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# Some Properties of Distal Actions on Locally Compact Groups

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**Abstract**

We show equivalence of distality and pointwise distality of certain actions. We also show that a compactly generated locally compact group of polynomial growth has a compact normal subgroup $K$ such that $G/K$ is distal and the conjugacy action of $G$ on $K$ is ergodic; moreover, if $G$ itself is (pointwise) distal then $G$ is Lie projective. We prove a decomposition theorem for contraction groups of an automorphism under a certain condition. We give a necessary and sufficient condition for distality of an automorphism in terms of its contraction group. We compare classes of (pointwise) distal groups and groups whose closed subgroups are unimodular. In particular, we study relation between distality, unimodularity and contraction subgroups.

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# 1 Introduction

Let $\Gamma$ be a (topological) semigroup acting on a Hausdorff space $X$ by continuous self-maps. We say that the action of $\Gamma$ on $X$ is *distal* if for any two distinct points $x, y \in X$, the closure of $\{(\gamma(x), \gamma(y)) \mid \gamma \in \Gamma\}$ does not intersect the diagonal $\{(a, a) \mid a \in X\}$ and we say that the action of $\Gamma$ on $X$ is *pointwise distal* if for each $\gamma \in \Gamma$, the action of $\{\gamma^n\}_{n\in\mathbb{N}}$ on $X$ is distal. The notion of distality was introduced by Hilbert (cf. Ellis [10]) and studied by many in different contexts: see Ellis [10], Furstenberg [11] and Raja-Shah [28] and the references cited therein.

Let $G$ be a locally compact (Hausdorff) group and let $e$ denote the identity of $G$. Let $\Gamma$ be a semigroup acting continuously on $G$ by endomorphisms. Then the $\Gamma$-action on $G$ is distal if and only if $e \not\in \overline{\Gamma x}$ for all $x \in G \setminus \{e\}$. The group $G$ itself is said to be *distal* (resp. *pointwise distal*) if the conjugacy action of $G$ on $G$ is distal (resp. pointwise distal).

It can easily be seen that the class of distal groups is closed under compact extensions. Abelian groups, discrete groups and compact groups are obviously distal. Nilpotent groups, connected groups of polynomial growth are distal (cf. [29]): recall that a locally compact group $G$ with left Haar measure $\lambda_G$ is said to be a group of polynomial growth if for each relatively compact neighborhood $U$ of $e$ in $G$ there is a $k \in \mathbb{N}$ such that $\{\frac{\lambda_G(U^n)}{n^k} \mid n \geq 1\}$ is bounded. It may also be noted that p-adic Lie groups of type $R$ and p-adic Lie groups of polynomial growth are pointwise distal (cf. Raja [24] and [25]): pointwise distal groups are called noncontracting in Raja [24] and Rosenblatt [29].

Clearly, distal actions are pointwise distal but there are pointwise distal actions which are not distal (see Jaworski-Raja [19] and Rosenblatt [29] for instance). The following types of groups were studied in [21] and [22].

A locally compact (resp. discrete) group $\Gamma$ is called a generalized $\overline{FC}$-group (resp. polycyclic) if G has a series $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n = \{e\}$ of closed normal subgroups such that $\Gamma_i/\Gamma_{i+1}$ is a compactly generated group with relatively compact conjugacy classes (resp. $\Gamma_i/\Gamma_{i+1}$ is cyclic) for $i = 0, 1, ..., n-1$. Note that polycyclic groups and compactly generated groups of polynomial growth are generalized $\overline{FC}$-groups. More detailed results on generalized $\overline{FC}$-groups may be found in [22].

Proposition 1 of [22] shows that a generalized $\overline{FC}$-group $G$ contains a compact normal subgroup $K$ such that $G/K$ is a Lie group. We improve on this result and show that if a generalized $\overline{FC}$-group $G$ is pointwise distal, then it is Lie projective, that is, $G$ has arbitrarily small compact normal subgroups $K_i$ such that $G/K_i$ is a Lie group (see Theorem 3.2). This would enable Lie theoretic considerations on generalized $\overline{FC}$-groups that are pointwise distal.

Distal actions of generalized $\overline{FC}$-groups were considered in [19], [27] and in [31]. We quote some related results: For the action of a polycyclic group $\Gamma$ on a totally disconnected group and for the action of a generalized $\overline{FC}$-group on a compact metrizable group, distality and pointwise distality are equivalent, (see Corollary 2.4 of [19] and Theorem 4.1 of [27]). We prove this equivalence for actions of generalized $\overline{FC}$-groups on any locally compact group (Theorem 3.5).

We next look at the contraction group of automorphisms. Let $\mathrm{Aut}(G)$ denote the group of bi-continuous automorphisms of $G$. For $\alpha \in \mathrm{Aut}(G)$ and for a $\alpha$-invariant compact subgroup $K$, we define the $K$-contraction group of $\alpha$ by

$$C_K(\alpha) = \{g \in G \mid \alpha^n(g)K \to K\}.$$

The group $C_{\{e\}}(\alpha)$ is denoted by $C(\alpha)$ and is called the contraction group of $\alpha$. It is easy to see that $C(\alpha) = \{e\}$ if the $\{\alpha^n\}_{n\in\mathbb{N}}$-action on $G$ is distal. It is an interesting question to look at the converse: if the $\{\alpha^n\}_{n\in\mathbb{N}}$-action on $G$ is not distal, is it possible to find a $x \neq e$ such that $\alpha^n(x) \to e$ as $n \to \infty$? Using exponential map of Lie groups and results by Abels in [2], the converse can easily be seen to hold for (connected) Lie groups and, it was proved in case of totally disconnected groups by Baumgartner and Willis (cf. [3]). Recently the converse is also proved in case of compact groups by Jaworski (cf. [18]). Here we first establish that an automorphism $\alpha$ is distal (i.e. $\{\alpha^n\}_{n\in\mathbb{Z}}$-action is distal) if and only if $C_K(\alpha^{\pm 1}) = K$ for any $\alpha$-invariant compact subgroup $K$ (see Theorem 4.3) and using the recent result of [18] on compact groups, we prove that $\{\alpha^n\}_{n\in\mathbb{N}}$-action is distal if and only if $C(\alpha) = \{e\}$, for any automorphism $\alpha$ on a locally compact group (see Corollary 4.13). Main ingredient in the proof of the above results is a decomposition theorem of the type $C_K(\alpha) = C(\alpha)K$(see Theorem 4.4).

We further explore properties of distal groups. It can easily be seen that pointwise distal groups are unimodular: recall that a locally compact group $G$ is called unimodular if the left Haar measure is also right-invariant. But obviously the converse need not be true as any non-compact semisimple connected Lie group is unimodular but it is not pointwise distal. But closed subgroups of pointwise distal groups are also pointwise distal and hence, any pointwise distal group has the property that all its closed subgroups are unimodular. We now attempt to see how far one could progress in the converse direction. Main result in Section 5 is Theorem 5.2 which relates pointwise distality, contraction groups and unimodularity of closed subgroups. It may also be noted that such class of unimodular groups arises in the classification of groups that admit recurrent random walks (see [12] for any unexplained notions) and study of contraction subgroups on such groups plays a crucial role in [12]. Here, we prove that $C_K(\alpha)$ is relatively compact for any inner automorphism $\alpha$ and for any compact $\alpha$-invariant subgroup $K$ of $G$ if and only if closed subgroups of $G$ are unimodular (see Theorem 5.2).

The following Proposition is useful in reducing the action on general groups to that on metrizable groups.

**Proposition 1.1** *Let $\Gamma$ be a $\sigma$-compact locally compact group acting on a $\sigma$-compact locally compact group $G$ by automorphisms. Then for each countable collection $\{U_n\}$ of neighborhoods of the identity $e$ in $G$ there exists a $\Gamma$-invariant compact normal subgroup $L$ in $G$ such that $L \subset \cap U_n$ and $G/L$ is metrizable with a countable basis for its open sets.*

**Proof** Consider the semidirect product $\Gamma \ltimes G$ whose underlying space is the product space $\Gamma \times G$ with binary operation given by $(\alpha, x)(\beta, y) = (\alpha\beta, x\alpha(y))$ for all $\alpha, \beta \in \Gamma$ and $x, y \in G$. Then $\Gamma \ltimes G$ is a $\sigma$-compact locally compact group. Let $\{U_n\}$ be a countable collection of neighborhoods of $e$ in $G$. Then $\{\Gamma \times U_n\}$ is a countable collection of neighborhoods of the identity in $\Gamma \ltimes G$, hence by Theorem 8.7 of [16] we get that there is a compact normal subgroup $K$ of $\Gamma \ltimes G$ such that $K \subset \cap(\Gamma \times U_n)$ and $(\Gamma \ltimes G)/K$ is metrizable with a countable basis for its open sets.

Let $L = G \cap K$ where $G$ is identified with $\{e\} \times G$. Then it can easily be verified that $L$ is a compact normal subgroup of $G$ such that $L \subset \cap U_n$. Let $\varphi \colon \Gamma \ltimes G \to (\Gamma \ltimes G)/K$ be the canonical projection. Since $K$ is compact, $\varphi$ is a closed map. In particular, $\varphi(G)$ is a closed subgroup of $(\Gamma \ltimes G)/K$, hence $G/L \simeq \varphi(G)$. This shows that $G/L$ is metrizable with a countable basis for its open sets.

Since $K$ and $G$ are normal subgroups of $\Gamma \ltimes G$, $L = G \cap K$ is a normal subgroup of $\Gamma \ltimes G$ and hence $L$ is $\Gamma$-invariant. $\square$

# 2 Actions on Compact Groups

In this section we discuss actions of semigroups on a compact group. Let $(X, \mathcal{B}, m)$ be a probability space. The action of a semigroup of measure preserving transformations $\Gamma$ on $X$ is said to be ergodic if for any $\Gamma$-invariant set $B \in \mathcal{B}$, we have $m(B) = 0$ or 1. For a compact group $K$ and $\Gamma \subset \mathrm{Aut}(K)$, the $\Gamma$-action on the homogeneous space $K/H$ for any closed $\Gamma$-invariant subgroup $H$, ergodicity is defined with respect to the $K$-invariant probability measure on $K/H$.

Throughout this section, let $K$ denote a compact group and let $\Gamma$ denote a topological semigroup acting (continuously) on $K$ by automorphisms.

The following is a generalization of Proposition 2.1 of [27] to not necessarily metrizable groups for semigroup actions. Also, note that the proof given here is different and somewhat simpler.

**Proposition 2.1** *There exists a unique minimal closed* (*resp. closed normal*) $\Gamma$*-invariant subgroup* $C$ (*resp.* $C_1$) *of* $K$ *such that the* $\Gamma$*-action on* $K/C$ (*resp.* $K/C_1$) *is distal. Moreover, the* $\Gamma$*-action on* $C$ (*resp.* $C_1$) *is ergodic and* $C \subset C_1$.

*Moreover, if* $K$ *is metrizable, then* $C$ *is normal in* $K$ *and* $C$ *is also the largest closed* $\Gamma$*-invariant subgroup such that the* $\Gamma$*-action on* $C$ *is ergodic.*

**Proof** Without loss of any generality, we may assume that $\Gamma \subset \mathrm{Aut}(K)$. Let $\mathcal{K}$ be the set of compact subgroups $L$ of $K$ such that the $\Gamma$-action on $K/L$ is distal. Here $\mathcal{K}$ is nonempty as $K \in L$. Let $C = \cap\{L \mid L \in \mathcal{K}\}$. Then $C$ is a compact $\Gamma$-invariant subgroup. Now we show that $\Gamma$ acts distally on $K/C$, i.e. $C \in \mathcal{K}$. Let $xC \in K/C$. Suppose $C$ belongs to the closure of $\{\gamma(x)C \mid \gamma \in \Gamma\}$ in $K/C$. Then for every $L \in \mathcal{K}$, since $C \subset L$, we have that $L$ belongs to the closure of $\{\gamma(x)L \mid \gamma \in \Gamma\}$ in $K/L$ and since the $\Gamma$-action on $K/L$ is distal, $x \in L$, and hence $x \in C$. Therefore, the $\Gamma$-action on $K/C$ is distal.

