Splitting of low rank ACM bundles on hypersurfaces of high dimension

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Abstract. Let $X$ be a smooth projective hypersurface. In this note we show that any rank 3 (resp. rank 4) arithmetically Cohen-Macaulay vector bundle over $X$ splits when $\dim X \geq 7$ (resp. $\dim X \geq 9$).

1. Introduction

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface where $n \geq 3$. By Grothendieck-Lefschetz theorem [7], we know the structure of the set of all line bundles on $X$. Vector bundles over $X$ are not so well understood. An obvious question about vector bundles on any projective variety is the splitting problem - When can we say that a given vector bundle is a direct sum of line bundles? The proper objects in the category of vector bundles over $X$ to look for the splitting behaviour are arithmetically Cohen-Macaulay bundles. We recall the definition,

**Definition 1.1.** An arithmetically Cohen-Macaulay (ACM) bundle on $X$ is a vector bundle $E$ satisfying

$$H^i(X, E(m)) = 0, \forall m \in \mathbb{Z} \text{ and } 0 < i < \dim X$$

The importance of this definition lies in a well known criterion of Horrocks [10] - ACM bundles are precisely the bundles on $\mathbb{P}^n$ that are split. Viewing $\mathbb{P}^n$ as a hypersurface of degree 1 in $\mathbb{P}^{n+1}$, one may ask if for hypersurfaces with degree $d > 1$, such a splitting holds. When $d > 1$, there exists indecomposable ACM bundles on hypersurfaces (see [13] for a specific example or [15] for a class of examples), though several splitting results are available for various degrees and ranks. In particular, fixing $d = 2$, the ACM bundles on quadrics have been completely classified, see [12]. The case of cubic surfaces in $\mathbb{P}^3$ has been investigated in [3].

In a different direction, we can fix the rank of the bundle and let degree vary. Here the general conjectural picture is that any ACM bundle of a fixed rank, over a sufficiently high dimensional hypersurface (irrespective of its degree) is split. The precise conjecture is,
Conjecture (Buchweitz, Greuel and Schreyer [2]): Let \( X \subset \mathbb{P}^n \) be a hypersurface. Let \( E \) be an ACM bundle on \( X \). If \( \text{rank} \ E < 2^e \), where \( e = \left\lfloor \frac{n - 2}{2} \right\rfloor \), then \( E \) splits. (Here \([q]\) denotes the largest integer \( \leq q \).

Splitting of ACM bundles of rank 2 on hypersurfaces have been understood fairly well. We summarize the results known. When \( d = 1 \), splitting follows by the Horrock’s criterion, so we assume \( d \geq 2 \). Let \( E \) be a rank 2 ACM bundle on \( X \), then \( E \) splits if,

1. \( \text{dim}(X) \geq 5 \) (see [11] and [13]).
2. \( \text{dim}(X) = 4 \) and \( X \) is general hypersurface and \( d \geq 3 \) (see [13] and [16]).
3. \( \text{dim}(X) = 3 \) and \( X \) is general hypersurface and \( d \geq 6 \) (see [14] and [16]).

The case of a general hypersurface of low degree in \( \mathbb{P}^4 \) and \( \mathbb{P}^5 \) have also been studied by Chiantini and Madonna in [4], [5], [6].

For rank \( \geq 3 \), very few results are known. To our knowledge, the only general splitting result in this direction is by Tadakazu [17] who found a splitting criterion for any rank \( k \) ACM bundle on a general hypersurface depending on the degree and the dimension of hypersurface.

The conjecture mentioned above predicts that any ACM bundle of rank 3 (resp. rank 4) over a hypersurface in \( \mathbb{P}^6 \) (resp. \( \mathbb{P}^8 \)) splits. In this note, we prove a weaker version,

**Theorem 1.2** (Corollary 3.3 + Corollary 3.4). Let \( E \) be an ACM bundle on a smooth hypersurface \( X \subset \mathbb{P}^{n+1} \). Then \( E \) splits if,

1. \( \text{rank} \ E = 3 \) and \( n \geq 7 \).
2. \( \text{rank} \ E = 4 \) and \( n \geq 9 \).

For rank 2 ACM bundles, our method gives another proof for splitting when \( n \geq 5 \).

2. **Preliminaries**

We will work over an algebraically closed field of characteristic zero.

Let \( X \subset \mathbb{P}^{n+1} \) be a hypersurface of degree \( d \geq 2 \). We set a conventional notation

\[
H^i_\ast(X, \mathcal{F}) := \bigoplus_{m \in \mathbb{Z}} H^i(X, \mathcal{F}(m))
\]

where \( \mathcal{F} \) denotes a coherent sheaf on \( X \).

