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# Operator decomposable measures and stochastic difference equation

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**Abstract**

We consider the following convolution equation or equivalently stochastic difference equation

$$\lambda_k = \mu_k * \phi(\lambda_{k-1}), \qquad k \in \mathbb{Z} \tag{1}$$

for a given bi-sequence $(\mu_k)$ of probability measures on $\mathbb{R}^d$ and a linear map $\phi$ on $\mathbb{R}^d$. We study the solutions of equation (1) by realizing the process $(\mu_k)$ as a measure on $(\mathbb{R}^d)^{\mathbb{Z}}$ and rewriting the stochastic difference equation as $\lambda = \mu * \tau(\lambda)$-any such measure $\lambda$ on $(\mathbb{R}^d)^{\mathbb{Z}}$ is known as $\tau$-decomposable measure with co-factor $\mu$-where $\tau$ is a suitable weighted shift operator on $(\mathbb{R}^d)^{\mathbb{Z}}$. This enables one to study the solutions of (1) in the settings of $\tau$-decomposable measures. A solution $(\lambda_k)$ of (1) will be called a fundamental solution if any solution of (1) can be written as $\lambda_k * \phi^k(\rho)$ for some probability measure $\rho$ on $\mathbb{R}^d$. Motivated by the splitting/factorization theorems for operator decomposable measures, we address the question of existence of fundamental solutions when a solution exists and answer affirmatively via a one-one correspondence between fundamental solutions of (1) and strongly $\tau$-decomposable measures on $(\mathbb{R}^d)^{\mathbb{Z}}$ with co-factor $\mu$. We also prove that fundamental solutions are extremal solutions and vice versa. We provide a necessary and sufficient condition in terms of a logarithmic moment condition for the existence of a (fundamental) solution when the noise process is stationary and when the noise process has independent $\ell_p$-paths.

## 1 Introduction

Stochastic/random difference equations arise in different contexts and studied by many since [Ke73], [Ts-75] and [Yo-92]. We consider the following stochastic difference equation

$$\eta_k = \xi_k + \phi(\eta_{k-1}), \qquad k \in \mathbb{Z} \tag{TY}$$

where $\eta_k$ and $\xi_k$ are $\mathbb{R}^d$-valued random variables and $\phi$ is a linear map on $\mathbb{R}^d$ (with the assumption that for each $k \in \mathbb{Z}$, $\{\xi_k, \eta_{k-i} \mid i \leq k\}$ are independent). The random variables $(\xi_k)$ are given and are called the noise process of the equation (TY). The random variables $(\eta_k)$ satisfying equation (TY) for given $(\xi_k)$ and $\phi$ will be called a solution of equation (TY). We wish to study the distributional properties of solutions of equation (TY). This pushes us to focus on the corresponding convolution equation

$$\lambda_k = \mu_k * \phi(\lambda_{k-1}), \qquad k \in \mathbb{Z} \tag{1}$$

where $\lambda_k$ and $\mu_k$ are the laws of $\eta_k$ and $\xi_k$ ($k \in \mathbb{Z}$) respectively. It may be noted that Lemma 4.3(ii) of [HiY-10] asserts that for a solution $(\lambda_k)$ of equation (1) there exists a solution $(\eta_k)$ of equation (TY) whose marginal laws are $(\lambda_k)$.

B. Tsirelson [Ts-75] considered the following stochastic difference equation on the real line

$$\eta_k = \xi_k + \mathrm{frac}(\eta_{k-1}) \qquad k \in -\mathbb{N} \tag{2}$$

with a given stationary Gaussian noise process $(\xi_k)$ to obtain his celebrated example of the stochastic differential equation

$$dX_t = dB_t + b^+(t, X)dt, \quad X(0) = 0$$

that has unique but not strong solution where $(B_t)$ is the one-dimensional Brownian motion (see [Ts-75] for more details). M. Yor [Yo-92] formulated Tsirelson's equation (2) in the form of equation (TY) on the one-dimensional torus $\mathbb{R}/\mathbb{Z}$ when $\phi$ is the identity automorphism and $(\xi_k)$ is a general noise process. In particular, [Yo-92] identified solutions of equation (TY) with measures on a quotient $(\mathbb{R}/\mathbb{Z})/M$ where $M$ is a closed subgroup of $\mathbb{R}/\mathbb{Z}$. When $\phi$ is the identity automorphism, equation (1) was considered on general compact groups [AkUY-08] and when the noise law $(\xi_k)$ is stationary, equation (1) was considered on abelian groups [Ta-09]. When the noise law is stationary, [Ra-12] considered equation (1) on general locally compact groups and provided a necessary and sufficient condition for existence of solutions, under further distal condition [Ra-12] completely classified solutions of equation (1).

Here, we study equation (1) by realizing the stochastic processes $(\eta_k)$ and $(\xi_k)$ in the space $(\mathbb{R}^d)^{\mathbb{Z}}$. Recall that $(\mathbb{R}^d)^{\mathbb{Z}}$ is the space of all bi-sequences $(v_n)_{n\in\mathbb{Z}}$ of vectors in $\mathbb{R}^d$ and $(\mathbb{R}^d)^{\mathbb{Z}}$ is a complete separable metric vector space (of infinite dimension). The metric $d$ on $(\mathbb{R}^d)^{\mathbb{Z}}$ is given by

$$d((v_n), (w_n)) = \sup_{n\in\mathbb{Z}} \min\{||v_n - w_n||, 1/|n|\}$$

and with this metric a sequence in $(\mathbb{R}^d)^{\mathbb{Z}}$ converges if and only if the sequence of each co-ordinates converges.

For any $k_1, \cdots, k_n$ in $\mathbb{Z}$, let $P_{k_1,\cdots,k_n}\colon (\mathbb{R}^d)^{\mathbb{Z}} \to (\mathbb{R}^d)^n$ be the projection defined by $P_{k_1,\cdots k_n}(x_k) = (x_{k_1}, \cdots, x_{k_n})$. Then $P_{k_1,\cdots,k_n}$ is a continuous linear map.

We now look at the realization of $(\lambda_k)$ and $(\mu_k)$ (that satisfy equation (1)) in $(\mathbb{R}^d)^{\mathbb{Z}}$. Using Kolmogorov consistency theorem we can find (unique) probability measures $\lambda$ and $\mu$ on $(\mathbb{R}^d)^{\mathbb{Z}}$ such that

$$P_{k_1,\cdots,k_n}(\lambda) = \lambda_{k_1} \times \cdots \times \lambda_{k_n} \quad \text{and} \quad P_{k_1,\cdots,k_n}(\mu) = \mu_{k_1} \times \cdots \times \mu_{k_n}$$

for any set of finite integers $k_1, \cdots, k_n$. In particular, $P_k(\lambda) = \lambda_k$ and $P_k(\mu) = \mu_k$ for all $k \in \mathbb{Z}$. In this situation, we sometime denote $\mu$ also by $(\mu_k)$.

In order to rephrase equation (1) in $(\mathbb{R}^d)^{\mathbb{Z}}$, we need the following weighted shift operator. Define $\tau\colon (\mathbb{R}^d)^{\mathbb{Z}} \to (\mathbb{R}^d)^{\mathbb{Z}}$ by

$$\tau(v_i) = (\phi(v_{i-1})), \qquad (v_i) \in (\mathbb{R}^d)^{\mathbb{Z}} \tag{L}$$

which is the composition of the right shift operator and the diagonal action of $\phi$ . It can easily be seen that $\tau$ is a continuous linear operator on the complete separable metric vector space $(\mathbb{R}^d)^{\mathbb{Z}}$ and $\tau$ is invertible if and only if $\phi$ is invertible. With the help of the weighted shift operator $\tau$ and the uniqueness part of Kolmogorov consistency theorem, equation (1) can be rewritten as the following convolution equation on $(\mathbb{R}^d)^{\mathbb{Z}}$

$$\lambda = \mu * \tau(\lambda). \tag{3}$$

Equation (3) motivates us to look at the so called operator decomposable measures on vector spaces studied by Siebert et al ([Si-91], [Si-92] and the references cited therein).

We now recall operator decomposable measures on complete separable metric vector spaces. Let $V$ be a complete separable metric vector space and $M^1(V)$ be the space of Borel probability measures on $V$. Given a continuous linear operator $T$ on $V$, $\rho \in M^1(V)$ is called $T$-decomposable if there is a $\nu \in M^1(V)$ such that

$$\rho = \nu * T(\rho)$$

and in this situation we also say that $\rho$ is $T$-decomposable with co-factor $\nu$.

In the language of decomposable measures, a solution $(\lambda_k)$ of equation (1) gives a $\tau$-decomposable measure $\lambda$ on $(\mathbb{R}^d)^{\mathbb{Z}}$ with co-factor $\mu$ such that $P_k(\lambda) = \lambda_k$ and $P_k(\mu) = \mu_k$ for all $k \in \mathbb{Z}$. We use this approach to study the distributional properties of solutions of the stochastic difference equation (TY). For instance, this approach is useful to understand when solutions of equation (1) can have atoms: it may be recalled that if $\mu_k$ has density, then $\lambda_k$ also has density that is even smoother.

**Proposition 1.1** *Let $(\mu_k)$ be a bi-sequence in $M^1(\mathbb{R}^d)$ and $\phi$ be a linear map on $\mathbb{R}^d$. Assume that $(\lambda_k)$ is a solution of equation (1). If $\inf_k \prod_{i=-k}^{k} \lambda_i(\{x_i\}) \neq 0$ for some sequence $(x_k)_{k\in\mathbb{Z}}$ in $\mathbb{R}^d$, then each $\mu_k$ is a dirac measure for all $k$.*

The above condition that $\inf_k \prod_{i=-k}^{k} \lambda_i(\{x_i\}) \neq 0$ may be viewed as $\lambda_k(\{x_k\}) \to 1$ faster as $|k| \to \infty$. Thus, conclusion of Proposition 1.1 may be read as $\lambda_k(\{x_k\})$ does not converge to 1 faster as $|k| \to \infty$ for any sequence $(x_k)$ unless every $\mu_k$ degenerates.

