 2.1 *Let $C_i, 1 \leq i \leq n$, be n pairwise disjoint ovals in a finite projective plane π of even order n. Then these ovals have a unique* *common tangent line. (Every other line is a tangent to at most one of these ovals.)*

(Note that, in this lemma, the plane π is not assumed to be desarguesian. Conceivably, n may not even be a power of two.)

Proof: Notice that there is a unique point x of π which is not covered by the given ovals (the total number of points of π is $n^2+n+1=n(n+1)+1$).

Let x_i be the nucleus of C_i and put $m=\{x,x_1,\ldots,x_n\}$. We do not assume a priori that these points are distinct. Thus, we have $\#(m)\leq n+1$. Notice that every line ℓ of π intersects m. (This is trivial if $x\in\ell$. Otherwise, the ovals C_i induce a partition of ℓ into singletons and doubletons. Since $\#(\ell)=n+1$ is odd, at least one of the cells in this partition is a singleton. That is, ℓ is a tangent line to at least one C_i. The nucleus x_i of this oval lies on ℓ as well as on m.) Therefore a standard argument shows that m is a line of π (and, indeed, $\#(m)=n+1$). Since the line m passes through the nucleus x_i of C_i, for each i, m is a common tangent line to all these ovals. No other line through x can be a tangent to any of these ovals. If ℓ is a line of π not passing through x, then ℓ meets the line m in a unique point x_i, and hence ℓ is a tangent to exactly one oval C_i. $\square$

As an immediate consequence of Lemma 2.1, we have:

Theorem 2.2 *Let $\mathcal{O}_i, 1\leq i\leq q+1$ be $q+1$, pairwise disjoint ovoids of $PG(3,q), q=2^e$. (Thus they partition the point set of $PG(3,q)$.) Then the set of common tangent lines to these $q+1$ ovoids form a regular spread of $PG(3,q)$ (and every line outside this spread is a tangent line to exactly one of these ovoids).*

Proof: Take any plane π in $PG(3,q)$. The ovoids $\mathcal{O}_i$ induce a partition of the $q(q+1)+1$ points of π into $q+1$ sets : q of size $q+1$ and one of size 1. Therefore π is a tangent plane to exactly one $\mathcal{O}_i$, and it meets the remaining q ovoids in ovals. The non-trivial sections of the given ovoids with π constitute a set of q pairwise disjoint ovals in the projective plane π of even order q. Lemma 2.1 implies that π contains exactly one common tangent line to the $q+1$ ovoids. The set Σ of common tangent lines (to the given ovoids) contain exactly one line from every plane of $PG(3,q)$. Therefore Σ is a spread of $PG(3,q)$. Corollary 3.4 below implies that this spread is regular.

Now let $\ell\notin\Sigma$ be a line. Since $\#(\ell)=q+1$ is odd and the given partition by ovoids induces a partition of ℓ into singletons and doubletons, it follows that ℓ is a tangent to at least one of the given ovoids. Fix a plane $\pi \supseteq \ell$. Since $\ell \notin \Sigma$, ℓ is not a common tangent to the non-trivial π sections of the $\mathcal{O}_i$'s. Therefore Lemma 2.1 implies that ℓ is a tangent to at most one of these ovoids. Thus, any line $\ell \notin \Sigma$ is a tangent to a unique $\mathcal{O}_i$. $\square$

Example 2.3 *(Butler, [3]) Take a regular spread Σ of $PG(3,q)$. Such a spread is unique up to isomorphism. Let H be the line-wise stabilizer in $PGL(4,q)$ of the spread Σ. It is a cyclic group of order $q+1$, acting regularly on each line from Σ. Take a linear complex $L \supseteq \Sigma$. (There are exactly $q+1$ such linear complexes, see Remark 3 below.) Now suppose $q = 2^e$, and let $\mathcal{O}$ be an ovoid of the copy of $W(q)$ with line set L. Then the H-images of $\mathcal{O}$ constitute a partition of the point-set of $PG(3,q)$ into $q+1$ ovoids, one of which is the given (arbitrary) ovoid $\mathcal{O}$ of $PG(3,q)$. The non-trivial sections of these ovoids by any plane of $PG(3,q)$ give a set of q disjoint ovals in a plane of order q.*

Question 2.4 *What is the odd-order analogue (if any) of Lemma 2.1 and Theorem 2.2?*

3 Klein correspondence

The following lemma is an immediate consequence of (the existence of) the Klein correspondence.