Now we show that the $\Gamma$-action on $C$ is ergodic. Suppose $\pi$ is an irreducible unitary representation of $C$ such that $\pi_\Gamma = \{\pi \circ \gamma \mid \gamma \in \Gamma\}$ is finite up to unitary equivalence. Then there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that for any $\gamma \in \Gamma$, $\pi \circ \gamma$ is unitarily equivalent to $\pi \circ \gamma_i$ for some $i$. This implies that the finite-dimensional unitary representation $\tilde{\pi} = \oplus \pi \circ \gamma_i$ is $\Gamma$-invariant. Let $C' = \mathrm{Ker}(\tilde{\pi})$ and let $n$ be the dimension of $\tilde{\pi}$. Suppose $\gamma_d(x)C' \to C'$ in $C/C'$ for some $x \in C$. Then $\tilde{\pi}(\gamma_d(x)) \to I_n$ where $I_n$ is the trivial operator on $\mathbb{C}^n$. Since $\tilde{\pi}$ is $\Gamma$-invariant, $\tilde{\pi} \circ \gamma_d = u_d^{-1} \tilde{\pi} u_d$ for some $u_d \in U_n(\mathbb{C})$, hence

$$\tilde{\pi}(\gamma_d(x)) = u_d^{-1} \tilde{\pi}(x) u_d \to I_n.$$

Since $U_n(\mathbb{C})$ is compact, we get that $\tilde{\pi}(x) = I_n$, that is $x \in \mathrm{Ker}(\tilde{\pi}) = C'$. This shows that the $\Gamma$-action on $C/C'$ is distal. Since the $\Gamma$-action on $K/C$ is distal, it easily follows that the $\Gamma$-action on $K/C'$ is also distal. This proves that $C' \in \mathcal{K}$, hence the minimality of $C$ in $\mathcal{K}$ implies that $C = C'$. Thus, $\tilde{\pi}$ is the trivial unitary representation of $C$. Now it follows from Theorem 2.1 of [4] that the $\Gamma$-action on $C$ is ergodic.

Now we want to show the existence of a unique minimal closed normal $\Gamma$-invariant subgroup $C_1$ such that the $\Gamma$-action on $K/C_1$ is distal and the $\Gamma$-action on $C_1$ is ergodic. Take $\Gamma_1 = \Gamma.\mathrm{Inn}(K)$. This is a closed subsemigroup of $\mathrm{Aut}(K)$ as $\mathrm{Inn}(K)$ is a normal compact subgroup of $\mathrm{Aut}(K)$. Then from above we get that there exists a unique minimal closed $\Gamma_1$-invariant subgroup $C_1$ of $K$ such that the $\Gamma_1$-action on $K/C_1$ is distal and the $\Gamma_1$-action on $C_1$ is ergodic. Here, $C_1$ is normal in $K$ as $\mathrm{Inn}(K) \subset \Gamma_1$. Also, $\Gamma$ acts distally on $K/C_1$. Moreover, any $\Gamma$-invariant normal subgroup is also $\Gamma_1$-invariant and hence $C_1$ is the unique minimal closed normal $\Gamma$-invariant group such that $\Gamma$ acts distally on $G/C_1$. It is easy to verify that for any inner automorphism $\alpha$ defined by $x \in K$, $\pi \circ \alpha$ is unitarily equivalent to $\pi$, where equivalence is given by $\pi(x)$. This implies that the $\Gamma$-action on $C_1$ is ergodic (see also Lemma 2.4 of [27]). Also, the group $C$ as above is contained in $C_1$.

Now suppose $K$ is metrizable. Then there exits a dense $\Gamma$-orbit in $C_1$. This implies that there exists a dense $\Gamma$-orbit in $C_1/C$. But since the $\Gamma$-action on $C_1/C$ is distal, we have that $C_1 = C$ and hence $C$ is normal. Let $C_2$ be a $\Gamma$-invariant subgroup of $K$ such that the $\Gamma$-action on $C_2$ is ergodic. We need to show that $C_2 \subset C$. Then the $\Gamma$-action on $C_2C/C$ is also ergodic and hence it has a dense $\Gamma$-orbit. But the $\Gamma$-action on $K/C$ is distal and hence $C_2 \subset C$. This completes the proof. $\square$

**Remark 2.2** 1. As we assume that $\Gamma$ is a semigroup contained in $\mathrm{Aut}(K)$, let $[\Gamma]$ be the subgroup generated by $\Gamma$ in $\mathrm{Aut}(K)$, From Theorem 1 of [10], it is obvious that the $\Gamma$-action on $K$ is distal if and only if the $[\Gamma]$-action on $K$ is distal (see also the proof of Theorem 3.1 in [28]). By Theorem 2.1 of [4], the same statement holds for ergodic actions on a compact group. In this case we get in Proposition 2.1 that the $\Gamma$-action on $K/C$ (resp. on $C$) is distal (resp. ergodic) iff the $[\Gamma]$-action on $K/C$ (resp. on $C$) is distal (resp. ergodic).

2. If $\Gamma$ (and hence $[\Gamma]$) is compactly generated then metrizability of $K$ is not essential in the second assertion because in this case, by Proposition 1.1 we get that $K$ has arbitrarily small compact normal $\Gamma$-invariant subgroups $K_d$ such that $K/K_d$ is metrizable and we can argue as above for the $\Gamma$-action on each $K/K_d$ and get the desired assertion for $K$. $\square$

**Corollary 2.3** *Let $K$ be a compact metrizable group, $\Gamma \subset \mathrm{Aut}(K)$ be a semigroup and let $L$ be a $\Gamma$-invariant closed subgroup of $K$. Then the $\Gamma$-action on $K/L$ is distal if and only if the $\Gamma$-action is not ergodic on $H/C$, for any pair of compact $\Gamma$-invariant subgroups $C$ and $H$ of $K$ such that $L \subset C \subset H$ and $C \neq H$.*

**Proof** Suppose there exist $\Gamma$-invariant subgroups $H$ and $C$ with $L \subset C \subset H$ and $C \neq H$ such that the $\Gamma$-action on $H/C$ is ergodic. Then as $K$ is metrizable, so is $H/C$ and hence there exists a dense $\Gamma$-orbit in $H/C$, (see Theorem 5.6 of [32] for single transformation and the proof works for any semigroup action also). This in turn implies that the $\Gamma$-action on $H/C \simeq (H/L)/(C/L)$ is not distal. Since any factor action of a distal action is distal, the $\Gamma$-action is not distal on $H/L$, hence on $K/L$ (cf. [10]). We now prove the converse. Suppose the $\Gamma$-action on $K/L$ is not distal. By Proposition 2.1, there exists a closed normal $\Gamma$-invariant subgroup $H'$ of $K$ such that the $\Gamma$-action on $H'$ is ergodic and it is distal on $K/H'$. Let $H = H'L$ and $C = L$. Then the $\Gamma$-action on $H/C \simeq H'/H' \cap L$ is ergodic. Here $H \neq C$, otherwise, $H' \subset L$ and since the $\Gamma$-action on $K/H'$ is distal, the $\Gamma$-action on $K/L$ is also distal, a contradiction to our assumption. This completes the proof. $\square$

We say that $(\Gamma, K)$ satisfies DCC (*descending chain condition*) if for each sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact $\Gamma$-invariant subgroups of $K$ such that $K_n \supset K_{n+1}$, $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $K_n = K_{n_0}$ for all $n \geq n_0$.

The following proposition may be proved along the lines of Theorem 3.15 of [20] but here we give a simpler proof using Proposition 2.1.

**Lemma 2.4** *Suppose $\Gamma$ is a group of automorphisms of a compact metrizable group $K$ such that $(\Gamma, K)$ satisfies DCC. Then there exists a compact normal $\Gamma$-invariant subgroup $C$ of $K$ such that $K/C$ is a real Lie group, the $\Gamma$-action on $K/C$ is distal and the $\Gamma$-action on $C$ is ergodic.*

**Proof** From Proposition 2.1, there exists a compact normal $\Gamma$-invariant subgroup $C$ such that the $\Gamma$-action on $K/C$ is distal and the $\Gamma$-action on $C$ is ergodic. It is easy to verify that $(\Gamma, K/C)$ also satisfies DCC. Hence it is enough to prove that if the $\Gamma$-action on $K$ is distal and $(\Gamma, K)$ satisfies DCC, then $K$ is a Lie group.

Since $\mathrm{Inn}(K)$ is a compact normal subgroup of $\mathrm{Aut}(K)$, $\Gamma.\mathrm{Inn}(K)$ also acts distally on $K$. Note that if $(\Gamma, K)$ satisfies DCC, so does $(\Gamma.\mathrm{Inn}(K), K)$. Hence we may also assume that $\Gamma = \Gamma.\mathrm{Inn}(K)$. Note that since $K$ is a metric group and the action of $\Gamma$ is distal, it is not ergodic on any nontrivial subgroup of $K$ (see Corollary 2.3 or see [26]).

Suppose $K$ is not a real Lie group. By Theorem 2.1 of [4], there exists a non-trivial finite-dimensional unitary representation $\pi$ of $K$ such that $\pi$ is $\Gamma$-invariant (see also [27]). Let $K_1 = \mathrm{Ker}(\pi)$. Then $K_1 \neq K$ is a $\Gamma$-invariant compact normal subgroup of $K$ such that $K/K_1$ is a real Lie group as $\pi$ defines an injection of $K/K_1$ into a finite-dimensional unitary group. Proceeding this way we obtain a strictly decreasing sequence $\{K_n\}$ of $\Gamma$-invariant closed (normal) subgroups of $K$ such that $K_0 = K$. This is a contradiction to the hypothesis that $(\Gamma, K)$ satisfies DCC. Hence $K$ is a real Lie group. $\square$

**Lemma 2.5** *Suppose* $\Gamma$ *is a subgroup of* $\mathrm{Aut}(K)$ *and there exists a* $\Gamma$*-invariant compact normal subgroup* $C$ *in* $K$ *such that* $K/C$ *is a Lie group. Suppose also that* $\gamma \in \mathrm{Aut}(K)$ *is such that* $\gamma$ *normalizes* $\Gamma$ *and* $\{\gamma^n\}_{n\in\mathbb{Z}}$ *acts distally on* $K$*. Then there exists a compact normal* $\Gamma$*-invariant subgroup* $C' \subset C$ *such that* $\gamma(C') = C'$ *and* $K/C'$ *is a Lie group. In particular* $C'$ *is invariant under the group generated by* $\gamma$ *and* $\Gamma$*.*

**Proof** Let $C' = \cap_{n\in\mathbb{Z}}\gamma^n(C)$. It is clearly $\gamma$-invariant and normal in $K$. Moreover, for any $\gamma' \in \Gamma$, for each $n$, let $\gamma_n = \gamma^{-n}\gamma'\gamma^n$ then $\gamma_n \in \Gamma$ and $\gamma'(\gamma^n(C)) = \gamma^n(\gamma_n(C)) = \gamma^n(C)$. Hence $\gamma'(C') = C'$.