Let \( E \) be a rank \( k \) ACM bundle on \( X \). We take a minimal (1-step) resolution of \( E \) on \( \mathbb{P}^{n+1} \),

\[
0 \to \widetilde{F}_1 \to \widetilde{F}_0 \to E \to 0
\]

where \( \widetilde{F}_0 \) is direct sum of line bundles on \( \mathbb{P}^{n+1} \). By Auslander-Buchsbaum formula, \( \widetilde{F}_1 \) is a bundle and by Horrock’s criterion it is also a split bundle on \( \mathbb{P}^{n+1} \).
Restricting (1) to $X$, we get,

$$0 \to \text{Tor}^{1}_{\mathbb{P}^{n+1}}(E, \mathcal{O}_X) \to F_1 \to F_0 \to E \to 0$$

where $F_i = \tilde{F}_i \otimes \mathcal{O}_X$ for $i = 0, 1$. To compute the Tor term, we tensor the short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \to \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_X \to 0$ with $E$,

$$0 \to \text{Tor}^{1}_{\mathbb{P}^{n+1}}(E, \mathcal{O}_X) \to E(-d) \to E \to E \otimes \mathcal{O}_X \to 0$$

The map $E \to E \otimes \mathcal{O}_X$ is an isomorphism, thus we get $\text{Tor}^{1}_{\mathbb{P}^{n+1}}(E, \mathcal{O}_X) \cong E(-d)$. Exact sequence (2) breaks up into 2 short exact sequences,

$$0 \to G \to F_0 \to E \to 0$$

(3)

$$0 \to E(-d) \to F_1 \to G \to 0$$

(4)

Since $H^0(X, F_0) \to H^0(X, E)$ is a surjection of graded rings, $H^1(X, G) = 0$. It follows that $G$ is also ACM.

3. Proof of the main results

**Lemma 3.1.** Let $E$ be any non-split bundle (not necessarily ACM) on a hypersurface $X \subset \mathbb{P}^{n+1}, n \geq 3$. Assume further that $H^1(X, E^\vee) = 0$. Let the exact sequence (3) be a minimal (1-step) resolution of $E$ on $X$, then $G$ does not admit a line bundle as a direct summand.

**Proof.** We will assume the contrary. Let $G = G' \oplus L$ where $L$ is a line bundle. By Grothendieck-Lefschetz theorem, $L$ is of the form $\mathcal{O}_X(a)$. There exists following pushout diagram,

$$\begin{array}{ccc}
0 & \to & G' \\
\downarrow & & \downarrow \\
G & \to & F_0' \\
\downarrow & & \downarrow \\
E & \to & 0
\end{array}$$

(5)

$$\begin{array}{ccc}
0 & \to & G \\
\downarrow & & \downarrow \\
F_0 & \to & E \\
\downarrow & & \downarrow \\
E & \to & 0
\end{array}$$

where $G \to G'$ is the natural projection and $F_0'$ is the pushout. Completion of the diagram (5) gives $\eta : 0 \to L \to F_0 \to F_0' \to 0$. Applying $\text{Hom}_X(-, L)$ to the top horizontal sequence gives,

$$\cdots \to \text{Ext}^1(E, L) \to \text{Ext}^1(F_0', L) \to \text{Ext}^1(G', L) \to \cdots$$

In the above sequence $\eta \to \eta'$ where $\eta' : 0 \to L \to G \to G' \to 0$ is split. By assumption $\text{Ext}^1(E, L) \cong H^1(X, E^\vee \otimes L) = 0$, thus $\eta$ splits. Therefore $F_0'$ is a direct sum of line bundles of rank $r - 1$, by Krull-Schmidt theorem [1]. This implies that $0 \to G' \to F_0' \to E \to 0$ is a 1-step resolution of $E$ which contradicts the minimality of the resolution (3). \qed
On any projective variety \( Z \), for a short exact sequence of vector bundles \( 0 \to F_0 \to F_1 \to F_2 \to 0 \) and any positive integer \( k \), there exists a resolution of \( k \)-th exterior power of \( F_2 \),

\[
0 \to \text{Sym}^k(F_0) \to \cdots \to \text{Sym}^{k-i}(F_0) \otimes \wedge^i F_1 \to \cdots \to \wedge^k F_1 \to \wedge^k F_2 \to 0
\]

We will call this resolution the \( \text{Sym} - \wedge \) sequence of index \( k \) associated to the given short exact sequence. There exists a similar resolution of \( k \)-th symmetric power of \( F_2 \) (by interchanging symmetric product and wedge product) which we will call \( \wedge - \text{Sym} \) sequence of index \( k \) associated to the given sequence.

\[
0 \to \wedge^k(F_0) \to \cdots \to \wedge^{k-i}(F_0) \otimes \text{Sym}^i F_1 \to \cdots \text{Sym}^k F_1 \to \text{Sym}^k F_2 \to 0
\]

For details see [18].

We will now prove a result from which Theorem 1.2 will follow.

**Theorem 3.2.** Let \( E \) be any rank \( k \) bundle (not necessarily ACM) on a smooth hypersurface \( X \subset \mathbb{P}^{n+1} \) with \( n \geq 2k + 1 \). Assume further that \( E \) satisfies the following two conditions,

1. \( H^i_*(X; E) = 0 \), \( i \in \{2, 3, \ldots, k+1\} \cup \{n-1\} \)
2. \( H^i_*(X; \wedge^m E) = 0 \), \( i = 2m - 1, 2m, \ldots, k+m \) for each \( m \in \{2, \ldots, k-1\} \)

Then \( E \) splits.