Among $T$-decomposable measures, strongly $T$-decomposable measures (that is, $T$-decomposable measure $\rho$ with $T^n(\rho) \to \delta_0$) and $T$-invariant measures (that is, $T(\rho) = \rho * \delta_x$ for some $x \in V$) are particular cases. Significance of these two particular classes of $T$-decomposable measures stems from the factorization theorem of Siebert [Si-92]: recall that factorization theorem of [Si-92] proves that any symmetric $T$-decomposable measure on a separable Banach space is a product of a $T$-invariant measure and a strongly $T$-decomposable measure and a similar result was proved for $T$-decomposable measures verifying certain nondegeneracy condition in [Si-91].

This motivates us to look for similar interesting classes among solutions of equation (1). At this time we note that $(\lambda_k * \phi^k(\rho))$ is also a solution of (1) if so is $(\lambda_k)$. In view of these reasons a solution $(\lambda_k)$ of (1) will be called a fundamental solution if to each solution $(\nu_k)$ of equation (1), there is a $\nu \in M^1(\mathbb{R}^d)$ such that $\nu_k = \lambda_k * \phi^k(\nu)$ for all $k \in \mathbb{Z}$. Before we proceed to establish the correspondence between fundamental solutions and strongly $\tau$-decomposable measures, we would look at the other component of the factorization theorem of [Si-92], the so-called strictly $\tau$-invariant measure $\lambda \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ (that is, $\tau(\lambda) = \lambda$). If $\lambda \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ is strictly $\tau$-invariant, then it is easy to see that for any $k \in \mathbb{Z}$, $P_k(\lambda) = \phi^k(P_0(\lambda))$. Thus, forced by the factorization theorem of [Si-92], we pose the question: is there any fundamental solution of equation (1) if equation (1) has a solution. We provide an affirmative answer via a one-one correspondence between fundamental solutions and strongly $\tau$-decomposable measures.

**Theorem 1.1** *Let $(\mu_k)_{k\in\mathbb{Z}}$ be a bisequence in $M^1(\mathbb{R}^d)$ and $\phi$ be an invertible linear map of $\mathbb{R}^d$. Let $\tau$ be the linear operator on $(\mathbb{R}^d)^{\mathbb{Z}}$ defined by the equation (L).*

*(i) If $\lambda \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ is a strongly $\tau$-decomposable measure with co-factor $\mu = (\mu_k) \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$, then $\lambda_k = P_k(\lambda)$ is a fundamental solution of equation (1).*

*(ii) If equation (1) has a solution, then there is a strongly $\tau$-decomposable measure on $(\mathbb{R}^d)^{\mathbb{Z}}$ with co-factor $\mu * \delta_v$ for some $v \in (\mathbb{R}^d)^{\mathbb{Z}}$ where $\mu = (\mu_k) \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$. In particular, equation (1) has a fundamental solution.*

For $\mu \in M^1(V)$, it is easy to verify that if there is a strongly $T$-decomposable measure with co-factor $\mu$, then it is unique (see 1.2 and 1.3 of [Si-91]). Since fundamental solution is defined so as to have similarities with strongly $\tau$-decomposable measures, we ask the question that could there be two fundamental solutions: before the answer is stated, it is worth to note that $(\lambda_k * \phi^k(\delta_v))$ is also a fundamental solution if so is $(\lambda_k)$.

**Proposition 1.2** *If $(\lambda_k)$ and $(\rho_k)$ are fundamental solutions of (1), then there is a $v \in \mathbb{R}^d$ such that $\rho_k = \lambda_k * \phi^k(\delta_v)$ for all $k \in \mathbb{Z}$.*

It may be easily seen that the set of all solutions of equation (1) is a convex set and any extreme point of this convex set is known as extremal solution of (1). We now look at the relation between extremal solutions and fundamental solutions.

**Theorem 1.2** *Let $(\mu_k)$ and $\phi$ be as in Theorem 1.1. Then $(\lambda_k)$ is a fundamental solution of equation (1) if and only if $(\lambda_k)$ is a extremal solution. In particular, the set of extremal solutions is either empty or can be identified with $\mathbb{R}^d$.*

Thus, it follows from Theorem 1.2 and Proposition 1.2 that having one extremal solution is sufficient to get all (extremal) solutions.

Siebert [Si-91] provides a necessary and sufficient logarithmic moment condition for a measure on Banach space to be a co-factor of strongly operator decomposable measure by generalizing a result of Zakusilo [Za-77]. Motivated by this result we provide a similar condition on the noise process for the existence of a (fundamental) solution of equation (1) when the noise is stationary (Theorem 4.1) and when the noise has independent $\ell_p$-paths (Theorem 5.1).

# 2 Preliminaries

We consider a complete separable metric vector space $V$. Let $M^1(V)$ denote the space of all Borel probability measures on $V$ with weak topology with respect to all continuous bounded functions on $V$.

One of the useful fact about the weak topology is the criterion for compact sets in $M^1(V)$ given by the following:

**Prokhorov's Theorem (cf. Chapter 2, Theorem 6.7 of [Pa-67]):** *A subset $\Gamma$ of $M^1(V)$ has compact closure or equivalently relatively compact if and only if for each $\epsilon > 0$ there is a compact set $K_\epsilon \subset V$ such that $\nu(K_\epsilon) > 1 - \epsilon$ for any $\nu \in \Gamma$.*

For a measure $\nu \in M^1(V)$, we define $\check{\nu} \in M^1(V)$ by $\check{\nu}(E) = \nu(-E)$ for any Borel subset $E$ of $V$. A measure $\nu \in M^1(V)$ is called symmetric if $\check{\nu} = \nu$.

For any $x \in V$, $\delta_x \in M^1(V)$ denotes the dirac measure supported at $\{x\}$. For any $\nu_1, \nu_2 \in M^1(V)$, we denote the convolution product of $\nu_1$ and $\nu_2$ by $\nu_1 * \nu_2$ and is defined by

$$\nu_1 * \nu_2(E) = \int \nu_2(x+E) d\nu_1(x)$$

for any Borel subset $E$ of $V$. The convolution product $\delta_x * \nu$ is known as shift of $\nu \in M^1(V)$ by $x \in V$.

We say that a measure $\nu_1 \in M^1(V)$ is a factor of a measure $\nu \in M^1(V)$ if there exists a $\nu_2 \in M^1(V)$ such that $\nu = \nu_1 * \nu_2$.

For a continuous linear operator $T$ on $V$ and $\nu \in M^1(V)$, we define $T(\nu) \in M^1(V)$ by $T(\nu)(E) = \nu(T^{-1}(E))$ for any Borel subset $E$ of $V$. Then it can easily be seen that $T(\nu * \nu') = T(\nu) * T(\nu')$ for any $\nu, \nu' \in M^1(V)$.

We will be studying solutions $(\lambda_k)$ of the convolution equation

$$\lambda_k = \mu_k * \phi(\lambda_{k-1}), \qquad k \in \mathbb{Z} \tag{1}$$

when $V = \mathbb{R}^d$ and $\phi$ is a linear transformation on $\mathbb{R}^d$.

We often make use of the following handy application of Prokhorov's theorem to convolution equations.

**Theorem (cf. Chapter 3 Section 2 of [Pa-67]):** *Let $(\rho_n), (\sigma_n)$ and $(\nu_n)$ be sequences in $M^1(V)$ such that $\rho_n = \sigma_n * \nu_n$ for all $n \geq 1$.*

*(a) If $(\rho_n)$ is relatively compact, then there is a sequence $(x_n)$ in $V$ such that $(\sigma_n * \delta_{x_n})$ is relatively compact.*

*(b) If two of the three sequences $(\rho_n), (\sigma_n)$ and $(\nu_n)$ are relatively compact, then so is the third one.*

For symmetric measures we have the following improvement of the above, proof which is essentially based on methods in 3.3 of [Si-92].

**Lemma 2.1** *Let $(\rho_n)$, $(\nu_n)$ and $(\sigma_n)$ be sequences in $M^1(\mathbb{R}^d)$. Suppose $\rho_n = \sigma_n * \nu_n$ for all $n$ and $(\rho_n)$ is relatively compact. If each $\sigma_n$ is symmetric and $\sigma_n$ is a factor of $\sigma_m$ for all $n \geq m$ or for all $m \geq n$, then $(\sigma_n)$ converges in $M^1(\mathbb{R}^d)$.*

**Proof** By Corollary 2.5.3 of [Li-86], we get that $(\sigma_n)$ is relatively compact in $M^1(\mathbb{R}^d)$. If $\sigma_n = \sigma_m * \varrho_{n,m}$ for all $n, m$ with $n > m$ for some $\varrho_{n,m} \in M^1(\mathbb{R}^d)$, let $\sigma$ and $\sigma'$ be two limit points of $(\sigma_n)$. Let $(l_m)$ be a subsequence such that $\sigma_{l_m} \to \sigma$. Fix $n \geq 1$ and consider $\sigma_{l_m} = \sigma_n * \varrho_{l_m,n}$ for all $l_m \geq n$. This implies that $(\varrho_{l_m,n})$ is relatively compact. Then there is a $\varrho'_n \in M^1(\mathbb{R}^d)$ such that $\sigma = \sigma_n * \varrho'_n$ for all $n$. This implies that $\sigma = \sigma' * \varrho$ for some $\varrho \in M^1(\mathbb{R}^d)$. Similarly, we get that $\sigma' = \sigma * \varrho'$ for some $\varrho' \in M^1(\mathbb{R}^d)$. This implies that $\sigma' = \sigma' * \varrho' * \varrho$, hence $\varrho' * \varrho = \delta_0$. Thus, $\varrho = \delta_x$, hence $\sigma = \sigma' * \delta_x$ for some $x \in V$. Since each $\sigma_n$ is symmetric, the limit points $\sigma$ and

$\sigma'$ of $(\sigma_n)$ are also symmetric. But $\sigma = \sigma' * \delta_x$, hence $\sigma = \sigma'$. Thus, $(\sigma_n)$ converges. Convergence of $(\sigma_n)$ can be proved in a similar way if $\sigma_n$ is a factor of $\sigma_m$ for all $n \geq m$.