Let $G = K/C$. Then $G$ is a Lie group and we define a map $\phi : K/C' \to G^{\mathbb{Z}}$ as follows: $\phi(g) = (g_n)_{n\in\mathbb{Z}}$, where $g_n = \gamma^n(g)C \in G$. It is easy to see that $\phi$ is a continuous bijective homomorphism from $K/C'$ onto $\phi(K/C') \subset G^{\mathbb{Z}}$, and $\phi \circ \gamma = \alpha \circ \phi$, where $\alpha$ denotes the shift map on $G^{\mathbb{Z}}$. Here, $(\{\alpha^n\}_{n\in\mathbb{Z}}, G^{Z})$ satisfies DCC (cf. [20]). Now since $\{\gamma^n\}_{n\in\mathbb{Z}}$ acts distally on $K$, it also acts distally on $K/C'$ (cf. [28], Theorem 3.1). Now by Lemma 2.4, $K/C'$ is a Lie group. $\square$

**Proposition 2.6** *Suppose* $\Gamma$ *is a generalized* $\overline{FC}$*-group and its action on* $K$ *is pointwise distal. Then there exist compact normal* $\Gamma$*-invariant subgroups* $K_d$ *of* $K$ *such that* $\cap_d K_d = \{e\}$ *and each* $K/K_d$ *is a Lie group.*

**Proof** Let $\pi : \Gamma \to \mathrm{Aut}(K)$ be the natural map. Then $\pi(\Gamma)$ is isomorphic to $\Gamma/\ker\pi$ where $\ker\pi = \{\gamma \in \Gamma \mid \gamma(k) = k \text{ for all } k \in K\}$ is a closed normal subgroup of $\Gamma$. Therefore, $\Gamma/\ker\pi$, and hence $\pi(\Gamma)$, is a generalized $\overline{FC}$-group. Also, any $\pi(\Gamma)$-invariant group is $\Gamma$-invariant. Without loss of any generality we may assume that $\Gamma \subset \mathrm{Aut}(K)$. Moreover, since $\mathrm{Inn}(K)$ is a compact normal subgroup of $\mathrm{Aut}(K)$, $\Gamma.\mathrm{Inn}(K)$ is also a generalized $\overline{FC}$-group. Hence we may also assume that $\Gamma = \Gamma.\mathrm{Inn}(K)$. Then $\Gamma^0 = \mathrm{Inn}(K^0)$, the group of inner automorphisms of $K^0$. In particular, $\Gamma^0$ is compact. There exists a compact normal subgroup $L$ in $\Gamma$ such that $\Gamma/L$ is discrete and it has a polycyclic subgroup of finite index (see Proposition 2.8 of [19]). In particular, $\Gamma/L$ is finitely generated. Let $F = \{\gamma_i \in \Gamma \mid i = 1, \ldots, n\}$

be such that the followings hold: $\Gamma$ is generated by $L$ and $F$, there exist $L = \Gamma_0 \subset \ldots \subset \Gamma_n = \Gamma$, each $\Gamma_i$ is a normal subgroup of $\Gamma$ and it is generated by $\gamma_i$ and $\Gamma_{i-1}$, $1 \leq i \leq n$.

Here, $L \ltimes K$ is compact and hence Lie projective and there exist compact normal subgroups $H_d$ in $L \ltimes K$ such that $\cap_d H_d = \{e\}$ and each $(L \ltimes K)/H_d$ is a Lie group. Let $C_d = K \cap H_d$. Then each $C_d$ is a $L$-invariant normal subgroup in $K$, each $K/C_d$ is a Lie group and $\cap_d C_d = \{e\}$. Applying Lemma 2.5 successively for $\Gamma_i$ and $\gamma_{i+1}$, $0 \leq i \leq n-1$, we get that there exist compact normal $\Gamma$-invariant subgroups $K_d \subset C_d$ such that $K/K_d$ is a Lie group. Clearly, $\cap_d K_d = \{e\}$. □

# 3 Distal and Pointwise Distal Groups

In this section we compare distality and pointwise distality of certain actions. We know from Rosenblatt [29] that distality, pointwise distality and polynomial growth are all equivalent properties for a connected Lie group. This is not true in general as there are abelian extensions of compact groups that are pointwise distal but not distal (see Example 2.5 of [19]). There are also examples for $\mathbb{Z}$-extensions of compact groups which are not poinwise distal. Our first result generalizes the result of Rosenblatt [29] mentioned above.

**Theorem 3.1** *Let $G$ be a compactly generated locally compact group of polynomial growth. Then $G$ has a compact normal subgroup $C$ such that $G/C$ is distal and the conjugacy action of $G$ on $C$ is ergodic.*

**Proof** We first assume that $G$ is Lie group such that $G^0$ has no nontrivial compact normal subgroup. Since $G/G^0$ is discrete, it is enough if we show that the conjugacy action of $G$ on $G^0$ is distal. For each $g \in G$, let $\alpha_g$ denote the automorphism of $G^0$ defined by the conjugation action of $g$ restricted to $G^0$ and let $\mathrm{d}\alpha_g$ denote the corresponding Lie algebra automorphism of the Lie algebra $\mathcal{G}$ of $G^0$. Since $G^0$ has no nontrivial compact normal subgroup, Theorem 1 of [21] implies that the eigenvalue of $\mathrm{d}\alpha_g$ are of absolute value 1. Hence by Theorem 1 of [1] and Theorem 1.1 of [2], the conjugacy action of $G$ on $G^0$ is distal.

Suppose $G$ is not a Lie group. Then $G$ has a maximal compact normal subgroup $K$ such that $G/K$ is a Lie group (see [21]). Since $(G/K)^0$ has no nontrivial compact normal subgroup, from above, $G/K$ is distal. By Proposition 2.1 and Remark 2.2 (2), we have that $K$ has a maximal compact

$G$-invariant (normal) subgroup $C$ such that conjugacy action of $G$ on $K/C$ and hence, on $G/C$ is distal and the conjugacy action of $G$ on $C$ is ergodic. $\square$

We know that distal groups are pointwise distal and we also know that the converse is not true (see for instance, Example 2.5 of [19]). As noted above, pointwise distal Lie groups are distal. We now compare distality and pointwise distality of a particular class of locally compact groups.

**Theorem 3.2** *Let $G$ be a generalized $\overline{FC}$-group. Suppose $G$ is pointwise-distal. Then $G$ is Lie projective. Moreover, $G$ is distal.*

**Proof** It is easy to see from Theorem 3.1 of [28] and Theorem 9 of [29] that any Lie projective pointwise distal group is distal. Thus, it is sufficient to prove that any pointwise distal generalized $\overline{FC}$-group is Lie projective.

Suppose $G$ is a generalized $\overline{FC}$-group. Then there exists a maximal compact normal subgroup $K$ in $G$ such that $G/K$ is a Lie group (cf. [22]). Since the conjugacy action of $G$ on $K$ is pointwise distal, by Proposition 2.6 there exist closed normal subgroups $K_d$ of $K$ invariant under the conjugacy action of $G$ such that each $K/K_d$ is a Lie group and $\cap_d K_d = \{e\}$. That is, each $K_d$ is normal in $G$ and since $G/K = (G/K_d)/(K/K_d)$, where $G/K$ and $K/K_d$ are Lie groups, we get that $G/K_d$ is a Lie group for every $d$. This shows that $G$ is Lie projective. $\square$

Since any compactly generated locally compact group of polynomial growth is a generalized $\overline{FC}$-group (see [22]) we have the following:

**Corollary 3.3** *Let $G$ be a compactly generated locally compact pointwise-distal group of polynomial growth. Then $G$ is Lie projective. Moreover, $G$ is distal.*

**Remark 3.4** From Theorem 3.1 and Corollary 3.3, it follows that any compactly generated locally compact group of polynomial growth has a compact normal subgroup $C$ such that $G/C$ is distal and Lie projective and the conjugacy action of $G$ on $C$ is ergodic. $\square$

We next compare distality and pointwise distality of an action of a generalized $\overline{FC}$-group on a locally compact group. Note that it is proved in case of metrizable compact groups in [26] by a different method.

**Theorem 3.5** *Let $G$ be a locally compact group and let $\Gamma$ be a generalized $\overline{FC}$-group acting on $G$ by automorphisms. Then the $\Gamma$-action on $G$ is distal if and only if the $\Gamma$-action on $G$ is pointwise distal.*

**Proof** One way implication "only if" is obvious. Now suppose the $\Gamma$-action on $G$ is pointwise distal. Let $G^0$ be the connected component of the identity in $G$. Then there is a maximal compact normal subgroup $K$ of $G^0$ such that $G^0/K$ is a Lie group. Since $K$ is maximal, $K$ is characteristic in $G$ and hence $\Gamma$-invariant. Since the $\Gamma$-action on $K$ is pointwise distal. By Lemma 2.6, $K$ has compact normal $\Gamma$-invariant subgroups $K_d$ such that $\cap K_d = \{e\}$ and $K/K_d$ is a Lie group. Since the $G^0$-action on $K$ is by inner automorphisms of $K$, each $K_d$ is normal in $G^0$ and hence $G^0/K_d$ is a Lie group. By Theorem 3.1 of [28], the $\Gamma$-action on each $G^0/K_d$ is pointwise distal and hence, it is distal by Theorem 1.1 of [2]. Since this is true for each $d$ and $\cap_d K_d = \{e\}$, we get that the $\Gamma$-action on $G^0$ is distal. Now it is enough to prove that the $\Gamma$-action on $G/G^0$ is distal. We know that the $\Gamma$-action on $G/G^0$ is pointwise distal by Theorem 3.3 of [28]. Hence we may assume that $G$ is a totally disconnected group. Since the connected component $\Gamma^0$ of $\Gamma$ acts trivially on $G$ and $\Gamma/\Gamma^0$ is also a generalised $\overline{FC}$-group and its action on $G$ is pointwise distal, we may assume that $\Gamma$ is totally disconnected. By [22], $\Gamma$ contains a compact normal subgroup $L$ such that $\Gamma/L$ is a Lie group. Since $\Gamma$, and hence, $\Gamma/L$ is also totally disconnected, $\Gamma/L$ is discrete. As $\Gamma/L$ is a discrete generalized $\overline{FC}$-group, $\Gamma/L$ has a polycyclic subgroup of finite index (cf. [22]), Hence by Lemma 2.3 of [19] (which is also valid for any non-metric group by [17], see also [19]), we get that the $\Gamma$-action of $G$ is distal. This completes the proof. □

# 4 Distality and Contraction Groups

In this section we get a necessary and sufficient condition for distality of an automorphism of a locally compact group in terms of its contraction group. Let $\alpha \in \mathrm{Aut}(G)$. We recall that $\alpha$ is distal on $G$ if the $\{\alpha^n\}_{n\in\mathbb{Z}}$-action on $G$ is distal.

Recall that for an $\alpha$-invariant subgroup $K$ of $G$, the $K$-contraction group of $\alpha$ is defined as $C_K(\alpha) = \{x \in G \mid \alpha^n(x)K \to K \text{ as } n \to \infty\}$ and we denote $C_{\{e\}}(\alpha)$ by $C(\alpha)$ which is known as the contraction group of $\alpha$.

It is evident that for any automorphism $\alpha$, if the $\{\alpha^n\}_{n\in\mathbb{N}}$-action is distal, then $C(\alpha)$ is trivial. We prove the converse in this section. Recently, the converse has been proved in case of compact groups (cf. [18]). We first note the converse for (connected) Lie groups.