Lemma 3.1 *Let L_1 and L_2 be two distinct linear complexes in $PG(3,q)$. Then $L_1 \cap L_2$ is one of the following:*

(i) the set of q^2+1 lines in a regular spread,

(ii) the set of q^2+q+1 lines in L_1 which meet a given line $\ell_0 \in L_1$ (including ℓ_0 itself)

(iii) the set of $(q+1)^2$ lines which meet a given pair of Skew lines of $PG(3,q)$.

Proof: Under the Klein correspondence, L_i goes to a non-degenerate quadric $Q_i \subseteq Q^+(5,q)$ of projective dimension 4. Let $Q_i = Q^+(5,q) \cap H_i$ where H_i is a hyperplane in $PG(5,q)$. Thus $L_1 \cap L_2$ goes to $Q_1 \cap Q_2 = F \cap Q^+(5,q)$, where $F = H_1 \cap H_2$ is a 3-dimensional projective subspace of $PG(5,q)$. Thus, $Q_1 \cap Q_2$ is a quadric in $H_1 \cap H_2$, possibly degenerate. Notice that Q_1 is of Witt index 2, i.e., it does not contain any plane. So, $Q_1 \cap Q_2$ cannot be the union of two planes. Therefore, $Q_1 \cap Q_2$ is either an elliptic quadric $Q^-(3,q)$ or a 3-dimensional cone (over a planar conic) or a hyperbolic quadric $Q^+(3,q)$. The pull back $L_1 \cap L_2$ of these three objects under the Klein correspondence is as in (i), (ii) and (iii) (respectively). $\square$

In view of the discussion in Section 1, this lemma has the following immediate consequence. (Note that Lemma 3.1 is valid for any prime power q, but the following Corollary is proved only for $q = 2^e$.)

Corollary 3.2 *Let $\mathcal{O}_1, \mathcal{O}_2$ be two ovoids of $PG(3,q)$, $q = 2^e$. Then either $W(\mathcal{O}_1) = W(\mathcal{O}_2)$ (i.e., $\mathcal{O}_1$ and $\mathcal{O}_2$ share all the $(q+1)(q^2+1)$ tangent lines) or the common tangent lines to $\mathcal{O}_1$ and $\mathcal{O}_2$ are as in (i), (ii) or (iii) of Lemma 3.1.*

For ovoids $\mathcal{O}_1, \mathcal{O}_2$ of $PG(3,q)$, let us introduce the notation:

$$I(\mathcal{O}_1, \mathcal{O}_2) := \{(x,\pi) : x \in \mathcal{O}_1 \cap \mathcal{O}_2, \pi \in \mathcal{O}_1^* \cap \mathcal{O}_2^*, x \in \pi\}, \text{ and } i(\mathcal{O}_1, \mathcal{O}_2) = \#(I(\mathcal{O}_1, \mathcal{O}_2)).$$

Notice that, when $W(\mathcal{O}_1) = W(\mathcal{O}_2)$, we trivially have $\#(\mathcal{O}_1 \cap \mathcal{O}_2) = i(\mathcal{O}_1, \mathcal{O}_2) = \#(\mathcal{O}_1^* \cap \mathcal{O}_2^*)$. On the other hand, when $W(\mathcal{O}_1) \neq W(\mathcal{O}_2)$, we shall have $\#(\mathcal{O}_1 \cap \mathcal{O}_2) = \#(\mathcal{O}_1^* \cap \mathcal{O}_2^*)$ by Theorem 1.2, but:

Theorem 3.3 *Let $\mathcal{O}_1, \mathcal{O}_2$ be distinct ovoids of $PG(3,q)$, q even, which do not share all their tangent lines. Then $i(\mathcal{O}_1, \mathcal{O}_2) \leq 2$. More precisely, $i(\mathcal{O}_1, \mathcal{O}_2) = 0, 1$ or 2 according as the common tangent lines to $\mathcal{O}_1$ and $\mathcal{O}_2$ are as in (i), (ii) or (iii), respectively.*

Proof: For any point x of $PG(3,q)$, let $\pi_x^{(1)}$ and $\pi_x^{(2)}$ be the union of the tangent lines through x to $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Thus $\pi_x^{(1)}$ and $\pi_x^{(2)}$ are planes through x. (They are the images of x under the symplectic polarity corresponding to $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively.) For $x \in \mathcal{O}_i$, $\pi_x^{(i)}$ is the tangent plane to $\mathcal{O}_i$ through x. Also, for any point x we have $\pi_x^{(1)} = \pi_x^{(2)}$ if, and only if, x is in $q+1$ common tangent lines to $\mathcal{O}_1$ and $\mathcal{O}_2$. And, when $\pi_x^{(1)} \neq \pi_x^{(2)}$, $\pi_x^{(1)} \cap \pi_x^{(2)}$ is the unique common tangent line to $\mathcal{O}_1$ and $\mathcal{O}_2$ through x. Thus, each point is in one or $q+1$ common tangent lines to $\mathcal{O}_1$ and $\mathcal{O}_2$. (Corollary 3.2 confirms this assertion.) More importantly, letting A denote the set of all points x which are in $q+1$ common tangent lines, we see that

$$I(\mathcal{O}_1, \mathcal{O}_2) = \{(x, \pi_x^{(1)}) : x \in A \cap \mathcal{O}_1\}.$$

Hence $i(\mathcal{O}_1, \mathcal{O}_2) = \#(A \cap \mathcal{O}_1)$.

But, if the common tangent lines to $\mathcal{O}_1$ and $\mathcal{O}_2$ are as in (i) then $A = \emptyset$, if they are as in (ii) then A is a tangent line to $\mathcal{O}_1$ (and also to $\mathcal{O}_2$). If they are as in (iii), then $A = \ell_1 \sqcup \ell_2$ where ℓ_1, ℓ_2 are the skew lines of $PG(3,q)$ such that the common tangent lines to $\mathcal{O}_1$ and $\mathcal{O}_2$ are precisely the tranversals to ℓ_1 and ℓ_2. In the last case, exactly one of the two lines ℓ_1, ℓ_2 meet $\mathcal{O}_1$ in two points and the other is disjoint from $\mathcal{O}_1$. (Indeed, ℓ_1, ℓ_2 are images of each other under the symplectic polarity corresponding to $\mathcal{O}_1$.) $\square$

As applications of Theorem 3.3, we get the following results.

Corollary 3.4 *Let $\mathcal{O}_1, \mathcal{O}_2$ be two disjoint ovoids of $PG(3,q)$, q even. Then the common tangent lines to $\mathcal{O}_1, \mathcal{O}_2$ form a regular spread.*

Proof: Theorem 1.6 implies that $W(\mathcal{O}_1) \neq W(\mathcal{O}_2)$. So that Theorem 3.3 applies. Since $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$, we have $i(\mathcal{O}_1, \mathcal{O}_2) = 0$. Therefore, by Theorem 3.3, the common tangent lines form a regular spread. $\square$

Corollary 3.5 *Let $\mathcal{O}_1, \mathcal{O}_2$ be two ovoids of $PG(3,q)$ which have exactly two common points $x \neq y$ and two common tangent planes π_x and π_y through x and y, respectively. Then the common tangent lines to $\mathcal{O}_1$ and $\mathcal{O}_2$ are precisely the transversals to the pair of lines $xy, \pi_x \cap \pi_y$.*