We also need a reformulation of Theorem 3.1 of [Cs-66] and this reformulation is proved by considering semidirect products. Given a continuous invertible linear operator $T$ on a complete separable metric vector space $V$, the semidirect product of $\mathbb{Z}$ and $V$ with respect to $T$ is denoted by $\mathbb{Z} \ltimes_T V$ whose underlying space is $\mathbb{Z} \times V$ and the group multiplication is given by: $(n, v)(m, w) = (n + m, v + T^n(w))$ for $n, m \in \mathbb{N}$ and $v, w \in V$. Then $\mathbb{Z} \ltimes_T V$ is a complete separable metric group. For any measure $\mu \in M^1(V)$ and $n \in \mathbb{Z}$, we define $n \otimes \mu$ by $(n \otimes \mu)(A \times B) = \delta_n(A)\mu(B)$ for any subset $A$ of $\mathbb{Z}$ and any Borel subset $B$ of $V$. Then $n \otimes \mu$ is a probability measure on $\mathbb{Z} \ltimes_T V$. Since $v \mapsto (0, v)$ is an isomorphism of $V$ onto its image, $V$ will be realized as a subgroup of $G$, hence any measure on $V$ is also realized as a measure on $G$. Thus, $0 \otimes \mu$ will be simply denoted by $\mu$ for any measure $\mu \in M^1(V)$. We now present a reformulation of Theorem 3.1 of [Cs-66] suitable for our study of equation (1).

**Lemma 2.2** *Let $(\mu_k)$ be a bi-sequence in $M^1(\mathbb{R}^d)$ and $\phi$ be an invertible linear map on $\mathbb{R}^d$. Suppose equation (1) has a solution. Then there is a sequence $(w_n)$ in $\mathbb{R}^d$ such that the sequence $(\prod_{i=0}^{n-1} \phi^i(\mu_{k-i}) * \phi^n(\delta_{w_{n-k}}))$ converges for any $k < 0$.*

**Proof** If there is a solution $(\nu_k)$ of equation (1), then $\nu_{-1} = \mu_{-1} * \phi(\nu_{-2}) = \cdots = \prod_{i=0}^{n-1} \phi^i(\mu_{-i-1}) * \phi^n(\nu_{-1-n})$. Then there is a $(a_n)$ in $\mathbb{R}^d$ such that $(\prod_{i=0}^{n-1} \phi^i(\mu_{-i-1}) * \delta_{a_n})$ is relatively compact.

Let $G = \mathbb{Z} \ltimes_\phi \mathbb{R}^d$ and for $n \geq 1$, let $\lambda_n = 1 \otimes \mu_{-n} \in M^1(G)$. Then

$$\lambda_1 * \cdots * \lambda_n * \delta_{(-n, \phi^{-n}(a_n))} = \mu_{-1} * \phi(\mu_{-2}) * \cdots * \phi^{n-1}(\mu_{-n}) * \delta_{a_n}$$

for all $n$. By Theorem 3.1 of [Cs-66], there exist $w_n$ in $\mathbb{R}^d$, $k_n \in \mathbb{Z}$ and $\nu_k \in M^1(G)$ such that $\lambda_{k+1} * \cdots * \lambda_n * \delta_{(k_n, w_n)} \to \nu_k$ as $n \to \infty$ for $k \geq 0$. This implies for each $k < 0$ that there are $\varrho_k \in M^1(\mathbb{R}^d)$ such that $\mu_k * \phi(\mu_{k-1}) * \cdots * \phi^{k+n-1}(\mu_{-n+1}) * \phi^{k+n}(\delta_{w_n}) \to \varrho_k$ or equivalently $\prod_{i=0}^{n-1} \phi^i(\mu_{k-i}) * \phi^n(\delta_{w_{n-k}}) \to \varrho_k$ as $n \to \infty$.

As our approach involves operator decomposable measures, we recall the following:

**Definition 1** For a continuous linear operator $T$ on $V$, a measure $\rho \in M^1(V)$ is called $T$-decomposable if there is a $\nu \in M^1(V)$ such that $\rho = \nu * T(\rho)$ and in this case $\nu$ is called co-factor. A $T$-decomposable $\rho$ is called strongly $T$-decomposable if $T^n(\rho) \to \delta_0$.

We now recall the other component of splitting/factorization Theorems of [Si-91] and [Si-92].

**Definition 2** For a continuous linear operator $T$ on $V$, a measure $\rho \in M^1(V)$ is called $T$-invariant (resp. strictly $T$-invariant) if there is a $x \in V$ such that $\rho = T(\rho) * \delta_x$ (resp. $T(\rho) = \rho$).

As explained in the introduction, by realizing the $\mathbb{R}^d$-valued stochastic processes in the space $(\mathbb{R}^d)^{\mathbb{Z}}$ of bi-sequences of vectors, we study the solutions to equation (1). It is easy see that $(\lambda_k)$ is a solution to equation (1) if and only if the measure $\lambda$ is $\tau$-decomposable with co-factor $\mu$ where $\lambda$ and $\mu$ are given by Kolmogrov consistency theorem with marginal laws given by $\lambda_k$ and $\mu_k$ respectively and $\tau(x_k) = (\phi(x_{k-1}))$. We first obtain the proof on the atoms of solution of equation (1).

**Proof of Proposition 1.1** By Kolmogrov consistency theorem and by its uniqueness part there are $\lambda$ and $\mu$ in $M^1((\mathbb{R}^d)^{\mathbb{Z}})$ such that $P_k(\lambda) = \lambda_k$, $P_k(\mu) = \mu_k$ on $\mathbb{R}^d$ and $\lambda = \mu * \tau(\lambda)$ where $\tau(v_i) = (\phi(v_{i-1}))$. It follows from Lemma 3.5 of [Si-91] that $\lambda(\{(x_k)\}) = 0$ for any $(x_k)$ or $\mu$ is a dirac measure. Now the result follows from

$$\lambda(\{(x_k)\}) = \lim_{k\to\infty} \prod_{i=-k}^{k} \lambda_i(\{x_i\}) = \inf_k \prod_{i=-k}^{k} \lambda_i(\{x_i\}), \qquad (x_k) \in (\mathbb{R}^d)^{\mathbb{Z}}.$$

As observed in the introduction, for any solution $(\lambda_k)$, $(\lambda_k * \phi^k(\rho))$ is also a solution for $\rho \in M^1(\mathbb{R}^d)$. This motivates us to make the following definition:

**Definition 3** A solution $(\lambda_k)$ of equation (1) is called a fundamental solution of (1) if to each solution $(\nu_k)$ there is a $\nu \in M^1(\mathbb{R}^d)$ such that $\nu_k = \lambda_k * \phi^k(\nu)$ for all $k \in \mathbb{Z}$.

As we also consider infinitely divisible measures and process with independent $\ell_p$-paths, we make the following formal definitions:

**Definition 4** A measure $\rho \in M^1(\mathbb{R}^d)$ is said to be infinitely divisible if to each $n \in \mathbb{N}$, there is a $\rho_n \in M^1(\mathbb{R}^d)$ such that $\rho_n^n = \rho$.

**Definition 5** A $\mathbb{R}^d$-valued stochastic process $(Y_k)$ (resp. a sequence of probability measures $(\rho_k)$ on $\mathbb{R}^d$) is said to have independent $\ell_p$-paths for some $p \in [1, \infty]$ if there is a $\mathbb{R}^d$-valued stochastic process $(Y_k')$ such that $Y_k$ and $Y_k'$ have the same law (resp. law of $Y_k'$ is $\rho_k$), $(Y_k')$ is an independent bi-sequence and $(Y_k'(\omega))$ is in $\ell_p$ a.s.

# 3 Fundamental solution

We now prove the results on fundamental solution. We first provide a useful sufficient condition for the existence of strongly $\tau$-decomposable measures (or fundamental solutions) using methods of tail idempotents of [Cs-66].

**Proposition 3.1** *Let* $(\mu_k)$ *be a bi-sequence of probability measures on* $\mathbb{R}^d$ *and* $\phi$ *is a linear map on* $\mathbb{R}^d$*. Define the linear operator* $\tau$ *on* $(\mathbb{R}^d)^{\mathbb{Z}}$ *by equation (L). Suppose* $\mu_{k,n} = \mu_k * \phi(\mu_{k-1}) * \cdots * \phi^n(\mu_{k-n}) \to \rho_k \in M^1(\mathbb{R}^d)$ *as* $n \to \infty$ *for all* $k$*. Then* $(\rho_k) \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ *is strongly* $\tau$*-decomposable with co-factor* $\mu = (\mu_k) \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$*.*

**Proof** By Kolmogorov consistency theorem there exists $\mu, \rho \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ such that $P_k(\rho) = \rho_k$ and $P_k(\mu) = \mu_k$ for all $k \in \mathbb{Z}$.