**Proposition 4.1** *Let $G$ be a real Lie group and let $\alpha \in \mathrm{Aut}(G)$. Then $C(\alpha) = \{e\}$ if and only if the $\{\alpha^n \mid n \in \mathbb{N}\}$-action on $G$ is distal. In particular, $\alpha$ is distal on $G$ if and only if $C(\alpha^{\pm 1}) = \{e\}$.*

**Proof** Suppose $C(\alpha) = \{e\}$. Since $G/G^0$ is discrete, the $\{\alpha^n \mid n \in \mathbb{N}\}$-action on $G/G^0$ is distal. Hence we may assume that $G$ is a connected Lie group. Let $\mathcal{G}$ be Lie algebra of $G$ and $\mathrm{d}\alpha$ be the differential of $\alpha$ on $\mathcal{G}$. We first consider the case when $G$ is a compact connected abelian group. Since $C(\alpha) = \{e\}$, eigenvalues of $\mathrm{d}\alpha$ are all of absolute value greater than or equal to one. Since $G$ is a compact abelian Lie group, $|\det(\mathrm{d}\alpha)| = 1$. This implies that eigenvalues of $\mathrm{d}\alpha$ are of absolute value one. Thus, $\alpha$ is distal on $G$ (cf. [1] and [2]).

Now assume that $G$ is any connected real Lie group. Then there are two $\alpha$-invariant subspaces $V_1$ and $V_0$ such that $\mathcal{G} = V_1 \oplus V_0$, $V_1 = C(\mathrm{d}\alpha^{-1})$ is the Lie algebra of $C(\alpha^{-1})$ and $\mathrm{d}\alpha$ restricted to $V_0$ has eigenvalues of absolute value one (see Proposition 3.2.6 of [15]). Since $\mathcal{G} = V_1 \oplus V_0$, $V_1$ is an ideal in $\mathcal{G}$ (see Proposition 3.2.6 of [15]). Let $H = \overline{C(\alpha^{-1})}$. Then $H$ is a closed connected nilpotent $\alpha$-invariant normal subgroup of $G$. Since eigenvalues of $\mathrm{d}\alpha$ are of absolute value one on $\mathcal{G}/V_1 \simeq V_0$, eigenvalues of $\mathrm{d}\alpha$ on the Lie algebra of $G/H$ (which is a factor of $V_0$) are of absolute value one. This implies that $\alpha$ is distal on $G/H$ (cf. [1] and [2]). So, it is sufficient to prove that the $\{\alpha^n \mid n \in \mathbb{N}\}$-action on $H$ is distal. Let $K$ be the maximal central torus in $H$. Then $K$ is $\alpha$-invariant and $C(\alpha^{-1})K$ is dense in $H/K$ which is a simply connected nilpotent Lie group. This implies that $H = C(\alpha^{-1})K$. Since $C(\alpha) = \{e\}$, it follows from the first case that $\alpha$ is distal on $K$. So, it is sufficient to prove that the $\{\alpha^n \mid n \in \mathbb{N}\}$-action is distal on $H/K$. Since $H = C(\alpha^{-1})K$, $H/K$ is a simply connected nilpotent Lie group such that $\alpha^{-n}(x) \to e$ as $n \to \infty$ for all $x \in H/K$. Let $V$ be the Lie algebra of $H/K$ and $\beta \in GL(V)$ be the differential of $\alpha$ on $H/K$. Since $H/K$ is a simply connected nilpotent Lie group, exponential is a diffeomorphism of $V$ onto $H/K$. So it is sufficient to prove that the $\{\beta^n \mid n \in \mathbb{N}\}$-action on $V$ is distal. Since $\alpha^{-n}(x) \to e$ as $n \to \infty$ for all $x \in H/K$, $\beta^{-n}(v) \to 0$ as $n \to \infty$ for all $v \in V$. This implies that $||\beta^{-n}|| \to 0$ as $n \to \infty$. Now for any $v \in V$,

$$\|v\| \leq (\|\beta^{-n}\|)(\|\beta^n(v)\|)$$

for all $n \in \mathbb{N}$. This implies that the $\{\beta^n \mid n \in \mathbb{N}\}$-action on $V$ is distal. $\square$

Before we proceed to consider the general case, we first look at following particular cases.

**Proposition 4.2** *Suppose $G$ is a closed subgroup of a linear group $GL(n, \mathbb{F})$ over a local field $\mathbb{F}$ or $G$ is a real Lie group. Then the following are equivalent:*

(1) *$C(\alpha) = \{e\}$ for any inner automorphism $\alpha$ of $G$;*

(2) *$G$ is pointwise distal;*

(3) *$G$ is distal.*

*In addition if $G$ is compactly generated and $\mathbb{F}$ is non-archimedian, then* (1)–(3) *are equivalent to*

(4) *$G$ contains arbitrarily small compact open invariant subgroups.*

**Proof** It is clear that for any locally compact group $G$, (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Thus, to prove (1)-(3) are equivalent, it is sufficient to prove that (1) $\Rightarrow$ (3). If $\mathbb{F}$ is an archimedian local field, then closed subgroups of $GL(n, \mathbb{F})$ are real Lie groups. Thus, it is sufficient to consider the case of real Lie groups and the case of closed subgroups of $GL(n, \mathbb{F})$ when $\mathbb{F}$ is a non-archimedian local field. Assume that $G$ is a real Lie group. Then equivalent of (1) and (2) follows from Proposition 4.1 and equivalence of (2) and (3) follows from Theorem 1.2 of [2].

We now assume that $\mathbb{F}$ is non-archimedian and $G$ is a closed subgroup of $GL(n, \mathbb{F})$. Then $G$ is a totally disconnected group, hence (1) $\Rightarrow$ (2) follows from Proposition 2.1 of [19]. We now prove that (2) $\Rightarrow$ (3). Assume that $G$ is pointwise distal. Let $\Phi\colon G \to M(n, \mathbb{F})$ be such that $\Phi(g) = g - I$ for all $g \in G$. Then $\Phi$ is a homeomorphism of $G$ onto $\Phi(G)$ endowed with the topology induced from $M(n, \mathbb{F})$. Let $V$ be the smallest subspace of $M(n, \mathbb{F})$ such that $V \cap \Phi(G)$ is a neighborhood of 0 in $\Phi(G)$. We now claim that for any $g \in G$, $V$ is $g$-invariant and $(g^m v g^{-m})_{m\in\mathbb{Z}}$ is relatively compact for all $v \in V$. Let $g \in G$. Since $\mathbb{F}$ is non-archimedean, $G$ is a totally disconnected locally compact group. Then by Proposition 2.1 of [19], there is a basis of compact open subgroups $\{K_i\}$ at $e$ in $G$ such that $gK_ig^{-1} = K_i$ for all $i \geq 1$. Since $V \cap \Phi(G)$ is a neighborhood of 0 in $\Phi(G)$, there exists a $i$ such that $\Phi(K_i) \subset V$. Let $W$ be the subspace of $V$ spanned by $\Phi(K_i)$. Then

$\Phi(K_i) \subset W \cap \Phi(G)$ is a neighborhood of 0 in $\Phi(G)$. Since $V$ is the smallest such subspace $V = W$. For any $v \in \Phi(K_i)$, $g^m v g^{-m} \in \Phi(K_i)$ for all $m \in \mathbb{Z}$ and hence $(g^m v g^{-m})_{m\in\mathbb{Z}}$ is relatively compact as $\Phi(K_i)$ is compact in $V$. Since $V = W$ is spanned by $\Phi(K_i)$, we get that $(g^m v g^{-m})_{m\in\mathbb{Z}}$ is relatively compact for all $v \in V$. Since $g\Phi(K_i)g^{-1} = \Phi(gK_ig^{-1}) = \Phi(K_i)$ and $W = V$ is spanned by $\Phi(K_i)$, $gVg^{-1} = V$. Thus, $V$ is $G$-invariant and $(g^m v g^{-m})_{m\in\mathbb{Z}}$ is relatively compact for any $g \in G$ and $v \in V$.

Let $x \in G$ be such that $g_m x g_m^{-1} \to e$. Then $g_m v g_m^{-1} \to 0$ in $\Phi(G)$ for $v = \Phi(x)$. Since $V \cap \Phi(G)$ is a neighborhood of 0 in $\Phi(G)$, $g_m v g_m^{-1} \in V \cap \Phi(G) \subset V$ for large $m$. Since $V$ is $G$-invariant, we get that $v \in V$.

Let $\Psi\colon G \to GL(V)$ be such that $\Psi(g)(w) = gwg^{-1}$ for all $g \in G$ and for all $w \in V$. Then Lemma 3.2 of [12] implies that there exist a compact subgroup $K$ and a unipotent subgroup $U$ of $GL(V)$ such that $K$ normalizes $U$ and $\Psi(G) \subset K \ltimes U$. Now $\Psi(g_m)(v) = g_m v g_m^{-1} \to 0$. Since $K$ is compact, 0 is a limit point of $U(v)$. Since $U$ is a unipotent group, by Kolchin's Theorem there exists a flag $0 = V_0 \subset \cdots \subset V_n = V$ of $U$-invariant subspaces of $V$, that is, dimension of each $V_i$ is $i$. This implies that $U$ is trivial on $V_i/V_{i-1}$ for $1 \leq i \leq n$. Considering the action of $U$ on $V/V_{n-1}$ and using 0 is a limit point of $U(v)$, we get that $v \in V_{n-1}$. Proceeding this way one can show that $v = 0$. This implies that $x = I$. This proves (3).

In addition if $G$ is compactly generated, then the statement that (1) implies (4) follows from Lemma 3.2 of [12]. $\square$

**Theorem 4.3** *Let $G$ be a locally compact group and $\alpha \in \mathrm{Aut}(G)$. Then the following are equivalent:*

(1) *$\alpha$ is distal;*

(2) *for any $\alpha$-invariant compact subgroup $L$ of $G$, $C_L(\alpha^{\pm 1}) = L$.*

We first prove a decomposition theorem for contraction groups, under certain conditions. Note that in a locally compact group $G$, there exists a unique maximal compact normal subgroup in $G^0$ and it is characteristic in $G$.

**Theorem 4.4** *Let $G$ be a locally compact group and let $\alpha \in \mathrm{Aut}(G)$. Then $C_L(\alpha) = C(\alpha)L$ for any $\alpha$-invariant compact subgroup $L$ of $G$ if any one of the following conditions is satisfied:*

1. $G^0$ *is a Lie group.*

2. *For the maximal compact normal subgroup* $K$ *of* $G^0$, *the* $\{\alpha^n\}_{n\in\mathbb{N}}$*-action on* $K$ *is distal.*

The above decomposition theorem generalizes results for (connected) Lie groups (cf. [14]) and for totally disconnected groups (cf. [3], [17]) as these groups satisfy condition (1) above. However, it is not true for all compact groups if the automorphism is not distal; see [18] for a counter example.

Towards the proof of the theorem, we prove following preliminary results.

**Proposition 4.5** *Let* $G$ *be a connected locally compact group,* $\alpha \in \mathrm{Aut}(G)$ *and let* $K$ *be the maximal compact normal subgroup of* $G$*. Suppose the* $\{\alpha^n\}_{n\in\mathbb{N}}$*-action on* $K$ *is distal. Then* $C(\alpha)$ *is closed and for any compact* $\alpha$*-invariant subgroup* $L$ *of* $G$, $C_L(\alpha) = L \ltimes C(\alpha)$.

**Proof** We know that $G/K$ is a Lie group without any compact cental subgroup of positive dimension. Let $\alpha'$ be the automorphism on $G/K$ induced by $\alpha$ which is defined as $\alpha'(xK) = \alpha(x)K$ for all $x \in G$. Then $C(\alpha')$ is closed in $G/K$ and it is a simply connected nilpotent group (cf. [9]). Therefore, $C_K(\alpha)$ is closed in $G$. Since the $\{\alpha^n\}_{n\in\mathbb{N}}$-action, and hence, the $\{\alpha^n\}_{n\in\mathbb{Z}}$-action on $K$ is distal, $C(\alpha)\cap K$ is trivial.