Proof: Since $\#(\mathcal{O}_1 \cap \mathcal{O}_2) = 2$, Theorem 1.6 implies that $W(\mathcal{O}_1) \neq W(\mathcal{O}_2)$. Therefore Theorem 3.3 applies, and we have the result since $i(\mathcal{O}_1, \mathcal{O}_2) = 2$ by hypothesis. $\square$

Corollary 3.6 *Let $\mathcal{O}_1, \mathcal{O}_2$ be two ovoids of $PG(3,q)$, q even, which share exactly one point x and also a tangent plane π_x through x. Then, either $\mathcal{O}_1$ and $\mathcal{O}_2$ have all tangents in common, or there is a line ℓ_0 such that $x \in \ell_0 \subseteq \pi_x$ and the common tangent lines to $\mathcal{O}_1$ and $\mathcal{O}_2$ are precisely the q^2+q+1 tangent lines to $\mathcal{O}_1$ meeting ℓ_0.*

Proof: If $\mathcal{O}_1, \mathcal{O}_2$ do not share all $(q+1)(q^2+1)$ tangent lines, then Theorem 3.3 applies. The result follows since $i(\mathcal{O}_1, \mathcal{O}_2) = 1$ by hypothesis. $\square$

Now, the following results are easy.

Theorem 3.7 *Let $\mathcal{O}_i, 1 \leq i \leq q-1$, be $q-1$ ovoids of $PG(3,q)$, q even, which share two common points $x \neq y$, and no two of which have any third point in common. Suppose also that these ovoids share* *tangent planes π_x, π_y at x, y. Then, these $q-1$ ovoids have exactly $(q+1)^2$ common tangent lines (namely the transversals to the pair $xy, \pi_x \cap \pi_y$ of lines). Every other line is tangent to at most one of these ovoids.*

Proof: Under the hypotheses, Corollary 3.5 describes the tangent lines common to $\mathcal{O}_i$ and $\mathcal{O}_j$ for any two indices $1 \le i \ne j \le q-1$, and this description is independent of the choice of i, j. $\square$

Similarly, Corollary 3.6 implies the following result. (However, we repeat that Corollary 3.4 by itself does not seem to imply Theorem 2.2.)

Theorem 3.8 *Let $\mathcal{O}_i, 1 \le i \le q$, be q ovoids of $PG(3,q)$, q even, pairwise intersecting at a common point x. Suppose also that these ovoids share a tangent plane π_x at x. Then, either all these ovoids have the same set of tangent lines, or they have exactly $q+1$ tangent lines (namely the lines ℓ such that $x \in \ell \subseteq \pi_x$), or there is a unique line $\ell_0, x \in \ell_0 \subseteq \pi_x$ such that the common tangent lines to these ovoids are precisely the q^2+q+1 tangent lines of any of them which meet ℓ_0.*

Example 3.9 *Take any ovoid $\mathcal{O}$ of $PG(3,q)$. Fix two points $x \ne y$ in $\mathcal{O}$. Let π_x, π_y be the tangent planes to $\mathcal{O}$ at x, y respectively. Let ℓ_1 be the line joining x and y, and let ℓ_2 be the intersection of π_x and π_y. Thus, ℓ_1, ℓ_2 are skew lines. The pointwise stabilizer in $PGL(4,q)$. of $\ell_1 \sqcup \ell_2$ is a cyclic group of order $q-1$. The images under H of the ovoid $\mathcal{O}$ constitutes a set of $q-1$ ovoids satisfying the hypothesis of Theorem 3.7.*

Remark 3.10 *The examples 2.3 and 3.9 show that the cases (i) and (iii) of Lemma 3.1 and Corollary 3.2 actually occur. Indeed, pursuing the argument in Lemma 3.1 a little further, it is easy to see that (a) each regular spread of $PG(3,q)$ is contained in exactly $q+1$ linear complexes, (b) the configuration consisting of the lines of $W(q)$ meeting a given line of $W(q)$ occurs in exactly q linear complexes, and (c) the set of transversals to a pair of skew lines in $PG(3,q)$ is contained in exactly $q-1$ linear complexes. Thus, all three cases occur. We do not know if all three cases of Theorem 3.8 actually occur.*

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