For any $k \in \mathbb{Z}$ and $0 \leq j \leq n$, we have $\mu_{k,n} = \mu_{k,j-1} * \phi^j(\mu_{k-j,n-j})$. Letting $n \to \infty$, we get that $\rho_k = \mu_{k,j-1} * \phi^j(\rho_{k-j})$ for any $j \geq 1$ and $k \in \mathbb{Z}$. For $j = 1$, $\rho_k = \mu_k * \phi(\rho_{k-1})$ for all $k \in \mathbb{Z}$. By the uniqueness part in Kolmogorov consistency Theorem, we get that $\rho = \mu * \tau(\rho)$ as $P_k\tau = \phi P_{k-1}$. Thus, $\rho$ is $\tau$-decomposable with co-factor $\mu$.

Using $\rho_k = \mu_{k,j-1} * \phi^j(\rho_{k-j})$ for any $j \geq 1$ and $k \in \mathbb{Z}$, we get that $(\phi^j(\rho_{k-j}))$ is relatively compact. If $\rho_{k,\infty}$ is a limit point of $(\phi^j(\rho_{k-j}))$, we have $\rho_k = \rho_k * \rho_{k,\infty}$. This shows that $\rho_{k,\infty} = \delta_0$. Thus, $\phi^j(\rho_{k-j}) \to \delta_0$ as $j \to \infty$, hence $\tau^j(\rho) \to \delta_0$ as $j \to \infty$. Thus, $\rho$ is strongly $\tau$-decomposable with co-factor $\mu$.

The following lemma is useful for two reasons: one reason is that it gives a solution of equation (1) in case each $\mu_k$ degenerates and other reason is that it explains why we have only strictly $\tau$-invariant measures in the factorization of solutions to equation (1), that is in the concept of fundamental solutions.

**Lemma 3.1** *Let* $(x_k)$ *be a given bi-sequence in* $\mathbb{R}^d$ *and* $\phi$ *be an invertible linear transformation on* $\mathbb{R}^d$*. Then there is a* $(y_k)$ *such that* $y_k = x_k + \phi(y_{k-1})$ *for all* $k \in \mathbb{Z}$*. In other words, for each* $x \in (\mathbb{R}^d)^{\mathbb{Z}}$ *there is a* $y \in (\mathbb{R}^d)^{\mathbb{Z}}$ *such that* $y = x + \tau(y)$*.*

**Proof** Fix $y_0 \in \mathbb{R}^d$. Define $y_k = \phi^k(y_0) + \sum_{i=1}^{k} \phi^{k-i}(x_i)$ for $k > 0$. Then $y_{k+1} = \phi^{k+1}(y_0) + \sum_{i=1}^{k+1} \phi^{k+1-i}(x_i) = \phi(\phi^k(y_0) + \sum_{i=1}^{k} \phi^{k-i}(x_i)) + x_{k+1} = x_{k+1} + \phi(y_k)$ for all $k \geq 0$.

Define $y_k = \phi^k(y_0) - \sum_{i=0}^{-k-1} \phi^{k+i}(x_{-i})$ for $k < 0$. Then $y_{k-1} = \phi^{k-1}(y_0) - \sum_{i=0}^{-k} \phi^{k-1+i}(x_{-i}) = \phi^{-1}(\phi^k(y_0) - \sum_{i=0}^{-k-1} \phi^{k+i}(x_{-i})) - \phi^{-1}(x_k) = \phi^{-1}(y_k - x_k)$, hence $y_k = x_k + \phi(y_{k-1})$ for all $k \leq 0$.

The next result enables us to work with shifted noise process.

**Lemma 3.2** *Let* $(\mu_k)$ *be a bi-sequence in* $M^1(\mathbb{R}^d)$ *and* $\phi$ *be an invertible linear map on* $\mathbb{R}^d$*. If* $(\rho_k)$ *is a solution of equation of (1) with noise* $(\mu_k)$*, then to each bi-sequence* $(v_k)$ *in* $\mathbb{R}^d$*, there is a bi-sequence* $(x_k)$ *in* $\mathbb{R}^d$ *such that* $(\rho_k * \delta_{x_k})$ *is a solution of equation (1) with noise* $(\mu_k * \delta_{v_k})$ *where* $x_k = v_k + \phi(x_{k-1})$*.*

*Further, if* $(\rho_k)$ *is a fundamental solution of equation of (1) with noise* $(\mu_k)$*, then* $(\rho_k * \delta_{x_k})$ *is a fundamental solution of equation (1) with noise* $(\mu_k * \delta_{v_k})$*.*

**Proof** Choose $(x_k)$ in $\mathbb{R}^d$ so that $x_k = v_k + \phi(x_{k-1})$ using Lemma 3.1. Define $\lambda_k = \rho_k * \delta_{x_k}$ for any $k \in \mathbb{Z}$. Then $\mu_k * \delta_{v_k} * \phi(\lambda_{k-1}) = \mu_k * \delta_{v_k} * \phi(\rho_{k-1}) * \phi(\delta_{x_{k-1}}) = \rho_k * \delta_{x_k} = \lambda_k$ for any $k \in \mathbb{Z}$.

Further, if $(\rho_k)$ is a fundamental solution of equation of (1) with noise $(\mu_k)$ and $(\lambda'_k)$ is a solution of equation (1) with noise $(\mu_k * \delta_{v_k})$. Then by first part, $(\rho'_k)$ is a solution of equation (1) with noise $(\mu_k)$ where $\rho'_k = \lambda'_k * \delta_{-x_k}$. Since $\rho_k$ is a fundamental solution to equation (1) with noise $(\mu_k)$, there is a $\nu \in M^1(\mathbb{R}^d)$ such that $\rho'_k = \rho_k * \phi^k(\nu)$, hence $\lambda'_k = \rho'_k * \delta_{x_k} = \rho_k * \phi^k(\nu) * \delta_{x_k} = \lambda_k * \phi^k(\nu)$. Thus, $(\lambda_k)$ is a fundamental solution to equation (1) with noise $(\mu_k * \delta_{v_k})$.

We now prove the main result of this section.

**Proof of Theorem 1.1** Suppose there is a strongly $\tau$-decomposable measure $\lambda \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ with co-factor $\mu$. Then $\lambda = \mu * \tau(\lambda)$ and $\tau^n(\lambda) \to \delta_0$. Define $\mu^{(n)} = \mu * \tau(\mu) * \cdots * \tau^{n-1}(\mu)$ for any $n \geq 1$. Then $\lambda = \mu^{(n)} * \tau^n(\lambda)$ for any $n \geq 1$. Since $\tau^n(\lambda) \to \delta_0$, we get that $\mu^{(n)} \to \lambda$. Let $\lambda_k = P_k(\lambda)$. Then it follows easily that $(\lambda_k)$ is a solution of equation (1). Take any other solution $(\lambda'_k)$ of equation (1). By Kolmogorov consistency theorem, there is a $\lambda' \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ such that $P_k(\lambda') = \lambda'_k$. By the uniqueness part of Kolmogorov consistency theorem, $\lambda' = \mu * \tau(\lambda')$: note that $P_k\tau = \phi P_{k-1}$. This implies that $\lambda' = \mu^{(n)} * \tau^n(\lambda')$ for any $n \geq 1$. Since $\mu^{(n)} \to \lambda$, $(\tau^n(\lambda'))$ is relatively compact. Let $\rho \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ be a limit point of $(\tau^n(\lambda'))$. Then $\tau^k(\rho)$ is also a limit point of $(\tau^n(\lambda'))$, hence $\lambda' = \lambda * \rho = \lambda * \tau^k(\rho)$ for any $k \in \mathbb{Z}$. Let $\rho_0 = P_0(\rho)$. Then for any $k \in \mathbb{Z}$, $\lambda' = \lambda * \tau^k(\rho)$ implies that $\lambda'_k = \lambda_k * \phi^k(\rho_0)$. This proves that $(\lambda_k)$ is a fundamental solution.

Suppose the equation (1) has a solution. By lemma 2.2, there are $w_k$ in $\mathbb{R}^d$ and $\varrho_k \in M^1(\mathbb{R}^d)$ such that $\prod_{i=0}^{n-1} \phi^i(\mu_{k-i}) * \phi^n(\delta_{w_{n-k}}) \to \varrho_k$ as $n \to \infty$ for any $k < 0$. Let $v_k = \phi(w_{-k+1}) - w_{-k}$ for $k < 0$. Then

$$\begin{aligned}\textstyle\sum_{i=0}^{n-1} \phi^i(v_{k-i}) &= \textstyle\sum_{i=0}^{n-1} \phi^i[\phi(w_{-k+i+1}) - w_{-k+i}] \\ &= \textstyle\sum_{i=0}^{n-1} \phi^{i+1}(w_{i-k+1}) - \sum_{i=0}^{n-1} \phi^i(w_{i-k}) \\ &= \phi^n(w_{n-k}) - w_{-k}\end{aligned}$$

for any $k < 0$ and $n \geq 1$.

Let $v_k = 0$ for all $k \geq 0$ and $\mu'_k = \mu_k * \delta_{v_k}$ for all $k \in \mathbb{Z}$. Then

$$\prod_{i=0}^{n-1} \phi^i(\mu'_{k-i}) = \prod_{i=0}^{n-1} \phi^i(\mu_{k-i}) * \phi^n(\delta_{w_{n-k}}) * \delta_{w_{-k}}$$

for all $k < 0$. Thus, $\prod_{i=0}^{n} \phi^i(\mu'_{k-i}) \to \varrho_k * \delta_{w_{-k}}$ for all $k < 0$.