We first assume that $G$ is a Lie group. Then $C_K(\alpha) = KC(\alpha)$, (cf. [14], Theorem 2.4). Then $\overline{C(\alpha)} \subset C_K(\alpha)$ as the latter group is closed. Moreover, $\overline{C(\alpha)}$ is a closed connected nilpotent Lie subgroup; let $H$ denote the maximal compact (central) subgroup of it. Then $\alpha(H) = H$ and it is normal in $C_K(\alpha)$. Moreover, since $C_K(\alpha)/K = C(\alpha')$ is simply connected and nilpotent, it has no non-trivial compact subgroups and hence $H \subset K$. Now take $A = \{x_t\}_{t\in\mathbb{R}}$ be a one-parameter subgroup in $C(\alpha)$. Then either $A$ is closed in $G$ or its closure $\bar{A}$ is compact (cf. [13]); in the latter case it is contained in $H$, but since $C(\alpha)\cap H \subset C(\alpha)\cap K = \{e\}$, $x_t = e$ for all $t \in \mathbb{R}$ and hence $\{x_t\}_{t\in\mathbb{R}}$ is closed. This implies that $C(\alpha)$ itself is closed (cf. [13]). Moreover, since $C(\alpha)$ is a closed simply connected nilpotent group, it does not contain any non-trivial compact subgroup, we have $C(\alpha)\cap L = \emptyset$ and hence $C_L(\alpha) = L \ltimes C(\alpha)$.

Suppose $G$ is a connected (not necessarily Lie) group. Since the action of $\Gamma = \{\alpha^n\}_{n\in\mathbb{Z}} \ltimes \mathrm{Inn}(K)$ on $K$ is distal, there exist $\Gamma$-invariant compact normal subgroups $K_i$ of $K$ such that $\cap_i K_i = \{e\}$ and $K/K_i$ are Lie groups (see Proposition 2.6). Since $G$ is connected and $K$ is a compact normal subgroup of $G$, the action of $G$ on $K$ is by inner automorphisms of $K$. This implies that

$K_i$'s are normal in $G$. Since $G/K$ and $K/K_i$ are real Lie groups, $G/K_i$'s are connected real Lie groups. From above, $C_{K_i}(\alpha)$ is closed for each $i$. It follows from Theorem 2.4 of [14] that for each $i$, $C_{LK_i}(\alpha) = C_{K_i}(\alpha)L$, which in turn is closed. Moreover, it is easy to see that since $\cap_i K_i = \{e\}$. $C(\alpha) = \cap_i C_{K_i}(\alpha)$ and $C_L(\alpha) = \cap_i C_{LK_i}(\alpha)$. Hence $C(\alpha)$ is closed and $C_L(\alpha) = C(\alpha)L$. Now we have from above that $(C(\alpha)K_i/K_i) \cap (LK_i/K_i) = \{K_i\}$ in $G/K_i$. Hence $C(\alpha) \cap L = \{e\}$. Therefore, $C_L(\alpha) = L \ltimes C(\alpha)$. In particular, since $K$ is normal in $G$, we get that $C_K(\alpha) = K \times C(\alpha)$. □

**Lemma 4.6** *Let $G$ be a locally compact group and $\alpha \in \mathrm{Aut}(G)$. Suppose there is a directed family $\{K_i\}_{i\in I}$ of compact $\alpha$-invariant subgroups such that $\cap_i K_i = \{e\}$ and $K_i \subset K_j$ for $i > j$. Then for any $\alpha$-invariant compact subgroup $L$ of $G$ such that $C_L(\alpha) \subset C_{K_i}(\alpha)L$ and $C_{K_i}(\alpha) \cap L \subset K_i$ for all $i \in I$, we have $C_L(\alpha) = C(\alpha)L$.*

**Proof** Let $g \in C_L(\alpha)$. Then $g = x_i a_i$ for $x_i \in C_{K_i}(\alpha)$ and $a_i \in L$. Passing to a subnet, we may assume that $a_i \to a$ in $L$ and hence $x_i \to x$ for some $x \in G$. Therefore we have that $g = xa$. We now claim that $x \in C(\alpha)$ which would complete the proof.

Let $U$ be a neighborhood of $e$ in $G$. Then there is a neighborhood $V$ of $e$ in $G$ such that $\overline{VV} \subset U$. Since $\cap_i K_i = \{e\}$, there is a $j \in I$ such that $K_j \subset V$. For $i \geq j$, $K_i \subset K_j$ and $x_i a_i = g = x_j a_j$, hence $C_{K_i}(\alpha) \subset C_{K_j}(\alpha)$ and $x_j^{-1}x_i = a_j a_i^{-1} \in C_{K_j}(\alpha)C_{K_i}(\alpha) \cap L \subset C_{K_j}(\alpha) \cap L \subset K_j$. This implies that $x_i = x_j b_i$ for some $b_i \in K_j$ for all $i \geq j$.

Since $K_j \subset V$, there is a $N$ such that $\alpha^n(x_j) \in V$ for all $n \geq N$. This implies for $i \geq j$, that $\alpha^n(x_i) = \alpha^n(x_j)\alpha^n(b_i) \in VK_j \subset VV$ for all $n \geq N$. Since $x_i \to x$, $\alpha^n(x) \in \overline{VV} \subset U$ for all $n \geq N$. Therefore, $\alpha^n(x) \to e$ and $x \in C(\alpha)$. □

**Lemma 4.7** *Let $G$ be a locally compact group such that $G^0$ is a Lie group and let $\alpha \in \mathrm{Aut}(G)$. Then there exists an open almost connected subgroup $H$ in $G$ such that $H = C \times G^0$, where $C$ is a compact totally disconnected subgroup contained in $Z(G^0)$ and the following holds: If $c \in C$ is such that $\alpha(c) \in H$, then $\alpha(c) \in C$.*

**Proof** Since $G^0$ is a Lie group, there exists a neighborhood $V$ of the identity $e$ in $G$ such that $V \cap G^0$ does not contain any nontrivial subgroup. For $\alpha \in \mathrm{Aut}(G)$, choose a neighborhood $U$ of $e$ in $G$ such that $U\alpha(U) \subset V$. Let $M$ be an open almost connected subgroup in $G$. Then $M$ is Lie projective and it has a compact normal subgroup $C$ contained in $U$ such that $M/C$ is a Lie group. Therefore, $CG^0$ is open and we take $H = CG^0$. Moreover, $C \cap G^0 \subset U \cap G^0 \subset V \cap G^0$, and hence $C \cap G^0 = \{e\}$. This implies that $C$ is totally disconnected and $H = C \times G^0$, hence $C \subset Z(G^0)$. Let $c \in C$ be such that $\alpha(c) \in H$. Then $\alpha(c) = c_1 g_1 = g_1 c_1$ for some $c_1 \in C$ and $g_1 \in G^0$. Then $g_1 = c_1^{-1}\alpha(c) = \alpha(c)c_1^{-1} \in C\alpha(C) \cap G^0$. Also, $g_1^n = c_1^{-n}\alpha(c^n) \in C\alpha(C) \cap G^0$. Hence $g_1$ generates a compact group in $C\alpha(C) \cap G^0$ which is contained in $U\alpha(U) \cap G^0$. Hence $g_1 = e$, i.e. $\alpha(c) = c_1 \in C$. This completes the proof. $\square$

**Proposition 4.8** *Let $G$ be a locally compact group and let $K$ be the maximal compact normal subgroup of $G^0$. Let $\alpha \in \mathrm{Aut}(G)$. Then $C_{G^0}(\alpha) = C_K(\alpha)G^0$, where $C_{G^0}(\alpha) = \{x \in G \mid \alpha^n(x)G^0 \to G^0 \text{ in } G/G^0\}$. Moreover, if $G^0$ is a Lie group, then $C_{G^0}(\alpha) = C(\alpha)G^0$.*

**Proof** We know that $K$ is characteristic in $G$ and $G^0/K$ has no non-trivial compact normal subgroup hence it is a Lie group. Also $C_K(\alpha)/K$ is the same as $C(\alpha')$, where $\alpha' \in \mathrm{Aut}(G/K)$ is the automorphism induced by $\alpha$. Hence it is enough to assume that $G^0$ is a Lie group and prove that $C_{G^0}(\alpha) \subset C(\alpha)G^0$ as this would imply both the assertions in the statement. Let $x \in C_{G^0}(\alpha)$. Then $\alpha^n(x)G^0 \to G^0$ in $G/G^0$ as $n \to \infty$. Let $H = C \times G^0$ be an open subgroup in $G$ as in Lemma 4.7. There exists $N \in \mathbb{N}$ such that $\alpha^n(x) \in H$ for all $n \geq N$. Let $y \in H$ be such that $y = \alpha^N(x)$. Then $y \in C_{G^0}(\alpha)$ and $\alpha^n(y) \in H$ for all $n$. Then $y = cg = gc$ for $c \in C$ and $g \in G^0$. Then $\alpha^n(y) = \alpha^n(c)\alpha^n(g) = \alpha^n(g)\alpha^n(c) \in H$. Since $G^0$ is $\alpha$-invariant, $\alpha^n(c) \in H$ and hence $\alpha^n(c) \in C$. Moreover, as $\alpha^n(y)G^0 \to G^0$, we have that $\alpha^n(c)G^0 \to G^0$ and hence $\alpha^n(c) \to e$ as $C \cap G^0 = \{e\}$. In particular, $c \in C(\alpha)$ and $y = cg \in C(\alpha)G^0$. Now $x = \alpha^{-N}(y) \in C(\alpha)G^0$. This completes the proof. $\square$

We also need the following simple result on compact groups which may be known.

**Lemma 4.9** *Let $C$ be a compact group and let $K$ be a closed normal subgroup of $C$. Suppose $K'$ is a closed normal subgroup of $K$ such that $K/K'$ is a Lie group. Then for $L = \cap_{c \in C} cK'c^{-1}$, $K/L$ is a Lie group.*

**Proof** Since $C$ is compact, it is Lie projective. Let $\{C_d\}$ be the collection of closed normal subgroups of $C$ such that $C/C_d$ are Lie groups and $\cap_d C_d = \{e\}$. Then $C'_d = K \cap C_d$ is normal in $C$ and $K/C'_d$ are Lie groups. Since $\cap_d C'_d = \{e\}$

and $K/K'$ is a Lie group, we get that $C'_d \subset K'$ for some $d$. This implies that $C'_d \subset L$, hence $K/L$ is a Lie group. $\square$

**Proof of Theorem 4.4** Let $G$, $\alpha$ and $L$ be as in the hypothesis. We also know that $C_{LG^0}(\alpha) = C_{G^0}(\alpha)L$ from the result on the totally disconnected group $G/G^0$ (see Theorem 3.8 of [3] which is valid for non-metrizable groups by [17]). If $G^0$ is a Lie group, from Proposition 4.8, $C_{G^0}(\alpha) = C(\alpha)G^0$, hence $C_L(\alpha) = C(\alpha)(C_L(\alpha) \cap G^0)L = C(\alpha)(C_{L\cap G^0}(\alpha) \cap G^0)L = C(\alpha)L$ from the result on Lie groups.