Despite the odd assumptions, the proof is very simple and we just use hypothesis of the theorem in \( \wedge - \text{Sym} \) sequence for various indices to prove certain cohomological vanishings (6), which is then used in a \( \text{Sym} - \wedge \) sequence to prove the theorem.

**Proof of theorem 3.2.** We write \( \wedge - \text{Sym} \) sequence of some index \( l \in \{2, \ldots, k\} \) for the short exact sequence (4),

\[
0 \to \wedge^l E(-d) \to \wedge^{l-1} E(-d) \otimes F_1 \to \cdots \\
\cdots \to E(-d) \otimes \text{Sym}^{l-1} F_1 \to \text{Sym}^l F_1 \to \text{Sym}^l G \to 0
\]

This breaks up into short exact sequences,

\[
0 \to J_{j-1,l} \to \wedge^{l-j} E(-d) \otimes \text{Sym}^j F_1 \to J_{j,l} \to 0
\]

where \( J_{0,l} = \wedge^l E(-d) \), \( J_{j,l} \) is defined inductively for \( j = 1, \ldots l - 1 \) and \( J_{l,l} = \text{Sym}^l G \).

By assumption in the theorem and the fact that \( F_1 \) is split, we get \( H^i_*(X; J_{j,l}) = 0 \), for \( i = 2l - j - 1, 2l - j, \ldots, k + l - j \). This implies

\[
H^i_*(X; \text{Sym}^l G) = 0 \text{ for } i = l - 1, l, \ldots, k
\]

Now we look at \( \text{Sym} - \wedge \) sequence of the index \( k (= \text{rank } E) \) for the sequence (4),

\[
0 \to \text{Sym}^k G \to \text{Sym}^{k-1} G \otimes F_0 \to \cdots G \otimes \wedge^{k-1} F_0 \to \wedge^k F_0 \to \wedge^k E \to 0
\]

\[\text{We were unable to find any standard terminology in the literature for the given resolution.}\]
This breaks up into short exact sequences,

$$0 \to M_{j-1} \to \operatorname{Sym}^{k-j}G \otimes \wedge^j F_0 \to M_j \to 0$$

where $M_0 = \operatorname{Sym}^k G$ and $M_j$ is defined inductively for $j = 1, \ldots, k$ as

$$M_j = \operatorname{coker}(M_{j-1} \to \operatorname{Sym}^{k-j}G \otimes \wedge^j F_0)$$

Note that $M_k = \wedge^k E = \mathcal{O}_X(e)$ for some $e \in \mathbb{Z}$. Using the vanishing given by (6) in sequence (7) (and the fact that $F_0$ are split bundles),

$$H^i_c(X, M_j) = 0 \text{ for } i = k - j - 1, k - j$$

Therefore the short exact sequence $0 \to M_{k-1} \to \wedge^k F_0 \to \wedge^k E \to 0$ splits. In particular, $M_{k-1}$ splits. This implies that the following sequence splits,

$$0 \to M_{k-2} \to G \otimes \wedge^{k-1} F_0 \to M_{k-1} \to 0$$

In particular, $G$ has a line bundle as a direct summand. Thus by lemma 3.1, $E$ splits.

\[ \square \]

**Corollary 3.3.** Let $E$ be a rank 3 ACM bundle on a smooth hypersurface $X$ with $\dim(X) \geq 7$, then $E$ splits.

**Proof.** We note that $\wedge^i E$ is ACM when $i = 1, 2, 3$. In particular, both the assumptions of theorem 3.2 are satisfied. Thus $E$ splits. \[ \square \]

**Corollary 3.4.** Let $E$ be a rank 4 ACM bundle on a smooth hypersurface $X$ with $\dim(X) \geq 9$, then $E$ splits.

**Proof.** As before, we note that $\wedge^i E$ is ACM when $i = 1, 3, 4$. By theorem 3.2, $E$ splits, if we can show that $H^i_c(X, \wedge^2 E) = 0$ for $i = 3, 4, 5, 6$. Since $E$ splits $\iff E^c(m)$ splits for some $m \in \mathbb{Z}$, so we can assume that $E^c$ is globally generated. Then there exists a section $s \in H^0(X, E^c)$ of proper codimension i.e. the zero locus $Z$ of $s$ has codimension 4 in $X$. This implies that there exists a resolution of $\mathcal{O}_Z$ (see [8], pp. 448),

$$0 \to \wedge^4 E \to \wedge^3 E \to \ldots \to E \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

We note that $Z$ is Cohen-Macaulay subscheme of $X$. A cohomological computation gives $H^i_c(X, \wedge^2 E) = 0$ for $i = 3, 4, 5, 6$ when $\dim(X) \geq 9$. \[ \square \]

**Remark:** It is easy to verify the hypothesis of theorem 3.2 for any rank 2 ACM bundle when $n \geq 5$ which provides another proof for this well known splitting result.

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