Let $\rho_k = \varrho_k * \delta_{w_{-k}}$ for $k < 0$ and $\rho_k = \prod_{i=0}^{k} \phi^i(\mu_{k-i}) * \phi^{k+1}(\rho_{-1})$ for $k \geq 0$. Then $\prod_{i=0}^{n} \phi^i(\mu'_{k-i}) \to \rho_k$ as $n \to \infty$ for all $k \in \mathbb{Z}$. This implies by Proposition 3.1 that

$\rho = (\rho_k) \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ is a strongly $\tau$-decomposable measure with co-factor $\mu' = \mu * \delta_v$ where $v = (v_k)$. Now second part of (ii) follows from (i) and Lemma 3.2.

The following improves Theorem 1.1 for symmetric noise which could be compared with factorization theorem of [Si-92].

**Corollary 3.1** *Let $(\mu_k)$, $\phi$ and $\tau$ be as in Theorem 1.1. Suppose each $\mu_k$ is symmetric. Then equation (1) has a solution if and only if there is a symmetric strongly $\tau$-decomposable measure $\lambda \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ with co-factor $\mu = (\mu_k)$.*

**Proof** It is sufficient to prove the only if part. Suppose equation (1) has a solution. Then by Theorem 1.1, there is a $v \in (\mathbb{R}^d)^{\mathbb{Z}}$ and $\rho \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ such that $\rho$ is strongly $\tau$-decomposable with co-factor $\mu * \delta_v$ where $\mu = (\mu_k) \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$. This implies that $\mu * \delta_v = \mu * \delta_v * \tau(\rho) = \mu^{(n)} * \delta_{v^{(n)}} * \tau^n(\rho)$ where $\mu^{(n)} = \prod_{i=0}^{n-1} \tau^i(\mu)$ and $v^{(n)} = \sum_{i=0}^{n-1} \tau^i(v)$ for all $n \geq 1$. Since each $\mu_k$ is symmetric, by the uniqueness part of Kolmogorov consistency theorem, we get that $\mu$ is symmetric and hence each $\mu^{(n)}$ is symmetric. By Lemma 2.1, we get that $(\mu^{(n)})$ converges to $\lambda \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$. Since $\mu^{(n+1)} = \mu * \tau(\mu^{(n)})$, we get that $\lambda = \mu * \tau(\lambda)$, hence $\lambda = \mu^{(n)} * \tau^n(\lambda)$. Since $\mu^{(n)} \to \lambda$, $(\tau^n(\lambda))$ is relatively compact and for any limit point $\nu$ of $(\tau^n(\lambda))$, we have $\lambda = \lambda * \nu$. This implies that $\nu = \delta_0$, hence $\tau^n(\lambda) \to \delta_0$. This proves that $\lambda$ is a symmetric strongly $\tau$-decomposable measure with co-factor $\mu$.

As a consequence of Theorem 1.1 we now relax the requirement for a solution to be fundamental.

**Proposition 3.2** *Let $(\mu_k)$ and $\phi$ be as in Theorem 1.1. Let $(\lambda_k)$ be a solution of equation (1). Then $(\lambda_k)$ is a fundamental solution if and only if to each solution $(\nu_k)$ there is a $\nu \in M^1(\mathbb{R}^d)$ such that $\nu_0 = \lambda_0 * \nu$.*

**Proof** It is sufficient to prove the "if" part. Since $(\lambda_k)$ is a solution of (1), Theorem 1.1 implies that equation (1) has a fundamental solution $(\rho_k)$. Then there is a $\lambda \in M^1(\mathbb{R}^d)$ such that $\lambda_k = \rho_k * \phi^k(\lambda)$ for all $k \in \mathbb{Z}$. Suppose to each solution $(\nu_k)$ there is a $\nu \in M^1(\mathbb{R}^d)$ such that $\nu_0 = \lambda_0 * \nu$. Then for $(\rho_k)$, there is a $\rho \in M^1(\mathbb{R}^d)$ such that $\rho_0 = \lambda_0 * \rho$. Thus, $\rho_0 = \lambda_0 * \rho = \rho_0 * \lambda * \rho$. This implies that $\lambda * \rho = \delta_0$ and hence $\lambda = \delta_v$ for some $v \in \mathbb{R}^d$. Therefore, $\lambda_k = \rho_k * \phi^k(\delta_v)$. Since $(\rho_k)$ is a fundamental solution, $(\lambda_k)$ is also a fundamental solution.

The next proof explores the uniqueness of fundamental solution.

**Proof of Proposition 1.2** There exist $\rho$ and $\lambda$ in $M^1(\mathbb{R}^d)$ such that $\rho_k = \lambda_k * \phi^k(\rho)$ and $\lambda_k = \rho_k * \phi^k(\lambda)$ for all $k$. This implies that $\lambda_0 = \lambda_0 * \rho * \lambda$, hence $\rho * \lambda = \delta_0$. Thus, $\rho = \delta_v$ for some $v \in \mathbb{R}^d$.

Now the proof of the relation between fundamental solutions and extremal solutions.

**Proof of Theorem 1.2** Let $(\lambda_k)$ be a fundamental solution of (1). Suppose there solutions $(\lambda_{1,k})$ and $(\lambda_{2,k})$ of (1) such that $\lambda_k = a\lambda_{1,k} + b\lambda_{2,k}$ for some constants $0 \leq a, b \leq 1$ with $a + b = 1$. Since $(\lambda_k)$ is a fundamental solution, there are $\rho_1, \rho_2 \in M^1(\mathbb{R}^d)$ such that $\lambda_{1,k} = \lambda_k * \phi^k(\rho_1)$ and $\lambda_{2,k} = \lambda_k * \phi^k(\rho_2)$. This implies that $\lambda_k = \lambda_k * [a\phi^k(\rho_1) + b\phi^k(\rho_2)]$ and hence $a\phi^k(\rho_1) + b\phi^k(\rho_2) = \delta_0$. Thus, $\rho_1 = \delta_0$ or $\rho_2 = \delta_0$, hence $\lambda_k = \lambda_{1,k}$ or $\lambda_k = \lambda_{2,k}$. This proves that any fundamental solution is a extremal solution of (1).

Suppose $(\lambda'_k)$ is a extremal solution. By Theorem 1.1, equation (1) has fundamental solutions. Let $(\lambda_k)$ be a fundamental solution of equation (1). Then there is a $\rho \in M^1(\mathbb{R}^d)$ such that $\lambda'_k = \lambda_k * \phi^k(\rho)$. Let $x \in \mathbb{R}^d$ be a point in the support of $\rho$. Fix $n \in \mathbb{N}$, let $B_n = \{v \in \mathbb{R}^d \mid ||v - x|| < \frac{1}{n}\}$ and $a_n = \rho(B_n)$. Then $a_n > 0$. Define $\sigma_n = \frac{1_{B_n}\rho}{a_n}$ and if $b_n = 1 - a_n > 0$, define $\sigma'_n = \frac{\rho - a_n\sigma_n}{b_n}$ otherwise define $\sigma'_n = \delta_0$. Then $\sigma_n$ and $\sigma'_n$ are in $M^1(\mathbb{R}^d)$ and $\rho = a_n\sigma_n + b_n\sigma'_n$. This implies that $\lambda'_k = a_n(\lambda_k * \phi^k(\sigma_n)) + b_n(\lambda_k * \phi^k(\sigma'_n))$. Since $\lambda_k$ is a solution, $(\lambda_k * \phi^k(\sigma_n))$ and $(\lambda_k * \phi^k(\sigma'_n))$ are also solutions. Thus, $(\lambda'_k)$ is written as a convex combination of two solutions. Since $(\lambda'_k)$ is a extremal solution, we get that $\lambda'_k = \lambda_k * \phi^k(\sigma_n)$ or $\lambda'_k = \lambda_k * \phi^k(\sigma'_n)$. If $\lambda'_k = \lambda_k * \phi^k(\sigma'_n)$, then $\lambda'_k = a_n(\lambda_k * \phi^k(\sigma_n)) + b_n\lambda'_k$, hence $\lambda'_k = \lambda_k * \phi^k(\sigma_n)$. Thus, in any case we have $\lambda'_k = \lambda_k * \phi^k(\sigma_n)$ for all $k \in \mathbb{Z}$ and all $n \geq 1$. Since $\sigma_n = \frac{1_{B_n}\rho}{a_n}$, $\sigma_n \to \delta_x$. Thus, $\lambda'_k = \lambda_k * \phi^k(\delta_x)$. This proves that any extremal solution is also a fundamental solution.

Let $\mathcal{E}$ be the set of all extremal solutions of equation (1). If $\mathcal{E} \neq \emptyset$, then by the second part we get that all solutions in $\mathcal{E}$ are fundamental solutions. Fix a solution $(\lambda_k)$ in $\mathcal{E}$. For any $v \in \mathbb{R}^d$, define $\lambda_{k,v} = \lambda_k * \phi^k(\delta_v)$. Then $(\lambda_{k,v})$ is also a fundamental solution, hence by the first part $(\lambda_{k,v})$ is in $\mathcal{E}$. By Proposition 1.2, the map $x \mapsto (\lambda_{k,v})$ is onto. Suppose for some $v, w$, $(\lambda_{k,v}) = (\lambda_{k,w})$. Then $\lambda_0 * \delta_v = \lambda_0 * \delta_w$, hence $\lambda_0 * \delta_{v-w} = \lambda_0$. This implies that $v = w$. Thus the correspondence $v \mapsto (\lambda_{k,v})$ between $\mathbb{R}^d$ and $\mathcal{E}$ is bijective.

Using results proved here we show that fundamental solutions are also infinitely divisible provided the noise process $(\mu_k)$ consists of infinitely divisible measures.