Now let $K$ be the maximal compact normal subgroup of $G^0$ and assume that the $\{\alpha^n\}_{n\in\mathbb{N}}$-action on $K$ is distal. Using $C_{G^0}(\alpha) = C_K(\alpha)G^0$ from Proposition 4.8, we get that $C_L(\alpha) \subset C_K(\alpha)G^0L$ and hence,

$$C_L(\alpha) = (C_K(\alpha)G^0 \cap C_L(\alpha))L. \tag{1}$$

Now suppose

$$C_K(\alpha) = C(\alpha)K. \tag{2}$$

Also, since the $\{\alpha^n\}_{n\in\mathbb{N}}$-action on $K$ is distal, from Proposition 4.5

$$C_L(\alpha) \cap G^0 = (C(\alpha) \cap G^0)(L \cap G^0) \subset C(\alpha)L. \tag{3}$$

Hence using (2), (3), as $K \subset G^0$, (1) gives

$$C_L(\alpha) = (C(\alpha)G^0 \cap C_L(\alpha))L = C(\alpha)(G^0 \cap C_L(\alpha))L = C(\alpha)L.$$

Thus, we only need to prove (2).

If $G^0$ is Lie group, then from above $C_K(\alpha) = C(\alpha)K$. Now we assume that $G$ is any locally compact group. Since $G^0/K$ is a Lie group, by Lemma 4.7, there exist an open subgroup $H$ in $G$ containing $K$ and a compact totally disconnected subgroup $C_1$ of $H/K$ such that $H/K = C_1 \times (G^0/K)$ and if $c \in C_1$ with $\alpha'(c) \in H/K$, then $\alpha'(c) \in C_1$ where $\alpha'$ is the automorphism induced by $\alpha$ on $G/K$. Let $C$ be the compact subgroup of $G$ containing $K$ such that $C/K = C_1$. If $c \in C$ is such that $\alpha(c) \in H$, then from above, we have that $\alpha(c) \in C$.

Let $x \in C_K(\alpha)$. As $H$ is open and $K \subset H$, there exists $N$ such that $\alpha^n(x) \in H$, for all $n \geq N$. Hence for $y = \alpha^N(x) \in H$, $\alpha^n(y) \in H$ for all $n \in \mathbb{N}$. It is enough to show that $y \in C(\alpha)K$ as $x = \alpha^{-N}(y)$ and $C(\alpha)K$ is $\alpha$-invariant. Here $y = cg$ where $c \in C$ and $g \in G^0$. Now $\alpha^n(y) = \alpha^n(c)\alpha^n(g) \in H$. Therefore, $\alpha^n(c) \in H$ as $G^0$ is $\alpha$-invariant. Now from above, $\alpha^n(c) \in C$, for all $n$. Hence $\alpha^n(y)K = \alpha^n(c)K\alpha^n(g)K \to K$ in $G/K$. Since $H/K = C_1 \times (G^0/K)$, we get that $\alpha^n(c)K \to K$ in $C_1 = C/K$ and $\alpha^n(g)K \to K$ in $G^0/K$. Now we have $y = cg$, $c \in C \cap C_K(\alpha)$ and $g \in G^0 \cap C_K(\alpha) = (C(\alpha) \cap G^0)K$ by Proposition 4.5. Replacing $y$ by $yg^{-1}$, we may assume that $y \in C \cap C_K(\alpha)$ such that $\alpha^n(y) \in C$ for all $n \in \mathbb{N}$ and $K \subset C$ and $C/K$ is totally disconnected and compact.

We first suppose that $G$ is second countable. Since the $\{\alpha^m\}_{m\in\mathbb{N}}$-action on $K$ is distal, we choose $K_n$ to be $\alpha$-invariant compact normal subgroups of $K$, $K_{n+1} \subset K_n$ and $K/K_n$ is a Lie group for all $n$, $\cap_n K_n = \{e\}$. Moreover, since the conjugacy action of $G^0$ on $K$ is by inner automorphisms of $K$, each $K_n$ is normal in $G^0$. Let $C''$ be the smallest closed subgroup of $C$ such that $K \subset C''$, $y \in C''$ and $\alpha^n(y) \in C''$ for all $n \in \mathbb{N}$. By Lemma 4.9, $L_n = \cap_{c\in C''} cK_nc^{-1}$ is normal in $C''$ and $K/L_n$ is a Lie group. Moreover, as above, since $L_n$ are normal in $K$, they are normal in $G^0$ and hence each $G^0/L_n$ is a Lie group. Also, since $\alpha(C'') \subset C''$, it is easy to see that

$$L_n \subset \alpha(L_n) = \cap_{c\in C''} \alpha(c)K_n\alpha(c^{-1}).$$

Hence $\alpha^{-m}(L_n) \subset L_n$ for all $m \in \mathbb{N}$. Note that the distality of the $\{\alpha^m\}_{m\in\mathbb{N}}$-action on $K$ is equivalent to the statement that the closure of $\{\alpha^m\}_{m\in\mathbb{N}}$ in $K^K$ is a group (see [10], Theorem 1 which is for group actions and it can easily be seen that the same proof works for semigroup actions). In particular, we have that $\{\alpha^{-m}\}_{m\in\mathbb{N}}$-action on $K$ is also distal and there exists $\{k_m\} \subset \mathbb{N}$ such that $\alpha^{-k_m} \to \alpha$ in $K^K$. This implies that $\alpha(L_n) \subset L_n$, i.e. $L_n$ is $\alpha$-invariant. Thus $L_n$ is also normalized by $\alpha^{-n}(y)$, $n \in \mathbb{N}$. Let $N(L_n)$ be the normalizer of $L_n$ in $G$ and $N = \cap N(L_n)$. Then since $L_n$'s are normalized by $G^0$ and $y$, $G^0 \subset N$ and $y \in N$. Since $L_n$ is $\alpha$-invariant, $N(L_n)$ is $\alpha$-invariant, hence $N$ is $\alpha$-invariant. Since $G^0 \subset N$, the connected component, say $N^0$ of $N$ is $G^0$. Also, we have $N^0/L_n = G^0/L_n$ are Lie groups for all $n \geq 1$ and $y \in N$. Now the result follows from the previous case and Lemma 4.6.

Now suppose that $G$ is not second countable. Let $x \in C_L(\alpha)$. We need to show that $x \in C(\alpha)L$. Let $H$ be the closed subgroup generated by $G^0L$ and $\{\alpha^n(x)\}_{n\in\mathbb{Z}}$ in $G$. Then $H$ is $\alpha$-invariant $\sigma$-compact group. Replacing $G$ by $H$, we may assume that $G$ itself is $\sigma$-compact. By Proposition 1.1, $G$ contains arbitrarily small compact normal $\alpha$-invariant subgroups $K_d$ such that $G/K_d$ is second countable. Now from above, we have that $C_C(\alpha) \subset C_{K_d}(\alpha)C$ for any compact $\alpha$-invariant subgroup $C$. Hence from Lemma 4.6, $C_L(\alpha) = C(\alpha)L$.

□

Combining our result Theorem 4.4 with Proposition 10 of [17], we obtain the next result generalizing the main result in [18] but as following example shows the converse part of the main result in [18] need not be true for locally compact groups which could also be inferred from Theorem 4.4.

**Example 4.10** Let $T = \{z \in \mathbb{C} \mid |z| = 1\}$, $K = \prod_{i \in \mathbb{N}} T$ and let $A$ be the group of all finite permutations. Then there is a natural shift action of $A$ on $K$. Let $G = A \ltimes K$. Then $G$ is a locally compact group with $K$ as an open subgroup. Take $\alpha$ to be any distal automorphism of $G$. Then $C(\alpha) = \{e\}$ and for any $\alpha$-invariant compact subgroup $L$ of $G$, $C_L(\alpha) = L$ (cf. [28]). This implies that $C_L(\alpha) = C(\alpha)L$ for any $\alpha$-invariant compact subgroup $L$ of $G$. Suppose there are small compact normal subgroups $K_i$ of $G$ such that $G/K_i$ are finite-dimensional. Then $L_i = K_i \cap K$ are normal $A$-invariant subgroups of $K$ and $K/K_i$ are compact connected abelian subgroup of finite dimension. Since $A$ is torsion, each $\tau \in A$ acts distally on $K/K_i$. By Theorem 5.15 of [26], $A$-action is distal on $K/K_i$. Since $\cap K_i = \{e\}$, $A$-action on $K$ is distal which is a contradiction. Thus, $G$ has no small compact normal subgroups $K_i$ such that $G/K_i$ are finite-dimensional. $\square$

**Corollary 4.11** *Let $G$ be a locally compact group and $\alpha \in \mathrm{Aut}(G)$. Suppose there are small compact normal $\alpha$-invariant subgroups $K_i$ of $G$ contained in $G^0$ such that each $G^0/K_i$ is a real Lie group. Then for any $\alpha$-invariant compact subgroup $L$ of $G$, $C_L(\alpha) = C(\alpha)L$.*

**Proposition 4.12** *Let $K$ be a compact group and $\Gamma$ be a $\sigma$-compact locally compact group acting on $K$ by automorphisms. If the $\Gamma$-action on $K$ has DCC, then $K$ is metrizable. Further if $\Gamma = \{\alpha^n \mid n \in \mathbb{Z}\}$ and both $C(\alpha)$ and $C(\alpha^{-1})$ are trivial, then $K$ is a real Lie group and $\alpha$ is distal on $K$.*

**Proof** We first claim that $K$ is metrizable. By Proposition 1.1 we get that there is a $\Gamma$-invariant compact normal subgroup $K_1$ of $K$ such that $K/K_1$ is metrizable and hence a countable collection $\{U_n\}$ of neighborhoods of $e$ such that $\cap_n U_n = K_1$. If $K$ is not metrizable, then $K_1 \neq \{e\}$, hence by Proposition 1.1, there is a $\Gamma$-invariant compact normal subgroup $K_2$ of $K$ such that $K/K_2$ is metrizable and $K_2 \subset \cap_n (U_n \setminus \{x\})$ for some $x \in K_1$ with $x \neq e$. Proceeding this way we get a sequence $\{K_m\}$ of compact normal $\Gamma$-invariant subgroups such that $K/K_m$ is metrizable and $K_{m+1} \subsetneq K_m$ for all $m$. This is a contradiction to the hypothesis that the $\Gamma$-action on $K$ has DCC. Thus, $K$ is metrizable.

Further assume that $\Gamma = \{\alpha^n \mid n \in \mathbb{Z}\}$. By Proposition 5.4 of [20], there exist a compact Lie group $G$ with the identity $e$ and a full subgroup $H$ of $G \times G$ (i.e. both the projections of $H$ on $G$ are surjective), such that $K$ is isomorphic to a full shift invariant subgroup $Y_H$ of $G^{\mathbb{Z}}$, where $Y_H = \{(x_i) \in G^{\mathbb{Z}} \mid (x_i, x_{i+1}) \in H \text{ for all } i \in \mathbb{Z}\}$ and the action of $\alpha$ on K corresponds to the shift action $(x_i) \mapsto (x_{i+1})$ for $(x_i) \in Y_H$. We may assume that $K = Y_H$ and $\alpha$ is the shift-action defined above.