**Proposition 3.3** *Let $(\mu_k)$ and $\phi$ be as in Theorem 1.1 and $(\lambda_k)$ be a fundamental solution of (1). Suppose each $\mu_k$ is infinitely divisible. Then each $\lambda_k$ is also infinitely divisible.*

**Proof** By Kolmogorov consistency theorem there is a $\mu \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ such that $P_k(\mu) = \mu_k$. Let $(\lambda_k)$ be a fundamental solution of equation (1) and $\tau$ be defined as in equation (L). Then by Theorem 1.1, there is a $v \in (\mathbb{R}^d)^{\mathbb{Z}}$ and $\rho \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ such that $\rho$ is strongly $\tau$-decomposable with co-factor $\mu * \delta_v$. This implies that $\mu * \tau(\mu) * \cdots * \tau^n(\mu) * \delta_{v^{(n)}} \to \rho$ where $v^{(n)} = \sum_{k=0}^{n} \tau^k(v)$. For any $k \in \mathbb{Z}$, considering the $k$-th projection, we get that $\mu_k * \phi(\mu_{k-1}) * \cdots * \phi^n(\mu_{k-n}) * \delta_{x_k} \to \rho_k$ where $x_k = P_k(v^{(n)})$ and $\rho_k = P_k(\rho)$. Suppose each $\mu_k$ is infinitely divisible. Since the set of infinitely divisible measures in $M^1(\mathbb{R}^d)$ is a closed set, closed under convolution product, we get that $\rho_k$ is infinitely divisible. Since $\rho$ is strongly $\tau$-decomposable with co-factor $\mu * \delta_v$ and $P_k(\rho) = \rho_k$, we get that $(\rho_k)$ is a fundamental solution of equation (1) for the noise $(\mu_k * \delta_{v_k})$ where $v_k = P_k(v)$. By Lemma 3.2, $(\rho_k * \delta_{w_k})$ is a fundamental solution of equation (1) for the noise $(\mu_k)$. Since $(\lambda_k)$ is a fundamental solution of equation (1) for the noise $(\mu_k)$, by Proposition 1.2, there are $y_k \in \mathbb{R}^d$ such that $\lambda_k = \rho_k * \delta_{w_k} * \delta_{y_k}$. Since each $\rho_k$ is infinitely divisible, each $\lambda_k = \rho_k * \delta_{w_k} * \delta_{y_k}$ is also infinitely divisible.

# 4 Noise is stationary

We now look at the situation when the noise process $(\xi_k)$ is stationary with common law $\mu$. Thus, equation (1) becomes

$$\lambda_k = \mu * \phi(\lambda_{k-1}), \qquad k \in \mathbb{Z} \tag{4}$$

where $\phi$ is a linear map on $\mathbb{R}^d$. In this case the well known contraction subspace associated to a linear map $\phi$ plays a crucial role: recall that contraction subspace of a linear map $\phi$ is denoted by $C(\phi)$ and is defined by

$$C(\phi) = \{v \in \mathbb{R}^d \mid \phi^n(v) \to 0 \ \text{ as } \ n \to \infty\}$$

and it is easy to see that $C(\phi)$ is a subspace of $\mathbb{R}^d$.

In case noise is stationary we obtain the following.

**Theorem 4.1** *Suppose the noise process $(\xi_k)$ is stationary with common law $\mu$ and $\phi$ is an invertible linear map on $\mathbb{R}^d$. Then the following are equivalent:*

1. *equation (4) has a (fundamental) solution;*
2. *the noise law $\mu$ is supported on a coset $u + C(\phi)$ of $C(\phi)$ for some $u \in \mathbb{R}^d$ and satisfies the following logarithmic moment condition*

$$\int \log(||v|| + 1) d\mu(v) < \infty.$$

*In this case, $(\nu * \delta_{u_k})$ is a fundamental solution where $(u_k)$ is given by Lemma 3.2 so that $u_k = u + \phi(u_{k-1})$ and*

$$\nu = \mu * \delta_{-u} * \phi(\nu) = \lim_{k \to \infty} \prod_{i=1}^{k} \phi^{i-1}(\mu) * \delta_{-u_k}$$

*in other words, $\nu$ is a strongly $\phi$-decomposable measure with co-factor $\mu * \delta_{-u}$.*

We first establish the relevance of $C(\phi)$.

**Lemma 4.1** *Suppose $C(\phi) = \{0\}$. Then equation (4) has a solution $(\lambda_k)$ if and only if $\mu$ is a dirac measure.*

**Proof** Suppose $(\lambda_k)$ is a solution of (4). Replacing $\lambda_k$ and $\mu$ by $\lambda_k * \check{\lambda}_k$ and $\mu * \check{\mu}$ respectively, we may assume that $\mu$ is symmetric. Now by iterating the equation (4), we get for fixed $k \in \mathbb{Z}$ that $\lambda_k = \prod_{i=1}^n \phi^{i-1}(\mu) * \phi^n(\lambda_{k-n})$ for all $n \geq 1$.

Let $\mu_n = \prod_{i=1}^n \phi^{i-1}(\mu)$. Then $\mu_n = \mu_k * \prod_{i=k+1}^n \phi^{i-1}(\mu)$ for all $n \geq k$, hence it follows from Lemma 2.1 that $\mu_n \to \nu \in M^1(\mathbb{R}^d)$. This implies that $\nu = \mu * \phi(\nu)$, hence $\nu = \mu_n * \phi^n(\nu)$ for all $n \geq 1$. Thus, $(\phi^n(\nu))$ is relatively compact and any limit point $\sigma$ of $(\phi^n(\nu))$ satisfies $\nu = \nu * \sigma$, hence $\sigma = \delta_0$. This implies that $\phi^n(\nu) \to \delta_0$ and hence $\nu(C(\phi)) = 1$. Since $C(\phi) = \{0\}$, $\nu = \delta_0$. Thus, $\mu = \delta_0$.

**Proof of Theorem 4.1** Suppose equation (4) has a solution $(\lambda_k)$. Then applying Lemma 4.1 to $V/C(\phi)$, we get that $\mu$ is supported on a coset $C(\phi)$. Let $\mu_s = \mu * \check{\mu}$ and $\lambda_{k,s} = \lambda_k * \check{\lambda}_k$. Then $\lambda_{k,s} = \mu_s * \phi(\lambda_{k-1,s}) = \prod_{i=0}^n \phi^i(\mu_s) * \phi^{n+1}(\lambda_{k-n-1,s})$. By Lemma 2.1 we get that $\prod_{i=0}^n \phi^i(\mu_s) \to \nu \in M^1(\mathbb{R}^d)$ as $k \to \infty$. Since $\mu$ is supported on a coset $C(\phi)$, $\mu_s$ is supported on $C(\phi)$. This implies by [Za-77] that $\int\int \log(||v-w||+1)d\mu(v)d\mu(w) < \infty$. By Fubini's Theorem $\int \log(||v-w||+1)d\mu(v) < \infty$ for some $w \in \mathbb{R}^d$. Since $v \mapsto \log(||v||+1)$ is subadditive, that is $\log(1+||v_1+v_2||) \leq \log(1+||v_1||) + \log(1+||v_2||)$, we get that $\int \log(||v||+1)d\mu(v) < \infty$.

Assume that $\mu$ is supported on a coset $u + C(\phi)$ and $\int \log(||v||+1)d\mu(v) < \infty$. Then as above using subadditivity of the map $v \mapsto \log(||v||+1)$ we get that $\int \log(||v-u||+1)d\mu(v) < \infty$. By [Za-77] we get that $\prod_{i=1}^k \phi^{i-1}(\mu * \delta_{-u}) \to \nu \in M^1(C(\phi))$. Then $\nu = \mu * \delta_{-u} * \phi(\nu)$. Take $\nu_k = \nu$ and $\mu_k = \mu * \delta_{-u}$ for all $k \in \mathbb{Z}$. Then $\phi^n(\nu_{k-n}) = \phi^n(\nu) \to \delta_0$. This implies that $(\nu_k) \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ is strongly $\tau$-decomposable with co-factor $(\mu_k) \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$. By Theorem 1.1, $(\nu_k)$ is a fundamental solution to equation (1) for the noise $(\mu_k)$. Using Lemma 3.1 we may find a sequence $(u_k)$ such that $u_k = u + \phi(u_{k-1})$ for all $k \in \mathbb{Z}$. Define $\lambda_k = \nu * \delta_{u_k}$ for all $k \in \mathbb{Z}$. Then By Lemma 3.2, we get that $(\lambda_k)$ is a fundamental solution to equation (1) for the noise $(\mu_k * \delta_u)$, that is equation (4) has a (fundamental) solution.

# 5 Noise with independent $\ell_p$-paths

We now look at the situation where the noise process has independent $\ell_p$-paths.

Recall that $\ell_p = \{(x_k) \in (\mathbb{R}^d)^{\mathbb{Z}} \mid \sum ||x_k||^p < \infty\}$ for $1 \leq p < \infty$. Then $\ell_p$ are proper dense subspaces of $(\mathbb{R}^d)^{\mathbb{Z}}$. But the spaces $\ell_p$ with norm $||\cdot||_p$ given by $||(x_k)||_p^p = \sum ||x_k||^p$ are separable Banach spaces for $1 \leq p < \infty$.

Fix $p \in [1, \infty)$ and $\phi$ be a linear transformation on $\mathbb{R}^d$. Let $\iota: \ell_p \to (\mathbb{R}^d)^{\mathbb{Z}}$ be the natural inclusion and $\alpha: \ell_p \to \ell_p$ be defined by $\alpha(v_k) = (\phi(v_{k-1}))$. It is easy to check that $\iota$ and $\alpha$ are continuous linear maps, $\tau \odot \iota = \iota \odot \alpha$. If the noise process $(\xi_k)$ has independent $\ell_p$-paths, then for $\mu = (\mu_k) \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$, we have $\mu(\iota(\ell_p)) = 1$. So, $\mu$ can be realized as a Borel probability measure on $\ell_p$. We follow this notations and realization in this section.

If the noise process $(\mu_k)$ has independent $\ell_p$-paths, then we introduce logarithmic moment $M_p$ by

$$M_p = \lim_{n \to \infty} \int_{(\mathbb{R}^d)^{2n+1}} \log((\sum_{-n \leq k \leq n} ||v_k||^p)^{\frac{1}{p}} + 1) d\mu_n'(v)$$

where $\mu_n'$ is the product measure $\mu_{-n} \times \cdots \times \mu_n$ defined on $(\mathbb{R}^d)^{2n+1}$. We now obtain the following result which is similar to the stationary noise situation: the main ingredients in the proof is the realization of the noise process in the Banach space $\ell_p$ and results of [Si-91] and [Si-92].

**Theorem 5.1** *Suppose the noise process $(\mu_k)$ has independent $\ell_p$-paths and $\phi$ is an invertible linear map on $\mathbb{R}^d$. Then the following are equivalent:*

1. *there exists $(x_k)$ in $\ell_p$ such that each $\mu_k$ is supported on the coset $C(\phi) - x_k$ and satisfies the logarithmic moment condition that $M_p < \infty$;*

2. *there is a (fundamental) solution $(\lambda_k)$ of equation (1) such that $(\lambda_k * \delta_{y_k})$ has independent $\ell_p$-paths for some $(y_k)$ such that $y_k = -x_k + \phi(y_{k-1})$.*

**Remark 5.1** Theorem 5.1 has a shift $(y_k)$ because the equation $y_k = x_k + \phi(y_{k-1})$ may not have a solution $(y_k)$ in $\ell_p$ for a given $(x_k)$ in $\ell_p$. For instance, fix $b > 1$ and define $\phi: \mathbb{R} \to \mathbb{R}$ by $\phi(t) = b^2 t$ for all $t \in \mathbb{R}$. Take $x_k = b^{-|k|}$ for all $k \in \mathbb{Z}$. Then for $1 \leq p < \infty$, $(x_k) \in \ell_p$ but there is no $(y_k) \in \ell_p$ such that $y_k = x_k + \phi(y_{k-1})$ for all $k \in \mathbb{Z}$. Because if there is a $(y_k) \in \ell_p$ such that $y_k = x_k + \phi(y_{k-1})$, then $y_0 = \sum_{i=0}^k \phi^i(x_{-i}) + y_{-k-1} = \sum_{i=0}^k b^i + y_{-k-1}$ for all $k \geq 1$. Since $(y_k) \in \ell_p$, $y_{-k} \to 0$ as $k \to \infty$. Thus, $y_0 = \sum_{i \geq 0} b^i$ which is a contradiction to $b > 1$. In general, the equation $y = x + \tau(y)$ has a solution in $\ell_p$ for any given $x \in \ell_p$ if and only if 1 is not in the spectrum of $\tau$ as $\tau$ has no fixed points in $\ell_p$.

**Example 5.1** We now give examples of noise processes that have independent $\ell_p$-paths and also satisfy the logarithmic moment condition.

1. Suppose $(\mu_k)$ is a bi-sequence in $M^1(\mathbb{R}^d)$ such that the corresponding absolute moments $m_k = \int ||x|| d\mu_k(x)$ are summable, that is $\sum m_k < \infty$. Then if $(X_k)$ is a independent bi-sequence of random variables such that law of $X_k$ is $\mu_k$, then

$\sum \int ||X_k|| < \infty$. This implies that $\int \sum ||X_k|| < \infty$ and hence $\sum ||X_n|| < \infty$ a.s. Thus, $(\mu_k)$ has independent $\ell_1$-paths. Since $\log(|t|+1) \leq |t|$, we get that $\int \log(||v||+1) d\mu_k \leq m_k$. Using the subadditivity of $t \mapsto \log(|t|+1)$, we can conclude that $(\mu_k)$ satisfies the moment condition of Theorem 5.1.

2. Fix $0 < a < 1$. For $n \geq 1$, let $\omega_n$ (resp. $\omega_{-n}$) be the uniform measure defined on the closed interval $[0, a^n]$ (resp. on $[-a^n, 0]$) and $\omega'_n$ (resp. $\omega'_{-n}$) be the uniform measure defined on the closed interval $[n, n+1]$ (resp. on $[-n-1, -n]$). Suppose $(X_n)$ is a bi-sequence of independent random variables such that the law of $X_n$ is given by

$$\mu_n = \begin{cases} (1 - a^{|n|})\omega_n + a^{|n|}\omega'_n & n \neq 0 \\ \delta_0 & n = 0 \end{cases}$$

for all $n \in \mathbb{Z}$. Then

$$\sum P(X_n^2 > 1) = \sum \mu_n(\mathbb{R} \setminus [-1, 1]) = \sum_{n \neq 0} a^{|n|} < \infty,$$

$$\sum E(X_n^2 1_{\{|X_n|^2 < 1\}}) = \sum (1 - a^{|n|}) \int_{-1}^{1} x^2 d\omega_n(x) \leq \sum a^{2|n|} < \infty$$

and

$$\sum |\mathrm{Var}(X_n^2 1_{\{|X_n|^2 < 1\}})| \leq 2 \sum a^{4|n|} < \infty.$$

By Kolmogorov three series theorem (cf. Theorem 5.3.3 of [Ch-01]), we get that $\sum X_n^2$ converges a.s. So the laws $(\mu_k)$ have independent $\ell_2$-paths. This noise process also verifies the logarithmic moment condition given in Theorem 5.1. Similar examples can be obtained using Kolmogorov three series theorem.

We first compare $||\alpha||_p$ and $||\phi||$.

**Lemma 5.1** $||\alpha||_p = ||\phi||$.

**Proof** For $v \in \ell_p$, we have $||\alpha(v)||_p \leq ||\phi|| ||v||$. This implies that $||\alpha||_p \leq ||\phi||$. For any $v_0 \in \mathbb{R}^d$ such that $||v_0|| = 1$, define $v_k = 0$ for all $k \neq 0$ and take $v = (v_k)$. Then $v \in \ell_p$ with $||v||_p = 1$ and $||\alpha(v)||_p = ||\phi(v_0)||$, hence $||\phi|| \leq ||\alpha||_p$. This proves that $||\alpha||_p = ||\phi||$.

We next show the relevance of $C(\phi)$.

**Lemma 5.2** *Assume that the noise process $(\mu_k)$ has independent $\ell_p$-paths. If $(\lambda_k)$ is a solution of (1) that has independent $\ell_p$-paths, then there exists a $(x_k) \in \ell_p$ such that each $\mu_k$ is supported on the coset $x_k + C(\phi)$.*

**Proof** We first show that $C(\alpha) \subset \prod C(\phi)$. For $v = (v_k) \in \ell_p$,

$$||\alpha^n(v)|| = (\sum_k ||\phi^n(v_{k-n})||^p)^{\frac{1}{p}} = (\sum_k ||\phi^n(v_k)||^p)^{\frac{1}{p}}$$

for all $n$. So, if $\alpha^n(v) \to 0$, then $||\phi^n(v_k)|| \to 0$ for all $k \in \mathbb{Z}$, hence $v_k \in C(\phi)$.

Suppose equation (1) has a solution $(\lambda_k)$ that has independent $\ell_p$-paths. Then by Kolmogorov consistency theorem and by its uniqueness part we get that there are $\mu$ and $\lambda$ in $M^1(\ell_p)$ such that $P_k(\mu) = \mu_k$, $P_k(\lambda) = \lambda_k$ and $\lambda = \mu * \alpha(\lambda)$. Replacing $\mu$ by $\mu * \check{\mu}$, we may assume that $\mu$ is symmetric and $\mu$ is a co-factor of a $\alpha$-decomposable symmetric measure. By 3.3 of [Si-92], there is a strongly $\alpha$-decomposable measure $\nu$ on $\ell_p$ with co-factor $\mu$. Now the result follows from Corollary 3.3 of [Si-91].

**Proof of Theorem 5.1** Assume $(\mu_k)$ has independent $\ell_p$-paths. By Kolmogorov consistency theorem there is a $\mu \in M^1(\ell_p)$ such that $P_k(\mu) = \mu_k$. We first observe the following:

$$\begin{aligned} \int \log(||v||_p + 1) d\mu(v) &= \int \log((\textstyle\sum_k ||v_k||^p)^{\frac{1}{p}} + 1) d\mu(v) \\ &= \lim_{n\to\infty} \int_{(\mathbb{R}^d)^{2n+1}} \log((\textstyle\sum_{-n\leq k\leq n} ||v_k||^p)^{\frac{1}{p}} + 1) d\mu'_n \end{aligned} \qquad (\mathrm{M}'_\mathrm{p})$$

for $p \in [1, \infty)$ where $\mu'_n$ is the product measure $\mu_{-n} \times \cdots \times \mu_n$ defined on $(\mathbb{R}^d)^{2n+1}$.

Let $X_0 = \{(v_k) \in \ell_p \mid v_k \in C(\phi)\}$. Then $X_0$ is a closed subspace of $\ell_p$ and $\alpha(X_0) = X_0$. Since $||\alpha||_p = ||\phi||$, it could easily be verified that the spectral radius of $\alpha$ restricted to $X_0$ is equal to the spectral radius of $\phi$ restricted to $C(\phi)$ which is less than one unless $C(\phi) = \{0\}$.

In order to prove the result, we assume that $C(\phi)$ is nontrivial. Thus, spectral radius of $\alpha$ restricted to $X_0$ is less than one.