Let $\phi\colon Y_H \to G$ be such that $\phi((x_i)) = x_0$. Then $\phi$ is a continuous homomorphism. We now prove that $\phi$ is injective. If $\phi((x_i)) = e$, that is $x_0 = e$ for some $(x_i) \in Y_H$. Then $(x_{-1}, e), (e, x_1) \in H$ and $(x_i, x_{i+1}) \in H$. Since $H$ is a group, $(e, e) \in H$. Define $a_i = x_i$ for $i < 0$ and $a_i = e$ for $i \geq 0$, hence $(a_i) \in Y_H$ but $(a_i) \in C(\alpha)$ which is trivial. This implies that $x_i = e$ for $i < 0$. Similarly we can show that $x_i = e$ for $i > 0$. Thus $\phi$ is injective. Since $Y_H$ is compact and $G$ is a real Lie group, we get that $K \simeq Y_H$ is also a real Lie group and the rest of the proof follows from Proposition 4.1. $\square$

**Proof of Theorem 4.3** If $\alpha$ is distal, then for any $\alpha$-invariant compact subgroup $L$, $C_L(\alpha^{\pm 1}) = L$ follows from Corollary 3.2 of [28]. Suppose for any $\alpha$-invariant compact subgroup $L$, we have $C_L(\alpha^{\pm 1}) = L$. Consider the factor $\bar{\alpha}$ of $\alpha$ on $G/G^0$. By Proposition 4.8, we get that $C(\bar{\alpha}^{\pm 1}) = G^0$ in $G/G^0$. This implies by [19] that $\bar{\alpha}$ is distal. Thus, it is sufficient to show that the restriction of $\alpha$ to $G^0$ is distal (see [28]). We will denote the restriction of $\alpha$ to $G^0$ also by $\alpha$. Since $G^0$ is connected, there is a maximal compact characteristic subgroup $K$ of $G^0$ such that $G^0/K$ is a connected Lie group. Since $K$ is characteristic, $K$ is $\alpha$-invariant. Since $C_K(\alpha) = K = C_K(\alpha^{-1})$, the contraction subgroups of factor automorphisms of $\alpha$ and $\alpha^{-1}$ on $G^0/K$ are trivial. This implies that the factor of $\alpha$ on $G^0/K$ is distal (see Proposition 4.1). Thus, it is sufficient to show the distality of the restriction of $\alpha$ on $K$ which will also be denoted by $\alpha$.

By Proposition 1.1 we get that each neighborhood $U$ of $e$ in $K$ contains a compact normal $\alpha$-invariant subgroup $K_U$ such that $K/K_U$ is metrizable. Also, $\alpha$ is distal on $K$ if and only if $\alpha$ is distal on $K/K_U$ for each $U$. Since the assumption that $C_L(\alpha) = L$ for any compact $\alpha$-invariant subgroup $L$ is valid on quotient groups $K/K_U$, we may assume that $K$ is metrizable. By Theorem 3.16 of [20], there exists a sequence $\{K_n\}$ of closed normal $\alpha$-invariant subgroups of $K$ such that $K_{n+1} \subset K_n$ for all $n$, $\cap_n K_n = \{e\}$ and the $\{\alpha^n \mid n \in \mathbb{Z}\}$-action on $K/K_n$ has DCC. By Proposition 4.12, $\alpha$ is distal on $K/K_n$. Since $\cap_n K_n = \{e\}$, $\alpha$ is distal on $K$. $\square$

It may be remarked that the subgroup $L$ in Theorem 4.3 may be assumed to be contained in the maximal compact normal subgroup of $G^0$. In [19], it is shown that $\alpha$ is distal on a totally disconnected group if and only if $C(\alpha^{\pm 1}) = \{e\}$. A recent result in [18] combined with our results yields the following corollary which strengthens and generalizes the result in [19] and extends the result in [18] on compact groups to all locally compact groups.

**Corollary 4.13** *Let $G$ be a locally compact group and $\alpha \in \mathrm{Aut}(G)$. Then the $\{\alpha^n\}_{n\in\mathbb{N}}$-action on $G$ is distal if and only if $C(\alpha) = \{e\}$. In particular, $\alpha$ is distal if and only if $C(\alpha^{\pm 1}) = \{e\}$.*

**Proof** If the $\{\alpha^n\}_{n\in\mathbb{N}}$-action on $G$ is distal, then $C(\alpha) = \{e\}$. We now assume that $C(\alpha) = \{e\}$. Let $G^0$ be the connected component of the identity in $G$ and $K$ be the maximal compact normal subgroup of $G^0$ such that $G^0/K$ is a Lie group. Since $K$ is maximal, it is characteristic. In particular, $K$ is an $\alpha$-invariant normal subgroup of $G$. Since $K$ is compact, $\alpha$ is distal on $K$ (cf. [18]). By Theorem 4.4, $C_K(\alpha) = C(\alpha)K = K$. Thus replacing $G$ by $G/K$, we may assume that $G^0$ is a Lie group. Since $G^0$ is a Lie group, Proposition 4.1 implies that the $\{\alpha^n\}_{n\in\mathbb{N}}$-action on $G^0$ is distal. Now using Proposition 4.8 and replacing $G$ by $G/G^0$ we may assume that $G$ is totally disconnected. Let $U$ be any compact open subgroup of $G$. Since $C(\alpha)$ is closed, by Theorem 3.32 of [3], $C(\alpha^{-1})$ is also closed. Then there is a $n$ such that $\cap_{i=0}^{n}\alpha^{-i}(U)$ is tidy for $\alpha^{-1}$ (cf. Theorem 3.32 of [3]). Let $V = \cap_{i=0}^{n}\alpha^{-i}(U)$. Then $V$ is a compact open subgroup of $G$ such that $V \subset U$. Since $C(\alpha)$ is trivial, Proposition 3.24 of [3] implies that the scale of $\alpha^{-1}$ is one, hence $\alpha^{-1}(V) \subset V$ (see Definition 2.1 of [3]). If $e$ is in the closure of $\{\alpha^n(x) \mid n \in \mathbb{N}\}$ for some $x \in G$, then $\alpha^n(x) \in V$ for some $n \geq 1$. This implies that $x \in \alpha^{-n}(V) \subset V \subset U$. Since $U$ is any compact open subgroup, we get that $x = e$. This proves that the $\{\alpha^n\}_{n\in\mathbb{N}}$-action on $G$ is distal. The second assertion easily follows from the first.

□

The following corollary follows easily from above and Theorem 1.1 of [31].

**Corollary 4.14** *Let $G$ be a locally compact group and $\alpha \in \mathrm{Aut}(G)$. Then the closure of the $\alpha$-orbit of $x$, $\overline{\{\alpha^n(x)\}}_{n\in\mathbb{Z}}$ is a minimal closed $\alpha$-invariant set for every $x \in G$ if and only if $C(\alpha^{\pm 1}) = \{e\}$.*

# 5 Unimodularity

In this section, we relate relative compactness of contraction groups and groups whose closed subgroups are unimodular. We first recall the following well-known result on Lie groups which is often used:

**Lemma 5.1** *Let $G$ be a real Lie group and $\alpha \in \mathrm{Aut}(G)$. Then $\alpha$ preserves the Haar measure of $G$ if and only if $|\det(\mathrm{d}\alpha)| = 1$ where $\mathrm{d}\alpha$ is the differential of $\alpha$ on the Lie algebra of $G$.*

**Proof** Let $G^0$ be the connected component of $G$. Since $G$ is a real Lie group, $G^0$ is an open $\alpha$-invariant subgroup of $G$. Then $\alpha$ preserves the Haar measure on $G$ if and only if $\alpha$ preserves the Haar measure on $G^0$. So, we may assume that $G$ is connected. Now the result follows from Proposition 55, Section 3.16, Chapter III of [5].

For a locally compact group, we consider the following conditions:

($i$) Closed subgroups of $G$ are unimodular;

($ii$) $C_L(\alpha)$ is relatively compact for any inner automorphism $\alpha$ of $G$ and for any $\alpha$-invariant compact subgroup $L$ of $G$;

($iii$) $C(\alpha)$ is relatively compact for any inner automorphism $\alpha$ of $G$;

($iv$) $G$ is pointwise distal.

We now obtain the following relations between ($i$)-($iv$).

**Theorem 5.2** *Let $G$ be a locally compact group. Then ($i$) and ($ii$) are equivalent and ($iv$) $\Rightarrow$ ($ii$) $\Rightarrow$ ($iii$).*

1. *Further, if the connected component in $G$ is a real Lie group, then ($i$)–($iii$) are equivalent.*

2. *Furthermore, if there is a continuous injection $\phi\colon G \to GL(n, \mathbb{F})$ where $\mathbb{F}$ is a local field or $G$ is an almost connected group, then ($i$)–($iv$) are equivalent.*

To prove Theorem 5.2 we need the following Lemmas.

**Lemma 5.3** *Let $G$ be a locally compact group, $\alpha \in \mathrm{Aut}(G)$ and let $H$ be a $\alpha$-invariant closed normal subgroup of $G$.*

*If Haar measures of $G$ and $H$ are $\alpha$-invariant, then the Haar measure of $G/H$ is also $\alpha$-invariant.*

*In particular, if the Haar measure of any $\alpha$-invariant closed subgroup of $G$ is $\alpha$-invariant, then the Haar measure of any $\alpha$-invariant closed subgroup of $G/H$ is also $\alpha$-invariant.*

For a locally compact group $L$, let $\lambda_L$ denote the left Haar measure on $L$ and $\Delta_L$ denote the modular function of $L$.

**Proof** Let $G$, $\alpha$ and $H$ be as above. Then the following formula for $\phi \in C_c(G)$,

$$\lambda_G(\phi) = \int d\lambda_{G/H}(\overline{u}) \int \phi(uh) d\lambda_H(h)$$

defines the left Haar measure on $G$; for $u \in G$, $\overline{u} = uH$ in $G/H$. For $\phi \in C_c(G)$, define $\phi_\alpha$ by $\phi_\alpha(g) = \phi(\alpha(g))$ for any $g \in G$. Then $\phi_\alpha \in C_c(G)$. Since $\lambda_G$ and $\lambda_H$ are $\alpha$-invariant, we have for any $\phi \in C_c(G)$,

$$\begin{aligned} \lambda_G(\phi) &= \lambda_G(\phi_\alpha) \\ &= \int d\lambda_{G/H}(\overline{u}) \int \phi_\alpha(uh) d\lambda_H(h) \\ &= \int d\lambda_{G/H}(\overline{u}) \int \phi(\alpha(u)h) d\lambda_H(h) \\ &= \Delta_{G/H}(\alpha) \int d\lambda_{G/H}(\overline{u}) \int \phi(uh) d\lambda_H(h) \\ &= \Delta_{G/H}(\alpha) \lambda_G(\phi). \end{aligned}$$

This implies that $\Delta_{G/H}(\alpha) = 1$. Thus, the Haar measure of $G/H$ is $\alpha$-invariant. $\square$

The next result is proved for compact Lie groups, however it can be proved for any compact metrizable group using [18] but here we include the simple proof for the Lie case which also makes the paper more self-contained.

**Lemma 5.4** *Let $K$ be a compact Lie group and $\alpha$ be an automorphism of $K$. Then we have the following:*

(1) $\overline{C(\alpha)} = \overline{C(\alpha^{-1})}$.