Suppose there is a $x = (x_k) \in \ell_p$ such that $\mu_k$ is supported on $C(\phi) - x_k$ and has the logarithmic moment condition that $M_p < \infty$. Then using $(M'_p)$ and the subadditivity of $v \mapsto \log(||v||_p + 1)$ we get that

$$\int \log(||v + x||_p + 1) d\mu(v) \leq \int \log(||v||_p + 1) d\mu(v) + \log(||x||_p + 1) < \infty.$$

Since $\mu_k$ is supported on $C(\phi) - x_k$, $\mu * \delta_x(X_0) = 1$. Since $\alpha$ restricted to $X_0$ has spectral radius less than one, 1.5 of [Si-91] implies that $\prod_{i=0}^{n-1} \phi^i(\mu * \delta_x) \to \lambda \in M^1(X_0)$. This implies that $\lambda = \mu * \delta_x * \alpha(\lambda)$ and $\alpha^n(\lambda) \to \delta_0$. It follows from Theorem 1.1 that $P_k(\lambda)$ is a fundamental solution to equation (1) for the noise $(\mu_k * \delta_{x_k})$. By Lemma 3.2, equation (1) has a fundamental solution for the noise $(\mu_k)$.

Suppose equation (1) has a solution $(\lambda_k)$ such that $(\lambda_k * \delta_{z_k})$ has independent $\ell_p$-paths. Then $\lambda_k = \mu_k * \phi(\lambda_{k-1})$ implies

$$\lambda_k * \delta_{z_k} = \mu_k * \phi(\lambda_{k-1}) * \delta_{z_k} = \mu_k * \delta_{a_k} * \phi(\lambda_{k-1} * \delta_{z_{k-1}})$$

where $a_k = z_k - \phi(z_{k-1})$. Since $\mu_k$, $\lambda_k * \delta_{z_k}$ have independent $\ell_p$-paths, $(a_k)$ is in $\ell_p$. Since $(\mu_k)$ has independent $\ell_p$-paths, $(\mu_k * \delta_{a_k})$ also has independent $\ell_p$-paths. Applying Lemma 5.2 to $\lambda_k * \delta_{z_k} = \mu_k * \delta_{a_k} * \phi(\lambda_{k-1} * \delta_{z_{k-1}})$, we get that each $\mu_k$ is supported on $x_k + C(\phi)$ for some $x = (x_k) \in \ell_p$. By Kolmogorov consistency theorem and by its uniqueness there is a $\lambda \in M^1(\ell_p)$ such that $\lambda = \mu * \delta_a * \alpha(\lambda)$ where $P_k(\lambda) = \lambda_k * \delta_{z_k}$ and $a = (a_k)$. This implies that $\lambda * \check{\lambda} = \mu * \check{\mu} * \alpha(\lambda * \check{\lambda})$. By Theorem 3.3 of [Si-92] we get that $\prod_{i=1}^{n} \alpha^{i-1}(\mu * \check{\mu}) \to \nu \in M^1(\ell_p)$ as $n \to \infty$. Since each $\mu_k$ is supported on $x_k + C(\phi)$, $\mu_k * \check{\mu}_k$ is supported on $C(\phi)$. So $\mu * \check{\mu}$ is supported on $X_0$. This implies by 1.5 of [Si-91] that $\int\int \log(||v-w||_p+1)d\mu(v)d\mu(w) < \infty$. By Fubini's theorem $\int \log(||v-w||_p + 1)d\mu(v) < \infty$ for some $w \in \mathbb{R}^d$. Since $v \mapsto \log(||v||_p + 1)$ is subadditive, we get that $\int \log(||v||_p + 1)d\mu(v) < \infty$. Now the result follows from $(M'_p)$.

# 6 Remarks

We now make a few remarks. The first one provides a counter-example to show that having moment condition on the individual marginals of noise process as in Theorem 4.1 is not sufficient for existence of (fundamental) solutions.

(1) Let $\mu_0$ be the uniform measure supported on $[0, 1]$ and $0 < a < 1$. Define

$$\mu_k = \phi^k(\mu_0)$$

for all $k \in \mathbb{Z}$ where $\phi(t) = at$ for all $t \in \mathbb{R}$. Then

$$\int \log(|t| + 1)d\mu_k(t) = \int_0^1 \log(|a^k t| + 1)dt \leq \log(a^k + 1) < \infty$$

for all $k$. But the equation $\lambda_k = \mu_k * \phi(\lambda_{k-1})$, $k \in \mathbb{Z}$ has no solution. Because, if there is a solution $(\lambda_k)$, then $\lambda_0 = \mu_0 * \phi(\lambda_{-1}) = \mu_0 * \phi(\mu_{-1}) * \phi^2(\lambda_{-2}) = \cdots = \prod_{k=0}^{n} \phi^k(\mu_{-k}) * \phi^{n+1}(\lambda_{-n-1}) = \mu^{n+1} * \phi^{n+1}(\lambda_{-n-1})$ for all $n \geq 1$. This implies that $(\mu^{n+1} * \delta_{x_n})$ is relatively compact for some sequence $(x_n)$ in $\mathbb{R}$, hence $\mu$ is a dirac measure which is a contradiction. Here, Theorem 5.1 can not be applied as it can be seen by using Kolmogorov three series theorem that $(\mu_k)$ does not have independent $\ell_p$-paths for any $p \in [1, \infty)$.

(2) *Gaussian noise:* Let $(\mu_k)$ be a bi-sequence in $M^1(\mathbb{R}^d)$ and $\phi$ be a linear transformation on $\mathbb{R}^d$. Suppose each $\mu_k$ is Gaussian with covariance operator $A_k$.

We now claim that equation (1) has a solution if and only if $\sum_{i=0}^{\infty}(\phi^i)^* A_{-i}\phi^i$ converges where $(\phi^i)^*$ denotes the adjoint of $\phi^i$. Applying Lemma 3.2, we may assume that $\mu_k$ is symmetric. Suppose equation (1) has a solution. Define $\tau$ by equation (L). By Corollary 4.1, there is a symmetric strongly $\tau$-decomposable measure $\lambda \in M^1((\mathbb{R}^d)^{\mathbb{Z}})$ with co-factor $\mu = (\mu_k)$. Then $\lambda = \lim_n(\prod_{k=0}^{n-1}\tau^k(\mu))$. Let $\lambda_k = P_k(\lambda)$. Then by Theorem 1.1, we get that $(\lambda_k)$ is a fundamental solution of equation (1) and $\lambda_k = \lim_n \prod_{i=0}^{n-1}\phi^i(\mu_{k-i})$. By considering the corresponding characteristic functions, we get that $\sum_{i=0}^{\infty}(\phi^i)^* A_{k-i}\phi^i$ converges and $\lambda_k$ is Gaussian with covariance operator $\sum_{i=0}^{\infty}(\phi^i)^* A_{k-i}\phi^i$. Conversely, if $\sum_{i=0}^{\infty}(\phi^i)^* A_{-i}\phi^i$ converges, then $\sum_{i=0}^{\infty}(\phi^i)^* A_{k-i}\phi^i = (\phi^k)^*(\sum_{i=-k}^{\infty}(\phi^i)^* A_{-i}\phi^i)\phi^k$ converges for all $k \in \mathbb{Z}$. By taking $\lambda_k$ to be the Gaussian with covariance operator $B_k = \sum_{i=0}^{\infty}(\phi^i)^* A_{k-i}\phi^i$, it may be easily verified that $\lambda_k = \lim \prod_{i=0}^{n-1}\phi^i(\mu_{k-i})$. Thus, by Proposition 3.1 and Theorem 1.1 we get that $(\lambda_k)$ is a fundamental solution. A similar result may be obtained for any noise consisting of infinitely divisible distribution using Levy-Khinchin representation (cf. 5.7 of [Li-86]) but for simplicity we considered the Gaussian case.

(3) *Gaussian and Poisson solution:* Suppose the equation $\lambda_k = \mu_k * \phi(\lambda_{k-1})$ has a solution $(\lambda_k)$ consisting of Gaussian measures $\lambda_k$. Then it follows from Cramér's theorem that each $\mu_k$ is Gaussian (see [LiO-77]). Thus, equation (1) has a solution consisting of Gaussian measures only if each $\mu_k$ is Gaussian. Similarly, we may conclude that equation (1) has a solution consisting of Poisson measures only if each $\mu_k$ is Poisson-use Raikov's theorem (see [LiO-77]). Using similar idea it can also be seen that if there is a solution of (1) consisting of Gaussian (resp. Poisson) measures, then there is a fundamental solution of (1) consisting of Gaussian (resp. Poisson) measures.

(4) *Gaussian noise with no solution:* If we take $\mu_0$ in (1) to be the Gaussian $N(0,1)$ and define $\phi$ and $\mu_k$ as in (1). Then $\mu_k$ is the Gaussian measure $N(0, a^{2k})$. As in (1), it may be verified that the equation $\lambda_k = \mu_k * \phi(\lambda_{k-1})$, $k \in \mathbb{Z}$ has no solution. It may also easily be seen that $\mu_k$ does not satisfy the condition in (2).

(5) *General noise:* We now provide an example to show that in general one may get a (fundamental) solution of equation (1) even if $C(\phi) = \{0\}$. Let $\phi$ be a linear transformation on $\mathbb{R}^d$. Take $\mu_k$ to be the Gaussian with covariance operator $A_k$ with $||A_k|| = a^k ||\phi||^{2k}$ for some $a > 1$. Then

$$||\sum_{i=0}^{n}(\phi^i)^* A_{-i}\phi^i|| \leq \sum_{i=0}^{n} a^{-i}||\phi||^{2i}||\phi||^{-2i} = \sum_{i=0}^{n} a^{-i} < \infty$$

as $a > 1$. Thus, by (2) equation (1) has a solution for the noise $(\mu_k)$.

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