(2) $C(\alpha) = \{e\}$ *if and only if* $C(\alpha^{-1}) = \{e\}$;

**Proof** It is enough to prove (1) as it implies (2). Since $K$ is a compact Lie group, both $\overline{C(\alpha)}$ and $\overline{C(\alpha^{-1})}$ are normal subgroups contained in $K^0$. Hence we may assume that $K$ is connected. Let $\beta$ denote the factor of $\alpha$ on $K' = K/\overline{C(\alpha^{-1})}$. By 3.2.13 of [14], $C(\beta^{-1}) = \{e\}$. Let $V$ be the Lie algebra of $K'$ and let $\mathrm{d}\beta$ be the differential of $\beta$ on $V$. Then the eigenvalues of $\beta$ are of absolute value less than or equal to one. Since $K'$ is compact, $|\det\,(\mathrm{d}\beta)| = 1$. This implies that eigenvalues of $\mathrm{d}\beta$ are of absolute value one. Thus, $C(\beta) = \{e\}$ and hence $C(\alpha) \subset \overline{C(\alpha^{-1})}$. Replacing $\alpha$ by $\alpha^{-1}$ we get that $C(\alpha^{-1}) \subset \overline{C(\alpha)}$. $\square$

**Lemma 5.5** *Let $\alpha$ be an automorphism of a real Lie group $G$. If $H = \overline{C(\alpha)}$ and the Haar measure of $H$ is $\alpha$-invariant, then $H$ is compact.*

**Proof** Since $G$ is a real Lie group, $H = \overline{C(\alpha)}$ is a connected nilpotent Lie group. Then $H$ contains a compact connected central subgroup $K$ such that $H/K$ is a simply connected nilpotent Lie group. Since $C(\alpha)K/K$ is a simply connected nilpotent Lie subgroup of $H/K$, $C(\alpha)K/K$ is a closed subgroup of $H/K$ and hence $H = C(\alpha)K$. In particular, $\alpha$ contracts $H/K$. If the Haar measure on $H$ is $\alpha$-invariant, then since $K$ is compact, the Haar measure on $H/K$ is $\alpha$-invariant (cf. Lemma 5.3). Thus, $\alpha$ contracts $H/K$ and the Haar measure on $H/K$ is $\alpha$-invariant. Hence $H = K$. $\square$

**Proposition 5.6** *Let $G$ be a real Lie group and let $\alpha$ be an automorphism of $G$. Then the following are equivalent:*

(1) *The Haar measure of any $\alpha$-invariant closed subgroup of $G$ is $\alpha$-invariant.*

(2) *$C(\alpha)$ and $C(\alpha^{-1})$ are relatively compact;*

**Proof** Suppose $C(\alpha)$ as well as $C(\alpha^{-1})$ is relatively compact. Then they normalize each other as groups. Since $G$ is a real Lie group, this implies that $K = \overline{C(\alpha)C(\alpha^{-1})}$ is a compact $\alpha$-invariant Lie subgroup. Applying Lemma 5.4 to $\alpha$ restricted to $K$, we get that $\overline{C(\alpha)} = \overline{C(\alpha^{-1})} = K$. Let $\mathcal{G}$ be the Lie algebra of $G$. Then $\mathrm{d}\alpha$, the differential of $\alpha$ defines a linear transformation on $\mathcal{G}$. Let $V \subset \mathcal{G}$ be the Lie algebra of $K$. Then eigenvalues of the factor of $\mathrm{d}\alpha$ on $\mathcal{G}/V$ have absolute value one and hence its determinant has absolute value one. Since $V$ is a Lie algebra of a compact connected Lie group, the determinant of $\mathrm{d}\alpha$ restricted to $V$ also has absolute value one. Thus, the absolute value of determinant of $\mathrm{d}\alpha$ on $\mathcal{G}$ is one. This proves that the Haar measure on $G$ is $\alpha$-invariant. Thus, we get that (1) implies (2).

Now assume (2). Let $H = \overline{C(\alpha)}$. Then the Haar measure of $H$ is $\alpha$-invariant. Then by Lemma 5.5, $H$ is compact. We can also replace $\alpha$ by $\alpha^{-1}$ and get that (2) implies (1). $\square$

We now prove the following version of Theorem 5.2.

**Theorem 5.7** *Let $G$ be a locally compact group and let $\alpha$ be any automorphism of $G$. Then the following are equivalent:*

*(i) The Haar measure of any $\alpha$-invariant closed subgroup of $G$ is $\alpha$-invariant.*

*(ii) $C_L(\alpha^{\pm 1})$ is relatively compact for any $\alpha$-invariant compact subgroup $L$ of $G$.*

**Proof** Let $G^0$ be the connected component of $e$ in $G$. Then $G^0$ is $\alpha$-invariant and $G/G^0$ is totally disconnected. We denote the factor of $\alpha$ on $G/G^0$ by $\beta$ which is an automorphism defined by $\beta(gG^0) = \alpha(g)G^0$ for all $g \in G$. Let $\pi\colon G \to G/G^0$ be the canonical projection. Then $\pi(\alpha(g)) = \beta(\pi(g))$ for all $g \in G$. Let $s_{G/G^0}\colon \mathrm{Aut}(G/G^0) \to \mathbb{N}$ be the scale function defined as in [3].

Assume that the Haar measure of any $\alpha$-invariant closed subgroup of $G$ is $\alpha$-invariant. We now prove that $C_L(\alpha)$ is relatively compact for any $\alpha$-invariant compact subgroup $L$ of $G$. By Lemma 5.3, the Haar measure of any $\beta$-invariant closed subgroup of $G/G^0$ is also $\beta$-invariant. Then by Proposition 3.21 of [3], we get that $s_{G/G^0}(\beta^{\pm 1}) = 1$. Now Proposition 3.24 of [3] implies that $C(\beta)$ is relatively compact. Let $C = \pi(L)$. Then from Theorem 3.8 of [3], we get that $C_C(\beta) = C(\beta)C$ is relatively compact. Let $G_1 = \pi^{-1}(\overline{C_C(\beta)})$. Then $G_1$ is a $\alpha$-invariant closed subgroup containing $C_L(\alpha)$ and hence $\overline{C_L(\alpha)} \subset G_1$. Since $G_1/G^0$ is compact, $G_1$ is almost connected. By [23], $G_1$ contains a maximal compact normal subgroup $K$ such that $G_1/K$ is a real Lie group. Since $K$ is maximal, it is a characteristic subgroup of $G_1$. In particular, $K$ is $\alpha$-invariant. It follows from Lemma 5.3 that (i) is valid for $G_1/K$. Also, $\overline{C_L(\alpha)} \subset \overline{C_{KL}(\alpha)}$. Replacing $G_1$ by $G_1/K$, we may assume that $G_1$ is a real Lie group. Since $C(\alpha) \subset G_1$, it follows from Proposition 5.6 that $C(\alpha)$ is relatively compact. Since $G_1$ is a real Lie group and $C_L(\alpha) \subset G_1$, Theorem 4.4 implies that $C_L(\alpha) = C(\alpha)L$ and hence $C_L(\alpha)$ is relatively compact. Using the fact that the Haar measure of any closed $\alpha$-invariant subgroup of $G$ is $\alpha$-invariant if and only if it is $\alpha^{-1}$-invariant, as above we can get that $C_L(\alpha^{-1})$ is relatively compact for any $\alpha$-invariant compact subgroup $L$ of $G$.

Suppose $C_L(\alpha^{\pm 1})$ is relatively compact for any $\alpha$-invariant compact subgroup $L$ of $G$. Then by Proposition 4.8, $C(\beta^{\pm 1})$ is relatively compact in $G/G^0$, hence by Proposition 3.24 of [3] we get that $s_{G/G^0}(\alpha^{\pm 1}) = 1$. This implies that there is a $\alpha$-invariant open subgroup $H$ of $G$ such that $G^0 \subset H$ and $H/G^0$ is compact. Therefore, $H$ is almost connected. By [23], $H$ has a maximal compact normal subgroup $K$ such that $H/K$ is a real Lie group. Since $K$ is maximal, it is characteristic. In particular, $K$ is $\alpha$-invariant. Let $\nu$ be the factor automorphism of $\alpha$ on $H/K$. Then $C_K(\alpha)/K = C(\nu)$ and hence $C(\nu)$ as well as $C(\nu^{-1})$ is relatively compact. Since $H/K$ is a real Lie group, by Proposition 5.6, the Haar measure of $H/K$ is $\nu$-invariant. Since $K$ is compact, the Haar measure of $H$ is $\alpha$-invariant. Since $H$ is an open subgroup of $G$, the Haar measure of $G$ is $\alpha$-invariant. Since any $\alpha$-invariant closed subgroup of $G$ also has (ii), the Haar measure of any $\alpha$-invariant closed subgroup of $G$ is $\alpha$-invariant. □

**Proof of Theorem 5.2** Applying Theorem 5.7 to inner automorphisms, we get that $(i)$ and $(ii)$ are equivalent. By Theorem 3.1 of [28], we get that $(iv)$ implies $(ii)$ and by taking $L = \{e\}$ we get that $(ii)$ implies $(iii)$.

If the connected component in $G$ is a real Lie group. Then $(iii) \Rightarrow (ii)$ follows from Theorem 4.4.

Suppose there is a continuous injection $\phi\colon G \to GL(n, \mathbb{F})$ where $\mathbb{F}$ is a local field. In order to prove $(i)$–$(iv)$ are equivalent it is sufficient to prove that $(iii)$ implies $(iv)$. Let $x \in G$ and define $\alpha\colon GL(n,\mathbb{F}) \to GL(n,\mathbb{F})$ by $\alpha(g) = \phi(x)g\phi(x^{-1})$ for all $g \in GL(n,\mathbb{F})$. Let $C(\alpha, GL(n,\mathbb{F})) = \{g \in GL(n,\mathbb{F}) \mid \alpha^n(g) \to e \text{ as } n \to \infty\}$ and $C(\alpha, G) = \{g \in G \mid x^n g x^{-n} \to e \text{ as } n \to \infty\}$. Then $C(\alpha, GL(n,\mathbb{F}))$ is a closed $\alpha$-invariant subgroup of $GL(n,\mathbb{F})$ and $\phi(C(\alpha,G)) \subset C(\alpha, GL(n,\mathbb{F}))$. Now $(iii)$ implies that $\overline{C(\alpha,G)}$ is a $\alpha$-invariant compact subgroup of $G$. Since $\phi$ is continuous, $\phi(\overline{C(\alpha,G)}) \subset \overline{\phi(C(\alpha,G))} \subset C(\alpha, GL(n,\mathbb{F}))$. Since $C(\alpha, GL(n,\mathbb{F}))$ has no $\alpha$-invariant nontrivial compact subgroup, $\phi(\overline{C(\alpha,G)})$ is trivial, hence $C(\alpha,G)$ is trivial as $\phi$ is an injection. Now $(iv)$ follows from Corollary 4.13.

Suppose $G$ is an almost connected group. Then $G$ has arbitrarily small compact normal subgroups $K_d$ such that $G/K_d$ is an almost connected real Lie group. By Corollary 4.11, $G/K_d$ satisfies $(iii)$ if $G$ satisfies $(iii)$. Thus, it is sufficient to prove $(iii)$ implies $(iv)$ for almost connected real Lie groups. Let $G$ be an almost connected real Lie group. Then the connected component of the identity in $G$ has finite index in $G$, hence to prove (*iii*) implies (*iv*), we may further assume that $G$ is a connected Lie group. Let $x \in G$. Define $\alpha\colon G \to G$ by $\alpha(g) = xgx^{-1}$ for all $g \in G$. Suppose $C(\alpha)$ and $C(\alpha^{-1})$ are relatively compact. We will now claim that $\alpha$ is distal. Let $\mathcal{G}$ be the Lie algebra of $G$ and Ad be the adjoint representation of $G$ on $\mathcal{G}$. Let $U = \{\tau \in GL(\mathcal{G}) \mid (\mathrm{d}\alpha)^n \tau (\mathrm{d}\alpha)^{-n} \to e \ \text{ as } \ n \to \infty\}$. Then $U$ is a closed subgroup of $GL(\mathcal{G})$. Since $\mathrm{Ad}(\alpha(g)) = \mathrm{d}\alpha \mathrm{Ad}(g)\, \mathrm{d}\alpha^{-1}$ for all $g \in G$, we get that $\mathrm{Ad}(\overline{C(\alpha)}) \subset U$. Since $U$ has no nontrivial compact subgroup invariant under conjugation by $\mathrm{d}\alpha$, $C(\alpha)$ is contained in the kernel of Ad. Since $G$ is a connected Lie group, kernel of Ad is the center of $G$. This shows that $C(\alpha)$ is contained in the center of $G$. Since $\alpha$ is inner, $C(\alpha)$ is trivial. Similarly we can show that $C(\alpha^{-1})$ is trivial. Now Proposition 4.1 implies that $\alpha$ is distal. Thus, (*iii*) implies (*iv*).

$\square$

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